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## 1st INTERNATIONAL SYMPOSIUM
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## ON OPERATIONAL RESEARCH

**September 22 - 24, 2011**
**Thessaloniki, Greece**
**University of Macedonia, Economic and Social Sciences**

# PROCEEDINGS
## Volume 2

**EDITORS**

**Vassilis Kostoglou**
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# FORWARD

The 1st International Symposium and 10th Balkan Conference on Operational Research (BALCOR 2011) is held in Thessaloniki, Greece in the premises of University of Macedonia from 22 to 24 September 2011. The Conference has been organized by the Macedonia – Thrace Branch of the Hellenic Operational Research Society (HELORS) and the University of Macedonia.

The international academic and business communities have shown significant interest for submitting their research work and participating in BALCOR 2011. One hundred (100) articles have been included after review in the conference proceedings, originating from twenty (20) countries. Due to the large number of articles, the conference proceedings consist of two volumes including 44 articles in the first volume and 56 articles in the second volume.

The present volume also includes the organizing and the international scientific committees, the secretarial and technical support teams, and the sponsors of the conference. The main section of the conference proceedings includes the lectures of the distinguished invited speakers, followed by the contributed articles, classified in main research topics and within them in ascending order of first author's surname.

We would like to express our appreciation to all conference delegates for the high level of their submitted articles which are the most reliable guarantee for the success of BALCOR 2011, as well as for their active participation. Without their significant interest the production of these proceedings would not have been feasible. Last but not least we would like to thank the sponsors of the conference.

The editors of the Conference Proceedings

Vassilis Kostoglou

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# 1st International Symposium
# &
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# CONFERENCE PROCEEDINGS
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# CONTRIBUTED ARTICLES

# STATISTICS – DATA MINING – SIMULATION

# Performance management system implementation: Case study from transitional country

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**Abstract**

The purpose of this paper is to make a contribution to the performance measurement system (PMS) literature by concentrating on the transitional country context. Providing a Serbian perspective on PMS implementation, the study aims to identify important factors that influence PMS implementation in transitional countries.

Performance management system (PMS) improves results of an organization. Developing countries and countries in a transition from planned to market economy, need much higher increase rate of GDP and greater effectiveness and efficiency then developed countries in order to raise the level of their economy and to become much closer to the economic level of developed countries. The purpose of this article is to describe the PMS application and analyse the consequences of implementation and obstacles for PMS implementation in one transitional country like Serbia.

**KEYWORDS**

Performance management system; Controlling; Serbia.

## 1. INTRODUCTION

Performance management system (PMS) is of vital importance for an overall successful work of an organization. Performance management is the process of quantifying action, where measurement as a process of quantification and action leads to performance (Neely et al., 1995). In Serbian organizations PMS has not been well developed and is still influenced by the transition from planned to market economy. For the effectiveness of companies in developing countries it is important to improve their PMS and in doing this it would be wise to consider the status of PMS implementation and influence of existing controlling systems on establishing adequate PMS system.

Lack of knowledge on the implementation of PMS in Serbia and the importance of strengthening the practice of performance measurement and control in domestic companies is therefore the underlying motivation of this paper. The specific aim of this paper is to present on analysed case studies reporting and controlling system as a vanguard of PMS implementation.

Contribution of this work is related to the analysis of PMS implementation in selected studied organizations, influence of controlling system, ISO9001 implementation, motivation for existing systems of reporting and obstacles for wider performance measurement implementation.

## 2. REVIEW OF THE LITERATURE

The development of internal capabilities is strongly influenced by competitive priorities such as cost, quality, flexibility and delivery, which are at the heart of an organization's operations strategy (Boyer and Pagell, 2000). To control the obtainment of satisfactory performance (of cost, quality etc) is therefore important for a company. Shi (2007) and Mc Ivor et al., (2009) found that most of the organizations have not developed systematic and effective controlling and performance measurement system. It was recognised by Garengo and Bititci (2007) that one of the main barriers in implementing PMS is lack of adequate Management Information System (MIS). Implementation of PMS is closely connected with different cultural and

economic heritages of specific country and therefore there is a lot of doubts about the possibility of existence of one universal system (Neely, 2005; Bititci et al., 2000; Kennerley and Neely, 2002; Moxham, 2009).

Organizations traditionally, especially in transitioning countries, design their PMS based on controlling system of accounting and finance parameters and basically on system for tracking change and trends in performance over time (Hayes et al, 2005; Fine and Snyder, 1999). Performance measures that are focused on financial metrics are used for a long period of time to provide operational control and external financial reporting in private sector organisations (Kuwaiti, 2004). New market development and globalisation force companies to consider their performance in terms of quality of service, flexibility, customization, innovation and rapid response (Neely, 1999).

There is a belief that what an organization should measure depends on what it is trying to achieve (Neely and Austin, 2002). The continual updating of relevant measures is viewed as essential (Kuwaiti, 2004). A vast number of performance measurement frameworks aimed at evaluating private sector performance have been developed (Moxham, 2009). The majority of private sector PMS are developed internally and rely on self-assessment, the results of which are disseminated through financial controlling and reporting mechanism (Kennerley and Neely, 2002). There is a lot of skepticism for the implementation of managerial processes and practice from private to public sector. For example in public sector in Serbia it is very hard to implement PMS which heavily depends on the politico-administrative structure and willingness of politicians (Šević, 2005; Moxham, 2009).

Large organizations are more likely to have time and resources to implement new knowledge and new methods to advance lead performance measurement and benchmarking (Anderson and Mc Adam, 2004). Research studies indicate that PMS can play a key role in supporting managerial growth in Small and Medium Enterprises (SME) (Garengo et al., 2005). Companies who have implemented ISO9001 standards and TQM principles also have a higher level of potentials for PMS implementation (Neely et al., 2001).

In the analyzed literature (Boyer and Pagel, 2009; Neely at al. 2005; Mc Adam et al. 2008; Bititci et al. 2005; de Brujin and van Helden 2006; Neely 1999; Forza and Salvador 2009; Franco-Santos et al. 2005; Najmi et al. 2005; Costa et al., 2004; Takim et al., 2003) various prescriptions considering PMS characteristics are presented like cost, quality, strategy, customer focus, improvement, flexibility, innovation, technology development, supplier development, customization etc.

## 3. METHODOLOGY

Given the exploratory purpose of the present work and the limited knowledge of PMS implementation in Serbia, we decided to use the multiple case study approach. The selected research methodology was in accordance with interpretative research approach (Yin, 2003). Three companies, as shown in table 1, were selected to provide a data for system controlling and PMS implementation. These companies were chosen as private SME within different business sectors and with some level of foreign investors influence. Since the relevance and influence on Serbian economy of SME is huge, the analyzed data is important for domestic entrepreneurial market. Interviews in analyzed companies were held with 7 managers most responsible for the main processes in analyses companies.

Table 1. Case organizations

| Type | Activity | Size (turnover in 2008 in mil Eur) | Ownership | Employees |
|---|---|---|---|---|
| Distribution | Sale and distribution for construction | 8,5 | Private 50% domestic, 50% foreign | 54 |
| Production | Production for construction | 2,5 | Private 25% domestic, 75% foreign | 12 |
| Construction | Execution of works | 2,0 | Private 100% domestic | 6 |

Information was collected by interviews with employed staff, top management and stakeholders. Documents from production, sale, purchasing, planning, product division, construction site were used for analysis. Analysis of information and documents were based on the need for better quality management systems and decision support system that top management and stakeholders in companies need for the improvement of market position and efficiency.

# 4. RESULTS

PMS have been developed in studied companies internally on the basis of financial metrics and reporting mechanism. Management accounting systems in all three companies were used to provide data for external financial reporting rather than those necessary for managing business. Therefore spreadsheets and diagrams from accounting systems were used as reports which had been obligatory by law. There were too many data and too little analysis in the previous period. After ISO9001 implementation and new management staff collected data were analysed and used for business improvement.

As traditional family companies, they expressed interest in establishing PMS, but tend not to adopt expensive and advanced PMS and IMS. This result is coherent with the findings of Garengo and Bititci (2007). The cost of measurement was an issue of considerable concern for managers. Lack of investments in MIS in three analysed companies was compensated with intellectual capital, creativity, ideas and implementation of most valuable theories and the development of particular systems for measurement and control with available tools.

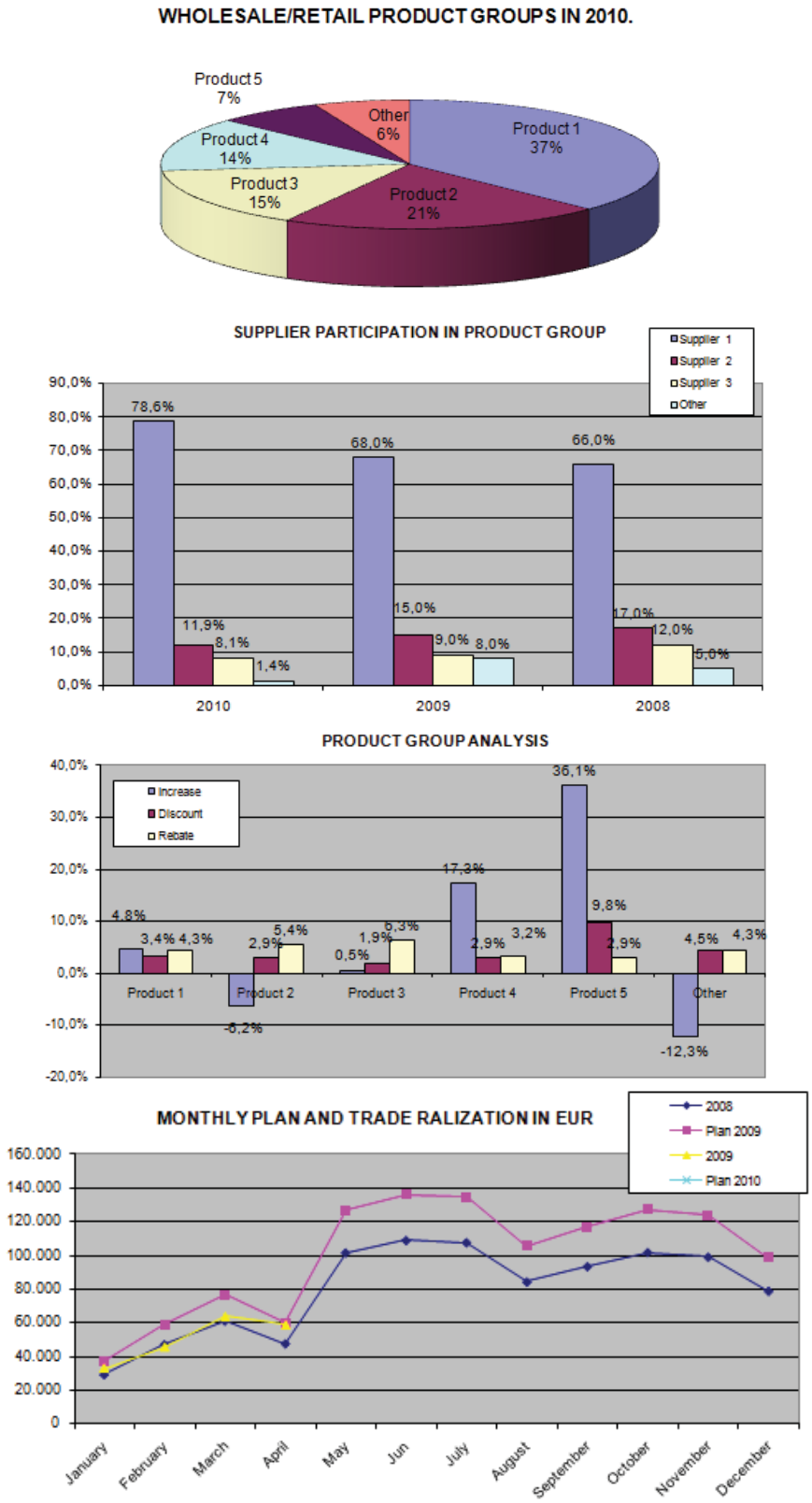
Details on performance measures, features for customer satisfaction, sources of information etc. were presented in the next part of this paper. The system created in these companies was based on regular monthly reports for the purpose of business controlling. Parameters were stored both in spreadsheets and in bookkeeping software developed commercially on domestic market. Both tracking systems were used for parallel control and comparison. The following part of this paper deals with the identification of the current characteristics of PMS implementation and management control, including the report system as well as the identification of usage and effectiveness of specific measures in different business sectors in Serbia.

## 4.1. Case – distribution.

Controlling system in distribution is tightly connected with results of product sale. Comparison of results with previous years is one of the parameters used by management to track their results. The best representation of this fact is table that is coined “to know where we are”. When we make interview with management they didn’t know anything about PMS and what elements it includes, but considering business they prepare some elements that they track. Figure 1 presents some of the graphs that were prepared for the management. Data are used for salesman salaries since their bonus on salaries was connected with results in sale of products. Before ISO9001 implementation parameters considering human resources and customers satisfaction were neglected. The only parameters that were interesting for management were financial.

Data were prepared by managers of outlets (existing 8 outlets all over Serbia), technical director and specialists responsible for technical support and sales of products according to different groups. Data are presented in datasheets and graphs. The stated parameters serve for internal comparison between sales points which makes a solid competition. A number of business decisions was made through regular monthly meetings of directors with sales outlets managers, a set of proposed measures towards suppliers was established, a position about sales of certain product on a given territory was taken, the questions of quantity of stocks was reviewed, trends were followed, problems and causes were established and possible ways of their removals, as well as business improvement were defined.

Figure 1. Reports for case distribution

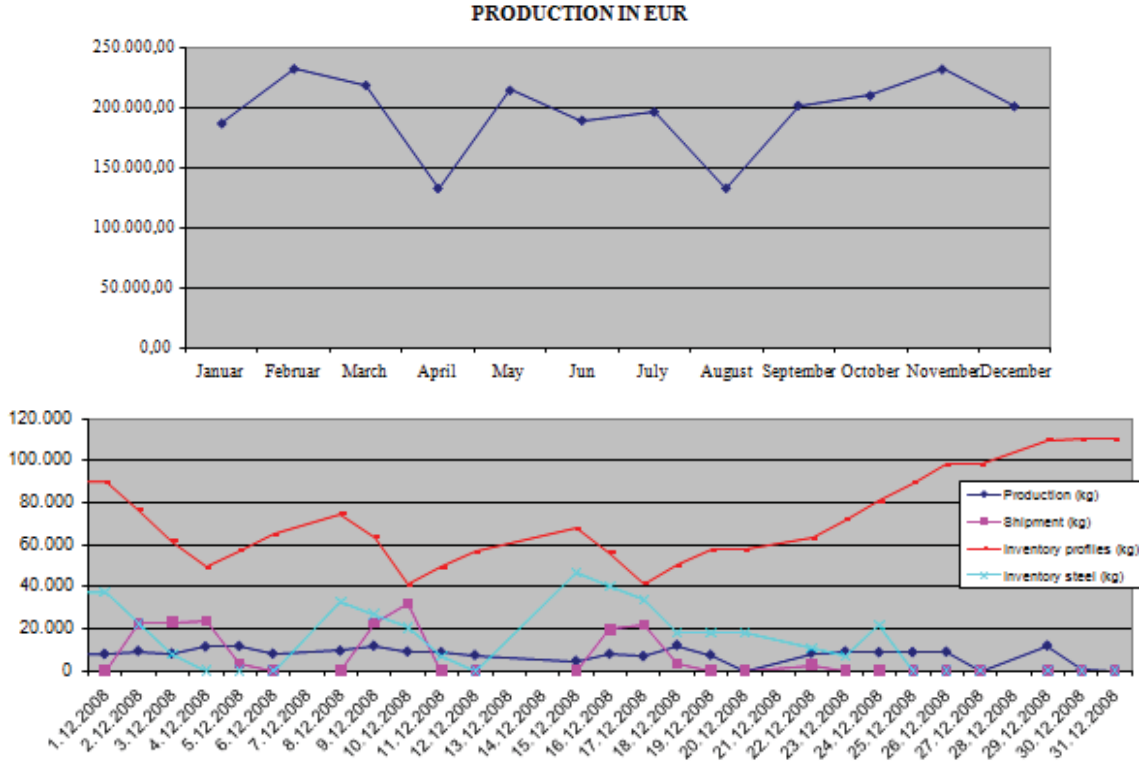


## 4.2. Case – production

Management in production Company is primarily interested in financial results and controlling of production quantities. Daily production data was prepared for graphs of production status. Status of raw material and finished stock products was of great interest for management. Cost of produced goods was tracked for the purpose of price calculation as well as monthly expenses of business process. In order to track customers' participation and sale on domestic and foreign market, reports include quantities of sold products per customer. Figure 2 presents graphs from monthly reports.

Data were prepared by the general manager, production manager and execution manager in charge of purchasing and selling. Collected data presented in graphics are used for management decisions. Shareholders (domestic and foreign) receive monthly reports with all prepared data. ISO standardization for company processes raise the level of PMS understanding especially in the field of non financial parameters.

Figure 2. Reports for case production

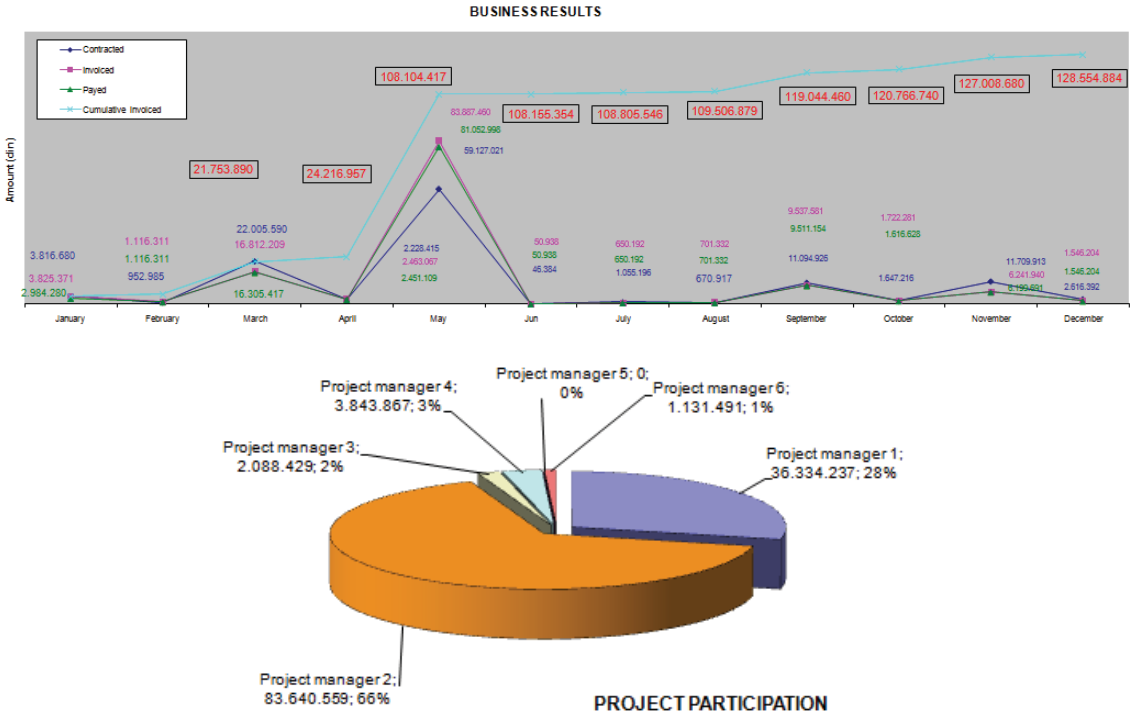



## 4.3. Case – construction.

Performance measures in execution of works are connected with controlling of financial parameters of project realization. All data are divided according do different projects and project managers that are responsible for project (figure 3). Collected data in datasheets represent project costs and total revenue per project. One of the most important elements was value contracted, invoiced and charged per every project and every project manager. Financial data were tracked and compared with previous years. In the measured performance there was a lack of non financial parameters and data from customer satisfaction. After ISO9001 certification some of these elements oriented toward customer satisfaction were improved.

Data were prepared by the general manager and the technical assistant. The basic follow up of parameters of business was conducted on realisation level of works done and invoices sent. Since there was a serious problem with delays in sending invoices – by checking the situation with regard to actual works done, and with payment, a higher degree of attention was paid to control of invoices and payment by projects.

Figure 3. Reports for case construction




## 5. DISCUSSION AND CONCLUSION

The lack of resources for PMS implementation, willingness and decisions has become an obvious problem in Serbian private and public sector. Use of financial data in order to control business status of company is the most common way of using performance measurement. Also, most of enterprises still use their old fashioned reporting templates which consist of few pages of narrative report on performance with a balance sheet. All other segments of usable parameters are neglected. When ISO 9001 is implemented these facts are modified. One of the motivated elements for business measurement and control is from the shareholder and top management view connected with business data transparency and possibility for some action. The information obtained is still not used for management purposes but just to fulfil the low demands. There is a resistance to strategic changes and very slow structural changes especially in the Serbian public sector (Šević, 2005). The lack of adequate information systems and ERP solutions also represents a problem in PMS implementation and efficient Decision Support System.

All three companies were issued an ISO9001 certificate by an internationally acknowledged certification company, which made a significant contribution in introducing process approach, defining and measuring business parameters, as well as additional analysis and improvements. Measurement of non-financial parameters raised higher level of attention. Benchmarking on the domestic market with regard to the sales scope, participation on the market, some new financial quantitative parameters, especially some non-financial parameters for all the three business sectors, was very difficult to conduct.

Implemented system of performance measurement and controlling in the studied companies improved the management without high expenses. The most complex implementation of PMS was encountered in the company for selling and distribution, while fewer difficulties in implementation were encountered in company for construction works and production. Presented system was based on companies needs for controlling and it was not based PMS theory and practice.

Parameters used in the analysed companies served for: production/sales scope; market position; costs/revenue analysis; financial results analysis per employer (project manager, sales manager); product and supplier analysis. In table 2 features for the company improvement are presented for all three cases.

Table 2. Features for the company improvement

| Case: Distribution | Case: Production | Case: Construction |
|---|---|---|
| Internal comparison between sales points (outlets) | Analysis of domestic/export sale (65% export) | Project managers realisations analysis |
| Set of proposed measures toward suppliers | Analysis of the structure of customers (92,5% customers within the group in region). Other customers neglected – conclusion need for area sales manager | Analysis of works execution (work done, invoiced, paid) |
| Position about sales of certain product on a given territory; Comparison to some companies in region (benchmarking) | Analysis of unit cost structure. The import raw material participates with 91% in the unit price | Financial control by projects conducted by project managers |
| Stock quantity; Analysis of problems and causes | Domestic currency influenced the competitiveness of the product on the world market | |

The paper presents implementation of PMS for three companies within different business sectors with low level of investment in this field. Although based on a limited number of companies, our findings suggest that every company with its own specialities, development, socio-economic, cultural, ethical and other conditions need specific answers which are not based on one concept, what is of great importance for the diversity of solutions and models. The paper contributes to the presentation of the importance of intellectual – human capital in circumstances of limited financial resources for the implementation of expensive solution of PMS. We conclude that ISO 9001 is a motivation and initiation for integrated PMS system which include financial and non financial parameters. A broader implementation of PMS in Serbia should improve efficiency and data transparency of domestic companies in both private and public sectors and improve business efficiency. Higher level of PMS implementation will also raise the level of foreign direct investments.

Limitation of this study is the fact that three analysed companies are owned by the same shareholder from Serbia and thereby with the same management system, so it does not present diversity of various managements and performance measurement models. Further research should include more companies from Serbia which have implemented different PMS, control and report systems.

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# Analysis of Competition in Banking Sector of Serbia

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**Abstract**

Banks play a vital role in the economy of a country and the evaluation of their overall performances is very important. In last ten years the banking sector in Serbia has gone through the period of turbulence, changes of the ownership structure caused by privatization, organizational structure changes inside the bank institutions, as well as the changes due to the law requirements.

In this paper will be presented and discussed results of the econometric analysis of competition in banking sector of Serbia in five-year period from the year 2005 to 2009. The Panzar-Rosse H-statistics is used.

**KEYWORDS**

Banking sector, competition H-statistics.

## 1. INTRODUCTION

Serbian industry went through a long period of recession in the 90s. Banking industry was faced with decreasing of performances, undeveloped financial market, absence of an appropriate legal framework, etc. In the last decade, comprehensive changes in industrial sector, legal system and institutions, and, consequently, in the banking system, have occurred. The process of economy revitalization and restructuring of the banking system began in 2001. Out of about fifty banks owned by the state, which operated under very bad and nontransparent conditions and without trust from clients and Serbia's population at that time, now, on the same market, there are over twenty foreign-owned banks and only few domestic, private or state-owned. All of them operate under significantly different conditions, under strong control by the National Bank of Serbia (NBS), which adopted many regulations that are being applied in the European Union as well.

Decrease of the number of banks is an expected trend in the following period, through further consolidation of the banking sector, under the influence of the country's strategy on the question of its share in the ownership of banks. It is expected that further growth of banks will influence the increase of competition.

An important characteristic of banking in Serbia is constant sharpening of competition. Besides the banks that are already operating on the market, there is an increased interest of foreign banks to enter, "the last vacant market with high growth potential" (Jeremić, 2007). The strategy of the NBS was to direct foreign investors towards purchase of domestic banks, so that between 2001 and 2008 no bank licenses were issued, not counting transformed savings banks. On the other hand, expansion of banks with majority of domestic capital which managed, in 2006, to strengthen its capital base by emission of stocks at Belgrade Stock Exchange, which were extremely current in stock exchange transactions and achieved significant growth, by which they successfully competed with foreign banks, is interesting. That can certainly make them even more interesting for foreign investors.

Due to constant tendency towards concentration of banking market, there is a fear of creation of oligopoly structure. That is not the case with Serbian banking sector. On the contrary, the banks in Serbia operate in healthy and strong competition. Confirmation of his quoted statement can also be found in annual reports published by the NBS, who regularly follows the concentration level in banking sector.

In this paper will be presented the results of quantitative analysis which confirms that Serbian banks operate in conditions of strong competition recently.

## 2. BANKING SECTOR OF SERBIA

Market concentration and market shares of banks in Serbia, at the end of the year, for the years from 2005 to 2009 are given in Table 1.

Table 1. Market concentration and market shares of banks in Serbia

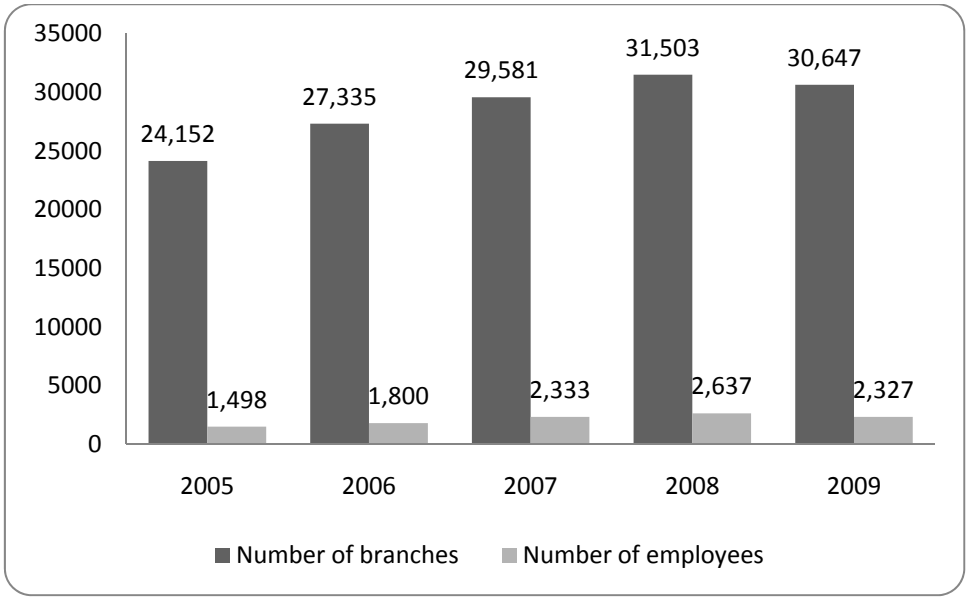
| Sector market share | 2005 | | 2006 | | 2007 | | 2008 | | 2009 | |
|---|---|---|---|---|---|---|---|---|---|---|
| | % | N° of banks | % | N° of banks | % | N° of banks | % | N° of banks | % | N° of banks |
| **More than 5%** | 52.3 | 5 | 52.6 | 6 | 52.77 | 6 | 51.62 | 6 | 57.71 | 7 |
| **2%-5%** | 18.2 | 5 | 25.8 | 8 | 35.79 | 11 | 36.27 | 11 | 29.83 | 10 |
| **Less than 2%** | 31.5 | 30 | 21.6 | 23 | 11.44 | 19 | 12.11 | 19 | 12.46 | 17 |
| **Total** | 100 | 40 | 100 | 37 | 100 | 36 | 100 | 36 | 100 | 34 |

It can be seen that the most intense competitive struggle occurs among medium banks, which managed to significantly increase their market share from 18.2% in 2005 to 36.27% in 2007. That growth was created at the expense of small banks whose share dropped from 31.5% to only 12.46%. Two medium banks even moved to the category of large banks, with 7 banks that make only 57.71% in 2009. It is clear that small banks are losing their pace more and more, while medium ones are scrambling for market share with large banks and among themselves.

As a prominent characteristic of banking and the type of competitive battle in the previous two years is also an expansive policy of bank network spreading. There was a question of how many branches would Serbian market be able to absorb. However, during 2009, due to a combination of market saturation and world financial crisis, the number of bank branches decreased by about 12%. Still, the increase of the number of branches compared to 2005 is over 50%. Competitive battle results in wide assortment and higher quality of banking services.

The number of banks' branches and employees in banking sector of Serbia at the end of the year, for the years from 2005 to 2009, are presented in Figure 1.

Figure 1. The umber of banks' branches and employees in banking sector of Serbia




Financial sector is the mirror of the condition of the whole economy, while banking sector represents its most important part. Positive correlation of banking sector development and the whole economy was proven empirically a long time ago.

## 2.1 Investigating Banking Competition

The conditions in which the banks in Serbia operate are one of the topics that are especially popular in the years of the banking sector revitalization. The main elements of the research presented in this paper are individual banks in Serbia, as participants on the observed market. Quantitative analysis of competition will be performed, by testing hypotheses on (non)existence of monopoly. The research is performed at the level of the whole sector, so that the banks that are different in their ownership structure, time spent on the market, size, business orientation, strategy, etc. are included in the analysis.

In the last 30 years in the world, many econometric models have been developed, which quantitatively evaluated conditions of modern economic systems. One of them is methodology for competition assessment using H-test, developed by Rosse-Panzar in 1980's (Panzar, J. C. and Rosse, J. N. 1982) and which leans on Chamberlain's theory of static economic equilibrium. Due to the great importance of the banking sector in the economy, as well as transparent and standardized data given by the banks, this methodology is most widely applied in that field.

The main idea of the methodology is to set a linear regression model whose parameters are evaluated and, based on which, the value of H-statistics is calculated. H-statistics is quantitative evaluation of market competition. If H is negative, the market is ruled by monopoly. Value 0<H<1 points to monopolistic competition. Some entities, banks, are fighting to be monopolists. In case of perfect competition, H=1.

Such defined H-statistic enables comparison of competition in shorter time sub-intervals of a longer period of time and in that way gives possibility to view the competition trend of the observed branch of economy during a time period.

The goal of this paper is to determine the degree of competition on the banking market in Serbia and its trend.

Revenue test by Rosse-Panzar methodology in the banking sector is based on empirical observation of two components:

- Impact of revenue at the level of one bank, depending on its input factors,
- Analysis of mutual influence of all banks on the observed market during a defined period of time

This theory was applied for the first time in banking in the works of Shafer in 1982. He modified the general equation of the model according to specific qualities of banking sector. As a response to criticism that Panzar-Rosse methodology is only efficient when the market is in long-term equilibrium, Shafer introduced a prerequisite for application of H-statistics, so-called E-statistic. It takes ROA as a dependent variable, instead of revenue; ROA depends both on total revenue and total expenses. E-statistic examines if the market is in equilibrium (E=0).

The structure of the models is as follows (Shaffer, S. 1982):

- Shafer's E-statistics:

$$lnROA_{it} = \alpha'_0 + \alpha'_1 lnPL_{it} + \alpha'_2 lnPK_{it} + \alpha'_3 lnPF_{it} + \beta'_1 lnRISKASS_{it} + \beta'_2 lnASSET_{it} + \beta'_3 lnBR_{it} + \gamma'_1 GROWTH_t + w'_{it} + \varepsilon'_i$$

$$E = \alpha'_1 + \alpha'_2 + \alpha'_3$$

- Panzar-Rosse H-statistics:

$$lnREV_{it} = \alpha_0 + \alpha_1 lnPL_{it} + \alpha_2 lnPK_{it} + \alpha_3 lnPF_{it} + \beta_1 lnRISKASS_{it} + \beta_2 lnASSET_{it} + \beta_3 lnBR_{it} + \gamma_1 GROWTH_t + w_{it} + \varepsilon_i$$

$$H = \alpha_1 + \alpha_2 + \alpha_3$$

Where, for the bank *i*, *i*=1,2,...,N and year *t, t*=1,2,...,J:

- *ROA* is the return on asset;
- *REV* is the ratio of revenue to total asset;
- *PL* is the personnel expenses to employees;
- *PK* is the ratio of capital asset to fixed asset;
- *PF* is the ratio of annual interest expanses to total loanable funds;
- *RISKASS* is the ratio of provisions to total assets;
- *ASSET* is bank size, measured by total asset;

- *BR* is the ratio of the bank's number of branches to the total number of branches for all banks;
- *GROWTH* is the rate of growth of GDP for the year *t*.

The models consist of 4 parts:

- Variables that refer to internal factors in a bank ($\alpha$ coefficients);
- Variables that refer to external factors that rule banking sector ($\beta$ coefficients);
- Variables that refer to the whole economy of a country ($\gamma$ coefficient) and
- Standard parts of all regression models (intercepts $\alpha'_0$, $\alpha_0$, observation errors that are dependent on mutual influence of banks $w'_{it}$, $w_{it}$ and errors of the models $\varepsilon'_i$ and $\varepsilon_i$).

For the needs of the analysis, it is necessary to gather data on all included entities and all indicators for all years included.

## 2.2 Competition in Banking Sector of Serbia

Due to problems with gathering consistent data that are necessary for competitive analysis, the research of banking competition in Serbia will include five-year period, from 2005 to 2009 (at the time of writing the paper, relevant data for 2010 are not available), and the sources is the NBS.

NBS, as the central bank in Serbia which is responsible for monetary policy in the country, has the role of the main supervisor of all banks. Banks are obliged to send quarterly financial reports on their business, which are then published on the NBS website. At the end of every year banks are obliged to file a report by an authorized external auditor, which represents the most reliable source of data.

On the NBS website all data from banks currently operating since 2003 are available. However, due to constant harmonization with the EU regulations, the format of the published data has been changing. Also, due to intense changes of bank owners in the period between 2003 and 2005 and merging of some banks, the analysis cannot include the period before 2005.

Out of 34 banks operating in Serbia at the end of 2009, from the analysis are excluded 4 banks for which the data are not available for all the years. Those banks are Kosovsko-metohijska bank and Jugobank Kosovska Mitrovica, which are local and do not have all data, and Moskovska and Opportunity banks started their work in 2007 and 2008. The sum of assets of all these banks is less than1% of total assets in the banking sector of Serbia in 2009, so that their exclusion from the analysis will not disturb the general image and it can be considered that the whole population is included. Finally, in the analysis presented in this paper five years from 2005 to 2009 and 30 banks operating in Serbia at the end of the considered period are considered. The analysis is conducted for the whole observed period, 2005-2009, and for two sub-periods, 2005-2006 and 2007-2009.

The analysis is performed in EViews software and fixed effects panel data models are used. The results for the tests of equilibrium (E-statistics) for the analysis performed on data for all three observed periods are given first, in Table 2. (t-ratios are given in parentheses).

The main result in table 2 is the equilibrium test, i.e. test of the hypothesis $H_0$ that E-statistics does not significantly differ from 0. Wald test is used and its results show that $H_0$ can be accepted at all significance levels for all three observed periods. The conclusion is that the banking sector of Serbia appears to be significantly in state of equilibrium. These results enable further research and application of Panzar-Rosse methodology in order to examine the competition in banking sector of Serbia.

The value of the contestability parameter H is given in the last row of the Table 3. In the same table are presented other relevant results of the analysis. For H-statistics are tested hypotheses $H_0$ (H=0) and $H_1$ (H=1) for the observed periods.

The value of H-statistics for the overall period, from 2005 to 2009, is 0.0767, positive, but near 0, and indicates that the market is not competitive. The fact that hypothesis $H_1$ for this period is rejected, for all significance levels, confirms that.

In order to analyze the problem in more details and to find trends of competition in the years of development of the banking sector, as mentioned above, the period is divided in two sub-periods and different results are obtained for them.

The value of H-statistics for the period 2005-2006 is -0.3444. Both hypotheses, H = 0 and H = 1, can not be rejected at none of the usual significance levels, but it can be concluded that the sector in this period is almost monopolistic. The absence of perfect competition and existence of monopoly in the observed period can be explained based on the data on banks' performances used in the analysis. Namely, two big banks

(Banka Intesa and Komercijalna Banka) out of 30 analyzed, stand out due to large assets and their widely spread networks. They have a share of over 30% of these indicators (additionally, shares of expenses in relation to total assets are significantly lower than for other banks). Besides, two other banks (Hypo and Raiffeisen) are singled out due to large assets, but less widely spread networks, as well as Vojvođanska banka, due to its very well developed network, but smaller assets.

Table 2. Tests of market equilibrium in Serbian banking sector

| Variable | 2005-2009 | 2005-2006 | 2007-2009 |
|---|---|---|---|
| Intercept | 0.283 | -0.317 | -2.749 |
| | (0.674) | (-0.209) | (-1.864) |
| lnPL | -0.021 | -0.054 | -0.005 |
| | (-0.731) | (-0.634) | (-0.058) |
| lnPK | 0.059 | 0.034 | 0.017 |
| | (5.034) | (2.172) | (0.669) |
| lnPF | -0.026 | 0.013 | 0.124 |
| | (-1.078) | (0.545) | (4.914) |
| lnRISKASS | 0.008 | 0.017 | -0.022 |
| | (0.503) | (0.573) | (-0.449) |
| lnASSET | -0.028 | -0.006 | -0.034 |
| | (-1.078) | (-0.068) | (-0.747) |
| lnBR | 0.014 | 0.023 | 0.137 |
| | (0.932) | (0.31) | (1.492) |
| lnGROWTH | 0.003 | 0.067 | 0.007 |
| | (2.060) | (1.042) | (3.3) |
| $R^2$ within | 0.501 | 0.749 | 0.726 |
| F-statistics | 3.153 | 1.903 | 3.895 |
| Ho : E=0 | F(1,113) = 0.1365 | F(1,23) = 0.0052 | F(1,53) = 2.1188 |
| Probability | 0.7125 | 0.94 | 0.151 |
| **E - statistics** | **0.0128** | **-0.0066** | **0.1352** |

Table 3. Tests of competition for Serbian banking sector

| Variable | 2005-2009 | 2005-2006 | 2007-2009 |
|---|---|---|---|
| Intercept | 8.533 | 10.804 | 3.223 |
| | (3.366) | (0.831) | (0.753) |
| lnPL | -0.167 | -0.974 | 0.348 |
| | (-0.961) | (-1.346) | (-1.347) |
| lnPK | 0.200 | 0.1494 | 0.212 |
| | 2.803 | (1.129) | (2.908) |
| lnPF | 0.043 | 0.481 | 0.136 |
| | 0.502 | (2.287) | (1.849) |
| lnRISKASS | -0.022 | -0.005 | 0.003 |
| | (-0.229) | (-0.018) | (0.018) |
| lnASSET | 0.349 | 0.107 | 0.741 |
| | (2.201) | (0.154) | (2.777) |
| lnBR | 0.041 | 0.633 | -0.118 |
| | (0.442) | (1.001) | (-0.882) |
| lnGROWTH | 0.011 | 0.632 | 0.003 |
| | (1.268) | (1.149) | (0.565) |
| R2 within | 0.9195 | 0.9703 | 0.9854 |
| F-statistics | 35.882 | 20.869 | 99.399 |
| Ho : H = 0 | F(1,113) = 0.1329 | F(1,23) = 0.1915 | F(1,53) = 6.6513 |
| Probability | 0.7161 | 0.6657 | 0.0127 |
| $H_1$ : H = 1 | F(1,113) = 12.239 | F(1,23) = 2.9184 | F(1,53) = 1.2843 |
| Probability | 0.000 | 0.101 | 0.2622 |
| **H - statistics** | **0.0767** | **-0.3444** | **0.6948** |

In the second period, 2007-2009, when all banks in previous years strengthened their positions and which witnessed much fiercer market fight, the value of H is 0.6948. The hypothesis H=0 can be rejected at the level 0.05 and the hypothesis that there exists the perfect competition (H=1) can be adopted at all the levels.

The situation on the market is completely different compared to previous two years. Coefficient with PL is now positive (by increasing expenses for salaries, i.e. by stimulation of working force, revenue increases as well) while others had a decrease, but are still positive.

By comparing values and tests' results for the three periods, some conclusions can be derived. Value H=0.0767 and the results of W- test, which confirms the hypothesis that in the period from the year 2005 to 2009 there exists monopoly, is in conflict with results from the last three years, then the hypothesis that there exists perfect competition on the market is accepted. The explanation of this paradox lies in the fact that the strength of monopoly from the first two years is much bigger from the strength of perfect competition from the last three years, so the end result H-statistics gravitates more towards monopoly and blurs the whole picture.

Still, the general trend is that H-statistics is rapidly growing with development of financial market in Serbia. Absolute change of value of H-statistics is 1.039. It is empirically impossible to achieve the maximum value of 1 and state of perfect competition.

The results obtained are consistent with similar researches that were conducted and that described certain rules for movements of H-statistics trend (2009). Average value of H for 25 analyzed countries in the research of Goddard, J. and J. Wilson is 0.528. For developing countries the average is 0.698, and for the developed ones 0.371. The value of H-statistic in Serbia for the second period is almost equal to the average of the developing countries, according to the mentioned research.

Based on the present research, general trend of movement of H- statistics can be seen. Shortly after structural reforms in the country, competition of banking market increased up to a point, after which there was a longer period of growth saturation and separation of 'big players' on the market. They gradually either oust or take over smaller players and, by that, decrease the strength of competition. Gradually oligopoly or monopoly appear, which can be proved by identifying decrease of competition indicators. Consequences of such trends are obvious during global financial crisis which started in 2008. However, with efficient legal mechanisms and strong institutions authorized to prevent absolute monopoly on financial markets, situations where the values of H-statistics in developed countries would reach negative values are very unlikely.

## 3. CONCLUSION

Transition as a complex process of movement from one system into another, better and more efficient system, has its advantages and disadvantages. Since transition is carried out in all levels of the society, it is difficult to scientifically measure progress in a defined period of time. Economy, as the most important part of a society, is encompassed by most extensive changes during transition. Financial sector, as the heart of economy, first started fierce reform, and it can be said that it ended it first in Serbia. Since the most has been done in transition, and U-cycle of transition is at its end, here are presented the results of the objective econometric analysis and testing of one of the most important qualities of the new system: competition.

After ten years of transition and questionable structural reforms in many spheres in Serbia, we can, with certainty, say that the banking sector in Serbia has been successfully reformed. The number of banks has almost been cut in half. The banks whose business was bad went into bankruptcy. Foreign banks arrived, which invested a lot into the development of the banking sector, as well as the economy of Serbia. NBS started successful application of standards that are in accordance with the EU regulations. It is certain that the image of the banking sector ten years ago and now are unbelievably different, and obviously, for the better. Concrete proofs for this claim are the results of Panzar-Rosse methodology for banking sector competition analysis.

Until Serbia enters the circle of developed countries, there should not be drastic decrease of competition, based on the experiences from other countries. The advantage of such market is stability of the whole system in case new participants enter, or if the existing participants crash.

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# Data Mining in Medicine: Comparison of the Results of Predictive Neural Networks on Medical Data Acquired in Three Different Time Moments

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**Abstract**

Data mining is the science of discovering new information, such as unknown patterns or hidden relationships, from huge databases. Neural networks have frequently been used to solve some predictive tasks, with high confidence. In this paper neural networks are applied on a specific medical problem. Data on patients who were hospitalized due to certain injuries are analyzed. They were diagnosed, on the hospitalization, with severe sepsis or SIRS (Systemic Inflammatory Response Syndrome). The problem of finding certain rules to determine the outcome of clinical treatment of patients is confronted. The idea is to compare predictions of the outcome, based on three different time moments, using neural networks. In the first experiment neural network is trained with data collected on the day of patient hospitalization. In the second experiment, the network is trained with data collected on the day of patient hospitalization and one day after, and in the third experiment it is trained with the data collected on the day of patient hospitalization and two days after. Data involve the results of laboratory analysis, which include blood tests, urine tests and patients age (age, hematocrit, hemoglobin, thrombocytes, leukocytes, segmentation, glycemia, urea, etc.). The goal is to find if there is any difference in those models, what are the differences if they exist and finally, which model has the best predictive accuracy.

**KEYWORDS**

Neural networks, data mining in medicine, prediction, SIRS, sepsis.

## 1. INTRODUCTION

Nowadays, increasing amount of data has been collected and real world problems have become more complex. It has become increasingly difficult for data analyst to build all the required models and manage them manually. One of the best techniques to help data analysts construct and manage large collections of models is certainly data mining.

Data mining is the science of discovering new information, such as unknown patterns or hidden relationships, from huge databases. Three ongoing projects illustrate different research challenges in data mining. First, data mining algorithms most often search humongous combinations of possibilities. A second challenge is to mine data with missing or wrong entries. A third challenge is the enormous complexity of today's databases.

Data mining is the application of specific algorithms for extracting patterns from data (Fayyad, Piatetsky-Shapiro, Smyth, 1996). Data mining is a process that consists of applying data analysis and discovery algorithms that, under acceptable computational efficiency limitations, produce a particular enumeration of patterns (or models) over the data. Note that the space of patterns is often infinite, and the enumeration of patterns involves some form of search in this space. Practical computational constraints place severe limits on the subspace that can be explored by data mining algorithms.

Since it had been invented, data dining was mostly associated with Computer Science and Business Intelligence. Those were the areas where it could be used to dig out some hidden patterns and information in lot of stored data, which could not be seen by a naked eye.

But since the Information Technologies (IT) has spread in almost every aspect of human life, the use and application of data mining has also spread along. So today we have application of data mining in marketing

and sales (Berry and Linoff, 1997), business and industry (Giudici and Figini, 2009), education (Ramaswami, Bhaskaran, 2010), crime (Wenke, Stolfo, Mok, 1999) etc.

Data mining is also widely used in medicine. Group of authors from University of Ljubljana, described various methods of data mining and explained their use in medical applications and problems (Lavrac, Zupan, 2005). There is a lot of research to prove this, so application of data mining can be found in cancer research (Hess et al., 1999), (Khan et al., 2001), orthopedic disabilities (Chan et al., 2006), clinical epidemiology (Marshall, 2001), trauma (Hill, Delaney, Roncal, 1997), cardiovascular medicine (Itchhaporia et al., 1996), etc.

Tasks supported by data mining include prediction, segmentation, dependency modeling, summarization, and change and deviation detection. Lavrac and Zupan (Lavrac, Zupan, 2005) propose that in medicine, data mining can be used for solving descriptive and predictive tasks. Descriptive data mining tasks are concerned with finding interesting patterns in the data, as well as interesting clusters and subgroups of data. Predictive data mining starts from the entire data set and aims to induce a predictive model that holds on the data and can be used for prediction or classification of unprocessed instances. Data mining has been frequently used to solve different predictive tasks (Ramaswami, Bhaskaran, 2010), (Chan et al., 2006), (Hill, Delaney, Roncal, 1997), (Khan et al., 2001).

Various data mining methods are used for predictive tasks, and the most popular are decision trees, neural networks, logistic regression, multiple regression, generalized linear models, discriminant models, support vector machine models, etc.

Neural networks have frequently been used to solve some predictive tasks. The goal is to predict with high confidence. When dozens of input parameters are included in the procedure of solving such sophisticated problem, it is clear that neural networks can be implemented (Cabarkapa et al., 2009). We find the application of neural networks in different areas of business, such as Forex market exchange rate forecasting (Medic, Krstic, Poledica, 2009), credit risk prediction (Cabarkapa et al., 2009), as well as application in medicine, such as prediction of cancers (Khan et al., 2001), in cardiovascular medical research (Itchhaporia et al., 1996), for medical decision support in clinical medicine (Forsström, Dalton, 1995), patients condition weight categorization (Simic, Simic, Milutinivic, 2009), etc.

## 2. PROBLEM DEFINITION

In this paper neural networks are applied on a specific medical problem. Data on patients who were hospitalized due to certain injuries are analyzed. They were diagnosed, on the hospitalization, with severe sepsis or SIRS.

Sepsis is a serious medical condition characterized by an inflammatory condition of the whole organism (which is called the Systemic Inflammatory Response Syndrome - SIRS) and the presence of known or presumed infection (Bone et al., 1992). Body develops inflammation in response to the microbes in the blood, urine, lungs, skin and other tissues. It appears by penetration of germs and bacteria in the bloodstream, where the bacteria live and multiply in the blood and cause blood poisoning or sepsis. In such cases, large amounts of bacteria can be found in the blood. Sepsis may be associated with clinical symptoms of systemic diseases, such as temperature, fever, malaise (general feeling of "uselessness"), low blood pressure, and psychological changes. Sepsis can be a serious condition, a disease which is life threatening. According to some data, sepsis is the third leading cause of death worldwide. SIRS is an inflammatory condition of the whole organism ("system"), without a proven source of infection. SIRS is a body condition before the sepsis diagnosis. Simply put SIRS + infection = sepsis (Bone, 1992).

Those patients were treated for seven days, and during those days, certain blood and urine tests were made (hemoglobin, hematocrit, sodium, thrombocytes, leukocytes, etc.). During those seven days patients have survived or died. Severe sepsis is a common cause of death in intensive care units, with a mortality rate exceeding 50% (Surbatovic et al., 2004).

Problem is to find certain rules to determine the outcome of clinical treatment of patients. The idea is to compare predictions of the outcome, based on three different time moments, using neural networks. In the first experiment neural network is trained with data collected on the day of patient hospitalization. In the second experiment, the network is trained with data collected on the day of hospitalization and one day after, and in the third experiment it is trained with the data collected on the day of hospitalization and two days

after. The data involve the results of laboratory analyses, which include blood tests, urine tests and patients age.

The goal is to find is there any difference in those models, what are the differences if they exist and finally, which model has the best predictive accuracy. First experiment involves next input variables: age, hematocrit, hemoglobin, thrombocytes, leukocytes, segmentation, glycemia, urea, creatinine, bilirubin, proteins, albumin, sodium, potassium, calcium, sodium phosphate, PCT (procalcitonin), C-reactive protein, partial thromboplastin time, D-dimer, plasminogen, ATIII (antithrombin), protein C. Those variables were measured on the day of patient hospitalization, so the network is trained solely with results based on the one day treatment. Second experiment involves the same variables, but both, measured on the day of hospitalization and one day after. The third experiment involves the same variables, but this time measured on the day of hospitalization and two days after. That way, the results of the analysis made in three different time moments will be available, so it will be possible to compare the results, check the differences in the models, check the differences in the accuracy of those models, and find the best prediction model. Brief preview of the data is given in table 1.

Table 1: Brief preview of input data (variables marked by 0 are data collected on the day of hospitalization, marked by 1 are data collected the day after, and marked by 2, two days after )

| Age | Group | Outcome | Hemato crit0 | Hemato crit1 | Hemato crit2 | Hemoglo bin0 | Hemoglo bin1 | Hemoglo bin2 | Thrombocyt es0 |
|---|---|---|---|---|---|---|---|---|---|
| 29 | 0 | 0 | .27 | .27 | .27 | 90.00 | 91.00 | 90.00 | 363.00 |
| 53 | 0 | 0 | .39 | .34 | .34 | 130.00 | 112.00 | 114.00 | 397.00 |
| 36 | 0 | 0 | .30 | .29 | .24 | 100.00 | 96.00 | 77.00 | 419.00 |
| 51 | 0 | 0 | .41 | .41 | .34 | 140.00 | 129.00 | 116.00 | 226.00 |
| 29 | 0 | 0 | .28 | .28 | .28 | 95.00 | 93.00 | 92.00 | 362.00 |
| 22 | 0 | 0 | .41 | .21 | .36 | 141.00 | 74.00 | 129.00 | 193.00 |
| 59 | 0 | 0 | .31 | .31 | .29 | 111.00 | 109.00 | 103.00 | 260.00 |
| 64 | 0 | 0 | .42 | .31 | .29 | 138.00 | 101.00 | 94.00 | 299.00 |
| 15 | 0 | 0 | .46 | .38 | .36 | 153.00 | 134.00 | 120.00 | 176.00 |
| 68 | 0 | 0 | .42 | .37 | .38 | 136.00 | 124.00 | 122.00 | 187.00 |
| 51 | 0 | 0 | .42 | .42 | .35 | 139.00 | 130.00 | 118.00 | 227.00 |
| 31 | 0 | 0 | .49 | .41 | .44 | 155.00 | 143.00 | 154.00 | 167.00 |
| 32 | 0 | 0 | .38 | .24 | .23 | 128.00 | 82.00 | 80.00 | 125.00 |
| 29 | 0 | 0 | .27 | .27 | .27 | 90.00 | 91.00 | 90.00 | 363.00 |
| 53 | 0 | 0 | .30 | .28 | .25 | 97.00 | 93.00 | 82.00 | 625.00 |
| 53 | 0 | 0 | .39 | .34 | .34 | 130.00 | 112.00 | 114.00 | 397.00 |
| 36 | 0 | 0 | .39 | .40 | .33 | 129.00 | 127.00 | 111.00 | 216.00 |
| 72 | 0 | 0 | .37 | .34 | .35 | 127.00 | 114.00 | 113.00 | 484.00 |
| 72 | 0 | 0 | .40 | .41 | .38 | 130.00 | 137.00 | 123.00 | 119.00 |
| 41 | 0 | 0 | .39 | .34 | .37 | 131.00 | 121.00 | 120.00 | 318.00 |
| 57 | 0 | 0 | .43 | .39 | .35 | 146.00 | 132.00 | 122.00 | 265.00 |
| 36 | 0 | 0 | .30 | .29 | .24 | 100.00 | 96.00 | 77.00 | 419.00 |

## 3. METHODS

Neural networks are one of the most popular methods for recognition, classification, evaluation, modeling, prediction and forecasting. They show good results in forecasting and modeling systems where physical processes are not clear, noisy or too complex. They are resilient to disturbances in the input data.

Architecture of simple neural network is shown in figure 1.

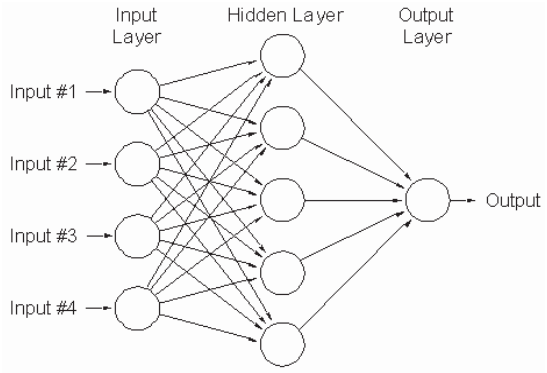



Figure 1: Simple neural network

In medical application, the most frequently used neural networks are feed-forward networks (Lavrac, Zupan, 2005). This is the type of network used in the analysis presented in this paper. They consist of number of inputs, which are interconnected forward in certain number of hidden layers. The last layer is called output layer. Each layer receives inputs from the previous layer and sends them to the next layer. Layers consist of processing elements called nodes or neurons. Neurons compute the weighted sum of inputs and filter it through a certain activation function to obtain the output, as shown in figure 2, where *i* is input, *w* is weight and *o* is output. Outputs of one layer serve as inputs to neurons of the next layer.

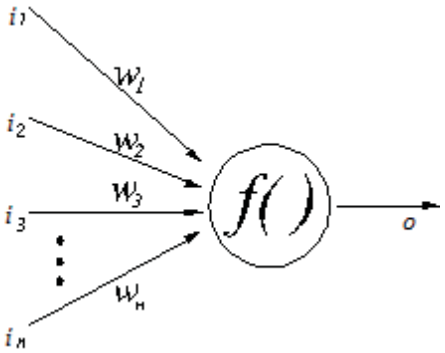


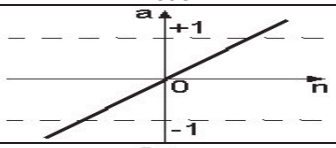
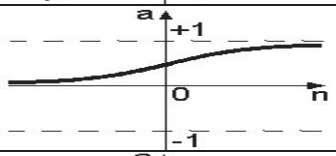
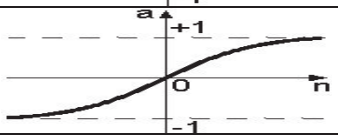

Figure 2: Processing element (neuron)

If we consider a feed-forward network with one hidden layer, the $j^{th}$ node in the hidden layer is defined as

$$n_j = f_j\left(\alpha_{0j} + \sum_{i \to j} w_{ij} x_i\right) \tag{1}$$

where $x_i$ is the value of the $i^{th}$ input node, $f_j$ is an activation function, $\alpha_{0j}$ is called the bias, the summation $i \to j$ means summing over all input nodes feeding to *j*, and $w_{ij}$ are the weights (Medic, Krstic, Poledica, 2009). Activation function can be different type of function, and the most common are step function, linear function, logistic (sigmoid) function, hyperbolic-tangent function, radial basis or radial distribution function, etc. Some of them are shown in table 2. In presented model, logistic (sigmoid) activation function was used.

Table 2: Different examples of activation functions, I-input

| Activation function | Model | Formula |
|---|---|---|
| Linear function |  | $f(I) = aI + b$ (2) |
| Logistic (sigmoid) function |  | $f(I) = \frac{1}{1+e^{-I}}$ (3) |
| Hyperbolic-tangent function |  | $f(I) = \frac{e^{I} - e^{-I}}{e^{I} + e^{-I}}$ (4) |

To obtain the output value for selected instance, its attribute values are stored in input neurons of the network, and in each next step, the outputs of the higher-level neurons are computed, until the result is obtained and stored in neurons at the output layer (Lavrac, Zupan, 2005). In our case, were we have two-class prediction problem, a neural network will have two output neurons, each modeling the probability of a specific class.

Weights that are associated with each neuron are determined from training instances (Lavrac, Zupan, 2005). The most popular learning algorithm for this is back-propagation, which initially sets the weights to some arbitrary value, and then considering one or several training instances at the time adjusts the weights so that the error (difference between the expected and the obtained value of neurons at the output level) is minimized. The activation function most commonly used in back-propagation algorithm is logistic (sigmoid) function. Such a training step is repeated until the overall prediction error across all of the training instances falls below some specified threshold (Lavrac, Zupan, 2005).

Neural networks are usually vague, and their interpretation is often very unclear and transcendental. It is often difficult to explain how or why a certain prediction was made. Method to relieve this limitation used in this paper is exhaustive pruning. It is based on pruning of the connections between neurons to procure rather accurate but less complex neutral network. It starts with a large network and prunes the weakest units in the hidden and input layers as training proceeds. Network training parameters are chosen to ensure a very comprehensive search of the space of possible models to find the best one. This method is usually the slowest, but it often gives the best results.

## 4. RESULTS

In the research presented in this paper, neural networks are used for prediction for several reasons. Firstly, they are a powerful modeling technique. Secondly, they are successful in dealing with noisy and missing data. Thirdly, they can model complex relationships. Neural networks were modeled in SPSS Clementine 12.0, a data mining software tool by SPSS Inc., an IBM company.

The type of all inputs, in all three experiments, is continuous. The type of output in all experiments is categorical. The target is a binary outcome and the predicted values are discrete with values 0 and 1. All inputs were normalized during the network training.

We used feed-forward neural network for all models, with back-propagation learning algorithm. Each of the three models randomly splits the data into separate training and validation set on one hand and testing set on the other, for purposes of model building. The network is trained on the training set, and accuracy is estimated based on the test set. The proportion of the data to be used for training is 70%, and the remainder of the data is used for validation. Method used for training neural networks is exhaustive pruning. This method starts with a large network and prunes the weakest units in the hidden and input layers as training proceeds.

## 4.1 Experiment #1

In the first experiment, architecture of the trained network consist of input layer with 23 neurons, two hidden layers with 30 and 16 neurons respectively, and output layer with 2 neurons. Importance of input variables is shown in table 3 (23 variables of which we show 9 with importance more than 0.04), and network accuracy in table 4.

Table 3: Variable importance

| No. | Nodes | Importance |
|---|---|---|
| 1 | Sodium0 | 0.1877 |
| 2 | Age | 0.1055 |
| 3 | Bilirubin0 | 0.0872 |
| 4 | Albumin0 | 0.0865 |
| 5 | SodiumPhosphate0 | 0.073 |
| 6 | CReactiveProtein0 | 0.0536 |
| 7 | Potassium0 | 0.0482 |
| 8 | PCT0 | 0.0479 |
| 9 | Segmentation0 | 0.0429 |

Table 4: Accuracy results for the first experiment

| **Results for output field Outcome** | | |
|---|---|---|
| | | |
| Comparing $N-Outcome with Outcome | | |
| Correct | 99 | 94.29% |
| Wrong | 6 | 5.71% |
| Total | 105 | |
| | | |
| Coincidence Matrix for $N-Outcome | | |
| | $N-Survived | $N-Died |
| Survived | 82 | 1 |
| Died | 5 | 17 |

Number of records is 105, out of which 99 records (94.29%) are predicted correctly, while 6 are predicted wrong. In Coincidence Matrix for $N-Outcome (predicted Outcome) it is shown that 82 patients who survived and 17 who died are correctly predicted. Death is falsely predicted for 1 patient who survived and survival is falsely predicted for 5 patients who died. Analysis accuracy is 94.29%.

## 4.2 Experiment #2

In the second experiment, architecture of the trained network consist of input layer with 43 neurons, two hidden layers with 30 and 20 neurons respectively, and output layer with 2 neurons. Importance of input variables is shown in table 5 (43 variables of which we show 13 with importance more than 0.03), and network accuracy in table 6.

Number of records is 105, out of which 101 records (96.19%) are predicted correctly, while 4 are predicted wrong. In Coincidence Matrix for $N-Outcome (predicted Outcome) it is shown that 80 patients who survived and 21 who died are correctly predicted. Death is falsely predicted for 3 patients who survived and survival is falsely predicted for 1 patient who died. Analysis accuracy is 96.19%.

Table 5: Variable importance

| No. | Nodes | Importance |
|---|---|---|
| 1 | Age | 0.0853 |
| 2 | SodiumPhosphate1 | 0.0788 |
| 3 | Sodium0 | 0.0618 |
| 4 | Glycemia0 | 0.06 |
| 5 | SodiumPhosphate0 | 0.0582 |
| 6 | Thrombocytes1 | 0.0407 |
| 7 | Thrombocytes0 | 0.0385 |
| 8 | Albumin0 | 0.0382 |
| 9 | ProteinC1 | 0.0357 |
| 10 | Leukocytes1 | 0.0331 |
| 11 | Ddimer0 | 0.0331 |
| 12 | Creatinine0 | 0.031 |
| 13 | Bilirubin1 | 0.0302 |

Table 6: Accuracy results for the second experiment

| **Results for output field Outcome** | | |
|---|---|---|
| Comparing $N-Outcome with Outcome | | |
| Correct | 101 | 96.19% |
| Wrong | 4 | 3.81% |
| Total | 105 | |

| Coincidence Matrix for $N-Outcome | | |
|---|---|---|
| | $N-Survived | $N-Died |
| Survived | 80 | 3 |
| Died | 1 | 21 |

## 4.3 Experiment #3

In the third experiment, architecture of the trained network consist of input layer with 18 neurons, two hidden layers, both with 1 neuron, and output layer with 2 neurons. Importance of input variables is shown in table 7 (18 variables of which we show 13 with importance more than 0.03), and network accuracy in table 8.

Table 7: Variable importance

| No. | Nodes | Importance |
|---|---|---|
| 1 | PCT2 | 0.1436 |
| 2 | ProteinC2 | 0.1215 |
| 3 | SodiumPhosphate1 | 0.0986 |
| 4 | Glycemia2 | 0.0871 |
| 5 | Creatinin2 | 0.0732 |
| 6 | CReactiveProtein0 | 0.0545 |
| 7 | Age | 0.0515 |
| 8 | Sodium0 | 0.0493 |
| 9 | Segmentation1 | 0.0468 |
| 10 | Segmentation0 | 0.0435 |
| 11 | Hemoglobin1 | 0.0395 |
| 12 | Creatinine0 | 0.0346 |
| 13 | Potassium2 | 0.0317 |

Table 8: Accuracy results for the third experiment

| **Results for output field Outcome** | | |
|---|---|---|
| | | |
| Comparing $N-Outcome with Outcome | | |
| Correct | 102 | 97.14% |
| Wrong | 3 | 2.86% |
| Total | 105 | |
| | | |
| Coincidence Matrix for $N-Outcome | | |
| | $N-Survived | $N-Died |
| Survived | 83 | 0 |
| Died | 3 | 19 |

Number of records is 105, out of which 102 records (97.14%) are predicted correctly, while 3 are predicted wrong. In Coincidence Matrix for $N-Outcome (predicted Outcome) it is shown that 83 patients who survived and 19 who died are correctly predicted. Survival is falsely predicted for 3 patients who died. Analysis accuracy is 97.14%.

# 5. CONCLUSION

The main results of the conducted research are the following:

Firstly, the results show that neural networks are promising tool for prediction of treatment outcome. In each experiment, prediction accuracy is found to be very high (94% - 97%). Coincidence matrix shows excellent results of the analysis as well. Those results are very good, and they confirm that neural networks should be considered as a powerful tool for prediction and forecasting in medical applications.

Secondly, the results of comparative analysis show that the first experiment, in which the network was trained with data collected on the day of patient hospitalization, gives us the accuracy of 94.29%. Second experiment, in which the network was trained with data collected on the day of hospitalization and one day after, gives us the accuracy of 96.19%. Third experiment, in which the network was trained with data collected on the day of hospitalization and two days after, gives us the accuracy of 97.14%. It is pretty clear that if more data are used for analysis, in the sense of more blood and urine medical reports, collected during the longer treatment period, more precise model will be created. Still, even though the accuracy is increasing, the question: “Is the accuracy add-value good and strong enough to counteract the expenses and costs of medical reports?” remains. In the first experiment the smallest amount of input data were used, and even if three times more data were available in the third experiment, the added value to the accuracy is only 2.85%. It certainly is significant from the outcome and the severity of the problem point of view, but is it significant from the the financial point of view? Further analysis can be conducted in the same direction.

Thirdly, the importance of input variables is not the same in every conducted experiment, for all measured blood and urine parameters. Sodium, sodium phosphate and age appear in each model on the top of importance ranking. The parameters of medical report that also appear often are glycemia, C reactive protein, PCT (procalcitonin), segmentation. For example, procalcitonin has shown to be the parameter that indicates sepsis (Ivancevic et al., 2007) so it may not be odd to refer to it as a death parameter. Further medical tests and analysis of these parameters can be suggested. If so, it could be explained why they are shown as the most important; what are the measures to be taken, in order to prevent the fatal outcome; attention could be concentrated on additional monitoring of these parameters; and mildly, certain expenses of medical analysis can be cut.

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# On Application of Discrete Event Simulation in Armoured and Mechanized Units Research*

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**Abstract**

The results of armoured and mechanized units research by means of discrete events simulation have been presented. The methodology and the research basic concepts: armed mobile platform and unexpected conflict have been first considered. A short survey of characteristic results of the research so far has been given: simulations of two tanks sudden conflict, impact of the main weapon choice to the conflict issue, ammunition expenditures during conflict, armoured battalion local area radio computer network operation and armoured battalion swarming. Finally, the basic directions of further research have been given. The research is underway at Union University School of Computing in Belgrade.

**KEYWORDS**

Discrete event simulation, armoured and mechanized units, research

## 1. INTRODUCTION

The armoured and mechanized units (AMU) research by means of discrete events simulation based methodology is underway at the Union University School of Computing in Belgrade. The basic motivation for the research is in the fact that AMU and main battle tank, as their representative paradigm, in spite of almost 100 years since their introduction in active service in armies throughout the world, have been and still are one of the key resources of the contemporary states armed forces, due to their great capabilities and versatility.

On the other hand, unconventional and asymmetric warfare, so prevailing in our time, have questioned usability of traditional armoured and mechanized units. That opens two directions of further activities in the area: development of new, essentially different generations of armed mobile platforms (AMPs), and research of new tactical and operational procedures in their use.

Finally, armoured and mechanized units are powerful, but very expensive resource. A huge amount of financial assets has already been invested in thousands of existing tanks, their production and maintenance evolve too many people, and all that represents an important economic factor, which should not be neglected.

For small countries, like Serbia, the research in this area can provide answers to many important questions, such as the issues of the existing armoured and mechanized units' rationalization, new generation of tanks development, production and maintenance, and particularly, conceptualizing of the new tactical and operational procedures, such as the swarming tactics. One should bear in mind that Serbia is one of few states which have relevant experts and facilities for upgrading certain types of armoured fighting vehicles and their provision with equipment appropriate for future war conflict challenges, which opens possibilities for introduction and competitiveness in the world market.

The aim of this paper is to present the methodology, basic concepts and some of the results of the research, achieved in the last decade, as well as to point out the main directions of further work in this area.

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## 2. MILITARY SYSTEMS RESEARCH METHODOLOGY

The authors' methodology for research and development of an armed mobile platform (AMP), as one of the most important ordnance systems, has been depicted in Fig.1. Computer simulation of the future AMP's mission is a central activity in the methodology, and the results of the experiments executed by means of the realized simulators should answer the question whether the satisfactory performances of the AMP are achieved, along with simultaneous fullfilment of technical and economic criteria, i.e. could the avaliable industry and the state power provide sustainable development and exploitation of such combat system.

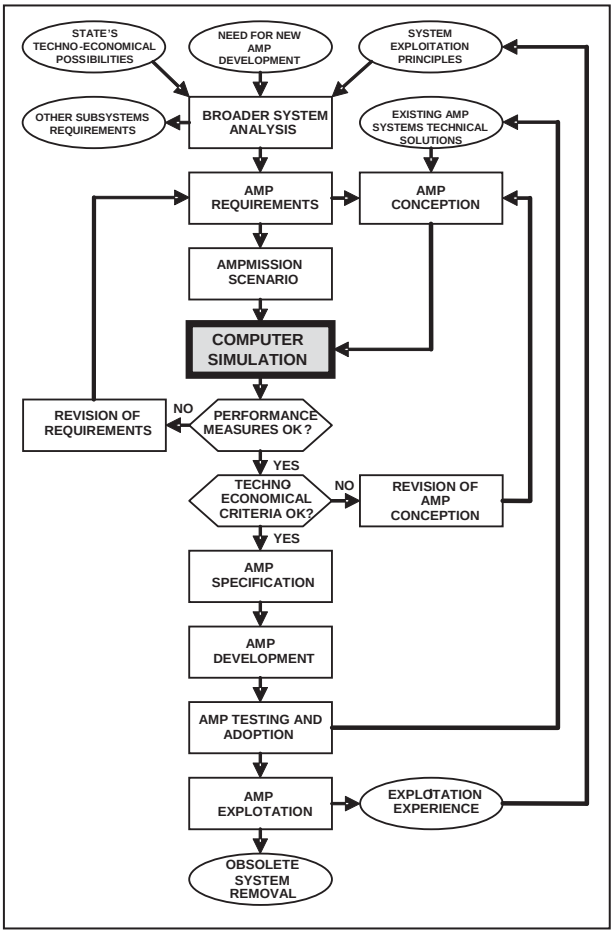


Fig.1 AMP systems /design methodology

One can go several times through different phases of the methodology, until the right solution, satisfying both groups of essential conditions, is achieved. It is only then when that the right result is obtained – the future system specification, which later goes through usual phases of development, testing and adoption in ordnance systems. Once adopted, the new AMP enriches the existing solutions knowledge base with new concepts, which has an important ipact to future solutions of such and similar complex military systems. It is of particular importance that the methodology includes and enriches the existing technical solutions set, both during and after the system's life cycle.

On the other hand, the experience acquired during research and development, as well as in course of testing and later exploitation, has been incorporated in the methodology, and as such influences in certain extent the critical analysis of the existing and creating new principles of complex military system combat use. Because of such widely stated goals and complexity of the research object, the methodology leans on the operations research techniques, and in particular discrete events simulation, having the central place in it.

## 3. THE ARMED MOBILE PLATFORM (AMP) CONCEPT

The armed mobile platform (AMP) concept has been introduced in the research, as an abstraction suitable for research and development of complex military system class, which has the following common properties: autonomous propulsion, man crew, armament and significant need for logistic support. The complex military systems that the AMP concept can be used for research and development may, for example, be: warship, armed helicopter, armed combat vehicle and many other.

The environment of an AMP, as military system represent: adversary, friendly forces and the space (territory, aquatory or air space in which AMP and different targets and threaths are moving). The AMP subsystems of interest for research and/or development are most often: propulsion, embedded command information system, armament and logistic support.

The AMP mission is the voyage to pass, from base to target and back to base. During that voyage, some other task execution is planned (some targets destruction, etc.). Target is adversary unit, system or device that can be destroyed or damaged by means of weapons at disposal to AMP, and threat is adversary unit, system or device that can destroy or damage the AMP. Target/Threat (T/T) is a concept which has been introduced, because the majority of threats can be targets at the same time. The unexpected conflict of two AMPs is a particular class of

extraordinary events in the simulation model of AMP mission, when there is conflict of interests of the AMP (to achieve the basic goal of the mission, i.e. to destruct all planned targets while preserving itself from destruction) and those of different adversary target/threats which tend to prevent the AMP, destroy or damage it. That concept is of particular importance for the research of armoured and mechanized units.

## 4. ARMED MOBILE PLATFORMS UNEXPECTED CONFLICT

The two armed mobile platforms unexpected conflict [3, 4] is a concept for researching at micro tactical level the situation in which the platforms of the armoured and mechanized units complement will be most often in contemporary and future conditions of unconventional and asymmetric warfare. Basic assumptions for developing of the simulation model of two armed mobile platforms (AMP-1 and AMP-2) sudden conflict are the following:

a) AMP-1 and AMP-2 are moving by azimuths $\alpha_1$ and $\alpha_2$, and velocities $V_1$ i $V_2$ . Until the sudden conflict, they are not aware of each other presence, and are accomlishing independent missions.
b) AMP-1 is in possession of sensor S-1, and AMP-2 is in possession of sensor S-2, which are intended for surveillance and fire directing. Sensors are defined by their maximal ranges, $D_{S-1}$ i $D_{S-2}$.
c) The AMP-1 platform is in a possesion of the weapon OR-1, which can hit AMP-2 with probability $p_{OR-1}$, and AMP-2 is in a possesion of the weapon OR-2, which can hit AMP-1 with probability $p_{OR-2}$. The hit probabilities depend on the distance $D$ between AMP-1 and AMP-2. The weapons are defined by their average action preparation times $T_{OR-1}$ and $T_{OR-2}$ , maximal ranges $D_{OR-1}$ and $D_{OR-2}$, velocities of projectile flights to targets $V_{L-1}$ i $V_{L-2}$ and sizes of combat sets $BK_1$ and $BK_2$ associated to weapons OR-1 and OR-2.
d) The coflict between AMP-1 and AMP-2 occurs when at least one of them detect the adversary by means of its sensor and if, due to further moving, their distance decreases till the range limit of at least one of the weapons, OR-1 and/or OR-2. The conflict stops when one of the AMPs is destroyed by the adversary's fire or when, due to further moving, distance increases out of range limits of weapons and sensors.

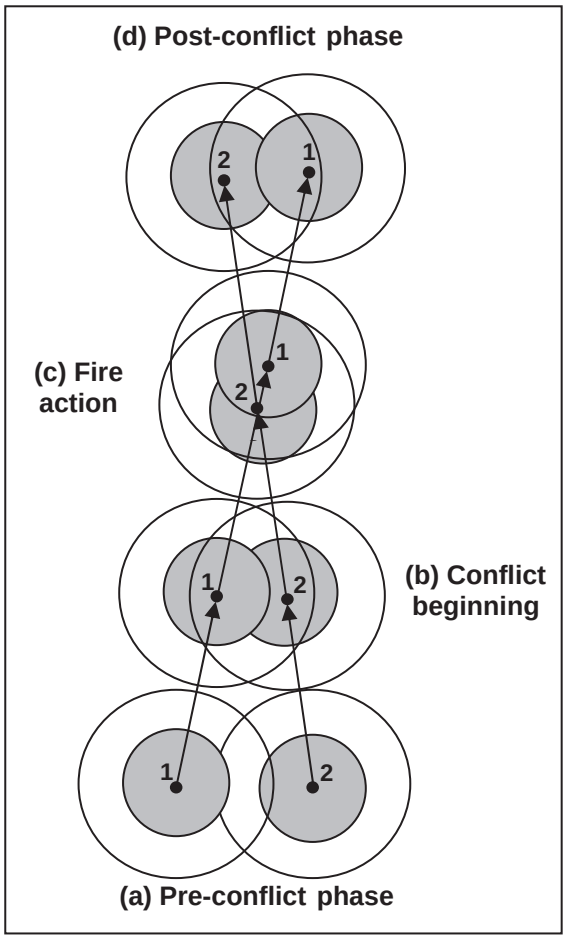

Fig. 2. Two AMPs unexpected conflict phases

In Fig. 2, the phases of two AMPs conflict have been depicted. The platforms AMP-1 and AMP-2 have been presented by their positions in the space they are moving in, zones of their detecting and fire directing sensors by clear circles, and zones of their weapons possible actions by shaded circles.

Armed mobile platforms in conflict are the discrete dynamic system. In the simulation model, armed mobile platforms AMP-1 and AMP-2 are moving in 2D space, which is represented by Cartesian coordinate system ($x$,$y$), in which the $y$ axe is oriented towards North. That assumption is valid in the majority of cases (always for warships, and almost always for armoured fighting vehicles). If the 3D moving AMPs are introduced in the model (aircrafts, helicopters or unmanned aerial vehicle), or if combat actions on the ground with great height differences are simulated, it can be introduced in already realized simulators without major difficulties.

During the conflict, the azimuths ($\alpha_1$ and $\alpha_2$) of the AMP-1 and AMP-2 directions of moving, as well as their velocities, do not vary. This has been assumed, because the armed mobile platforms sudden conflict is rather

random than planned event, while the intention of every AMP is to continue its mission which has been going on so far. On the other hand, duration of the conflict itself and the weapon characteristicts nearly exclude the posibility of manoeuvring during the conflict. Current positions of armed mobile platforms are represented by points AMP-1 ($x_1$,$y_1$) and AMP-2 ($x_2$,$y_2$), and the trajectories of their movements by equations of straight lines, or any other law of motion chosen. The simulation model of AMP-1 and AMP-2 conflict is discrete and dynamic one, oriented to events. Activities in the system are represented by pure time delays.

Simulation of two AMPs conflict is one of important activities in the research, because the main purpose of AMP, as complex military system is combat itself. i.e. conflict with adversary AMP. The goal of such simulation models development is to explore, by means of executing experiments with simulators, the impacts of various parameters of AMP as a system to the course and issue of the conflict and to make adequate decisions on AMP's technical solutions and priciples of combat use.

# 5. SOME OF THE RESEARCH RESULTS SO FAR

## 5.1 Two mainbattle tanks unexpected conflict simulation

Tank is an offensive combat vehicle, which means that its front side is best protected, lateral sides somewhat less, and the rear side is least protected [6, 7]. The engine and transmission are located in the rear part of the tank, and is mostly exposed at the rear side, less at lateral sides, and minimally at the front side. The propulsion mechanism (tracks, drive sprocket, road wheels) is least exposed at the front side, less exposed at rear side and most exposed at lateral sides. The turret drive is most exposed in the area between the hull and the turret. Having in mind tank dimensions, layout of the areas particularly important for tank operation and their sensitivity to hits, the tank in simulation has been firstly approximated by means of a parallelepiped which can contain its most visible parts (Fig. 3). If the tanks in simulation are presented by means of parallelepipeds with dimensions *a*, *b* and *c*, then the main weapon of the T-1 tank at the distance *d* should hit the T-2 tank total exposed area, which depends on its motion direction and position relative to T-1. In the improved simulation model [8, 9], besides tank dimensions, introduced through side areas of the parallelepiped approximating it, form factors have also been considered, defined by the expression:

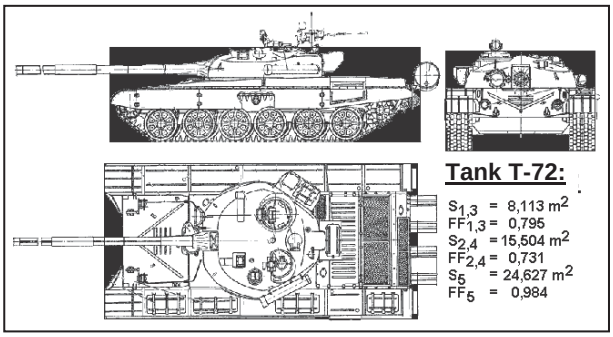

Fig.3 Tank aproximate areas and form factors

$$FF_{i,j} = \frac{S_{Ti,j}}{S_{i,j}} \tag{1}$$

Where $S_{Ti,j}$ is real area of the tank's side, and $S_{i,j}$ is the area of the corresponding approximate parallelepiped side. An example of determining form factors for well known T-73 tank has been depicted in figure 4 (shaded areas represent the difference between sides of approximate parallelepiped and corresponding real areas of the tank). Some well known contemporary tanks approximate areas and form factors [8,9] have been given in table 1. Index 1,3 from the table refers to front and rear side of tank, index 2,4 to lateral sides, and index 5 to tank's upper surface.

Table 1. Approximate areas and form factors

| Tank type | Country of origin | $FF_{1,3}/S_{1,3}$ [m$^2$] | $FF_{2,4}/S_{2,4}$ [m$^2$] | $FF_5/S_5$ [m$^2$] |
|---|---|---|---|---|
| **ABRAMS M1A2** | **USA** | 0.857 / 8.711 | 0.795/22.888 | n.a. /28.987 |
| **ARIETE 6** | **Italy** | 0.832/ 9.025 | 0.839/18.975 | 0,993/27.400 |
| **LEOPARD 2** | **Germany** | 0.856/11.100 | 0.849/23.100 | 0,936/28.490 |
| **T-90** | **Russia** | 0.808/7.908 | 0.739/16.000 | 0,931/25.510 |
| **M-84** | **Russia/ Serbia** | 0.795/8.113 | 0.731/15.504 | 0,984/24.627 |

In simpler versions of the simulator, the winning probabilities of T-1 and T-2 tanks in conflict have been expressed only through the exposed areas hitting probabilities of the main weapons $p_1$ and $p_2$, as functions of distance. In the improved simulation model, after introducing the impacts of tanks' dimensions and form factors, the winning probabilities of the tanks T-1 and T-2 have been expressed as following:

$$P_{T-1} = \frac{FF_{2/1,3}S_{2/1,3}|\cos\beta_2| + FF_{2/2,4}S_{2/2,4}|\sin\beta_2|}{FF_{1/1,3}S_{1/1,3}|\cos\beta_1| + FF_{1/2,4}S_{1/2,4}|\sin\beta_1|}\, p_1 \tag{2}$$

$$P_{T-2} = \frac{FF_{1/1,3}S_{1/1,3}|\cos\beta_1| + FF_{1/2,4}S_{1/2,4}|\sin\beta_1|}{FF_{2/1,3}S_{2/1,3}|\cos\beta_2| + FF_{2/2,4}S_{2/2,4}|\sin\beta_2|}\, p_2 \tag{3}$$

Where $\beta_1$ and $\beta_2$ are angles of motion directions of the T-1 and T-2 tanks relative to adversary weapon's sighting lines. In course of the research, many experiments have been executed by means of the programs-simulator implemented in the GPSS World simulation language, with intention to determine various factors impacts to the issue of the two tanks unexpected conflict.

### 5.2 Simulation of the main weapon choice impact to two tanks conflict issue

We shortly present here characteristic results of the tanks T-1 and T-2 unexpected conflict simulation [10, 11]. T-1 is armed with anti-armour laser beam guided missiles and T-2 with gun firing fast armour-piercing disposing sabot shells. The tanks characteristics have been given in table 1 (approximate areas and form factors) and table 2 (other characteristics included in the model). The tanks are moving towards each other, with velocities $V_1$ and $V_2$, and without awareness of each other until the very beginning of the conflict.

Table 2. Comparative characteristics of tanks in conflict

| **Characteristic** | **T-1** | **T-2** |
|---|---|---|
| Maximal speed $V_i$ [m/s] | 18 | 18 |
| Azimuth $\alpha_i$ [$^0$] | $\alpha_1$ | $\alpha_1$+180 |
| Observation-sighting set range $D_{OSS\text{-}i}$ [m] | 5000 | 5000 |
| Main weapon range $D_{W\text{-}i}$ [m] | 5000 | 2000 |
| Hit probability $p_i$ [%] | 80 | 70 |
| Time of main weapon preparation for fire action $T_{W\text{-}i}$ [s] | 10 ± 4 | 8 ± 2 |
| Main weapon projectile speed $V_{L\text{-}i}$ [m/s] | 300 | 1800 |
| Basic load (amount of ammunition associated to weapon) | 4/ 8/ 16 | 45 |

The chosen platforms are typical representatives of two different approaches in contemporary mainbattle tanks conceptions, and the goal of the simulation was to point out the advantages of considered main armaments and the impact of dimensions and form factors to such conflict issue. The characteristic result of such simulation, probability of victory (i.e. first hit probability, depending on starting distance of the tanks in conflict), has been presented in Fig. 4.

The results indicate that at close ranges, the T-1 tank, anti-armour laser beam guided missiles, has advantage, due to greater preciseness of the missile, and smaller own silhouette. That advantage reduces with range increase, and advantage of the T-2 tank, armed with classical gun firing fast armour-piercing disposing sabot shells, increases, until its effective range limit, because due to less velocity of the T-1 tank's missile, the T-2 tank in that range of starting distances has possibility to fire more shells and increase the chance to win in the conflict. The changing zone is at range from 2000 to 2500 m, when the conflict issue is uncertain, and at greater distances, the T-1 tank has an absolute advantage, due to considerably greater effective range.

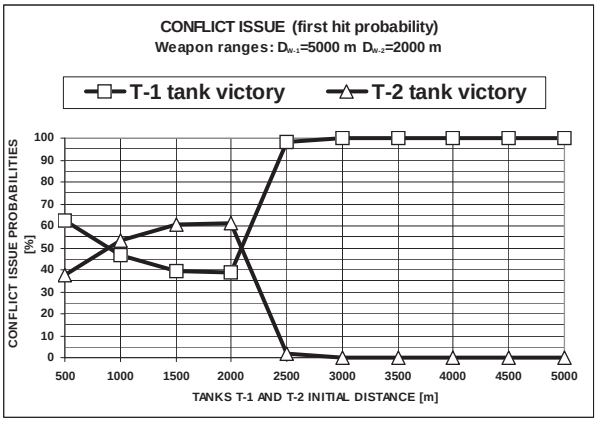


Fig.4 Unexpected conflict issue probabilities

### 5.3 Simulation of the ammunition expenditure during two tanks conflict

The ammunition expenditures of main armaments of the tanks in conflict ($UM_1$, the number of expended anti-armour laser beam guided missiles, and $UM_2$, the number of fired armour-piercing disposing sabot shells) have also been analysed in course the research so far [13,14,15]. One of the characteristic results has been presented in table 3: mean values of ammunition expenditures $UM_1$ and $UM_2$, depending on the starting distance of the tanks T-1 and T-2 in an unexpected conflict and the size of basic load of the T-1 tank, which in that consideration could have values of 4 or 8 anti-armour laser beam guided missiles.

Table 3. Mean values of ammunition expenditures $UM_1$ and $UM_2$

| $D_0$ [m] | Basic load 1 = 4 missiles | | Basic load 1 = 8 missiles | |
|---|---|---|---|---|
| | $UM_1$ | $UM_2$ | $UM_1$ | $UM_2$ |
| 500 | 1.589 | 2.079 | 1.689 | 1.926 |
| 2000 | 1.764 | 2.512 | 1.859 | 2.375 |
| 2500 | 2.108 | 2.076 | 2.252 | 1.853 |
| 5000 | 2.948 | 1.528 | 3.529 | 0.957 |

The results of the executed experiments have shown that the BK1 combat set of 4 missiles has not been sufficient, because it has happened that in spite of great starting distances, the T-1 tank has launched all 4 missiles and missed, which has given the T-2 tank chance to come closer and win in the conflict.

### 5.4 Simulation of armoured battalion local area computer network operation

For the sake of information support of the armoured and mechanized units' combat actions in contemporary warfare, it is necessary to develop command information systems of armed mobile platforms groups (CIS GAMP). The CIS GAMP primary function is regular reporting of every AMP of the group of its current position and other information of interest, for the sake of updating the electronic maps and the situation assessment in real time. For that reason, every AMP has to be equipped with computer with an electronic map on its display, GPS receiver as a sensor for determining its own position and VHF radio communication device, so that all computers of the AMPs could be connected in local area network.

The mobile local area computer network (LAN) represents one of the most important CIS GAMP subsystems, because it severely influences the system overall performance. During the RLAN research [16, 17, 18] simulation has been used to explore that influence, so that the adequate decisions could be made on technical solutions of the AMP group command-information system,. CIS GAMP operates effectively if it delivers regular AMP reports before their obsolescence. The obsolete time $t_z$ [s] has been defined by the expression:

$$t_z = \frac{P_{CIS}}{V_{AMP}} \qquad (4)$$

where $P_{CIS}$ [m] is CIS GAMP preciseness (assigned displacement of the AMP due to further motion relative to previous position which could be tolerated as no movement at all), and $V_{AMP}$ [m/s] is the platform's speed of movement within the group.

The $C^4I$ system preciseness, chosen for armoured and mechanized units, has been $P_{CIS}$ = 20m, which correspond to 3 lengths of contemporary tank.

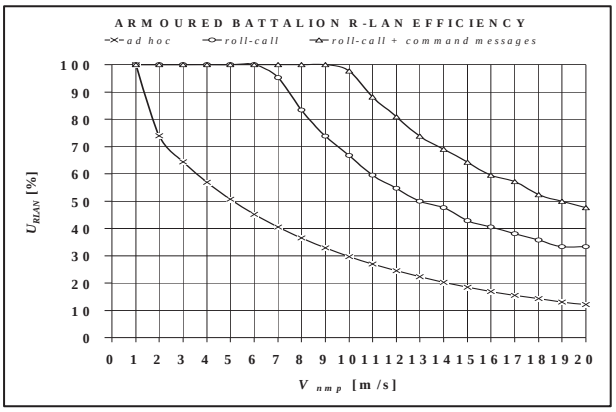


Fig.5 Armoured battalion LAN algorithm efficiency

Three different algorithms of the RLAN operation has been explored: *ad hoc* RLAN network with standard CSMA/MD algorithm [18], RLAN network with roll-call of participants [19] and RLAN network with roll-call of participants and two kinds of messages: command and reporting [20]. The characteristic result of simulation experiments has been presented in the Fig. 5.

### 5.5 Simulation of armoured battalion swarming

In military sense of the word, *swarming* [20] is a tactics by which military forces attack an adversary from many different directions, and then regroup. Repeated actions of many small, manoeuvrable units are going on, circling constantly through the following four phases of *swarming*: disperse deployment of units in battlefield, gathering (concentration) of many units on common target, action (strike or fire) at a target from all directions and dispersion of units. The way of swarming application is depicted in Fig.6, and its basic characteristics have been given in table 4.

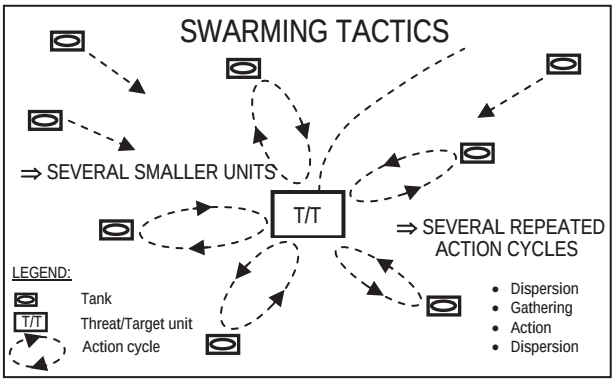


Fig.6 Armoured and mechanized units swarming

Table 4. Swarming: basic characteristics

| | |
|---|---|
| 1. | Autonomous/semiautonomous units, engaged in concentrated attack at common target |
| 2. | Amorphous, coordinated attack from all directions by continuous "pulsed" fire/shock assaults |
| 3. | Many small, space dispersed mutually networked units. |
| 4. | Integrated surveillance, sensors, and $C^4$ISR systems for upper level situation assessment. |
| 5. | Units' action capabilities, from distance as well as in direct contact. |
| 6. | Continuous attacks with aim of breaking adversaries' cohesion. |

The simulation model of the armoured battalion swarming [21, 22, 24, 25] is discrete and dynamic, oriented to events. In the model, system activities are represented by pure time delays. The moving entities in the model are: units of the battalion (AMP-*i*, *i*=1, 2…N), threat/target units and messages of the $C^4$ISR system.

The simulation goal has been to explore impact of the variable defended territory occupancy density by the armoured battalion to success of its swarming. It has been achieved by changing the dimensions of the territory in experiments, along with retaining fixed number of AMPs in the group. In the simulation, the group of the armoured battalion size has been considered, consisting of 43 armed mobile platforms (tanks, armoured personnel carriers, etc.).

The armed mobile platforms of the battalion get information of the threat/target units and other AMPs of the group motion in time expiring intervals $\Delta t$, and give reports about their own current positions. Based on that information, the armed mobile platforms, AMP-*i*, direct themselves toward threat/target unit, with the goal to reach, as soon as possible, the position enabling them to perform successful swarming, for the sake of destroying, disabling, or preventing the adversary in accomplishment of its mission.

The threat/target unit is accomplishing its own mission and, contrary to AMP-*i*, has no access to information of the $C^4$ISR system, so its primary goal is to fulfil its own task, which has been represented by motion on given trajectory between points A and B in the model, according to functional dependencies of its coordinates of time, $x_{t/t}(t)$ and $y_{t/t}(t)$. When the period of expiration elapses and $C^4$ISR system dispatches the report of the new threat/target position ($t = \Delta t$), threat/target has moved to its new position, T ($\Delta t$). Until then, AMP-*i* have moved to their new positions, 1($\Delta t$), 2($\Delta t$), 3($\Delta t$) and 4($\Delta t$), following the directions from the previous time interval, directed their velocity vectors towards new position of the threat/target, and then the process continues.

In order that individual swarming participating AMP could act upon T/T unit, the following must be fulfilled:

a. AMP-i must dispose of the main weapon MW-i, compatible with the adversary.

b. The distance between AMP-*i* and the adversary unit *j* must be in the main weapon MW-*i* range limit, i.e.:

$$D_{ij} = \sqrt{(y_j(t) - y_i(t))^2 + (x_j(t) - x_i(t))^2} \le D_{MW-i} \qquad (5)$$

c. The a. i b. conditions must be satisfied by enough other AMPs of the group, so that their total cumulative effect on threat/target, $KU_j$ should be equal or greater than critical threshold of the multiple AMPs cumulative effect, $PKU_j$, specific to threat/target unit $P_j$, i.e.:

$$KU_j = \sum_{i=1}^{N} A_{ij} \cdot K_{ij} \cdot U_{ij} \geq PKU_j \qquad (6)$$

where: $A_{ij}$ is the assignment coefficient (0/1), intended to assignment of the P-*j* threat/target to AMP-*i* in the multitarget swarming models, $K_{ij}$ is the main weapon MW-*i* compatibility coefficient with the P-*j* threat/target (0/1) and $U_{ij}$ is the possible effect of the main weapon MW-*i* on the P-*j* threat/target.

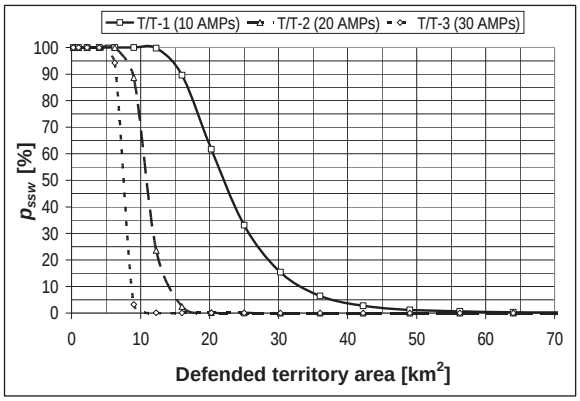


Fig.7 Successful swarming probability, function of defended territory area

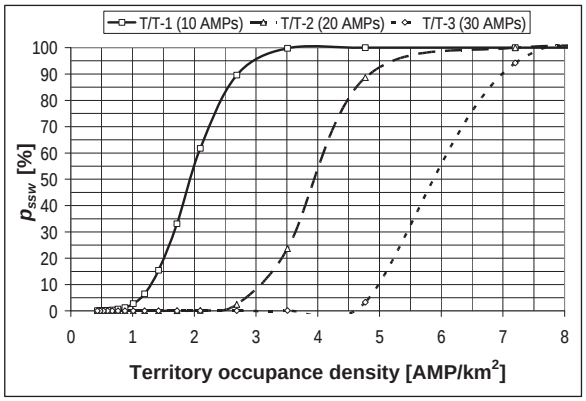


Fig.8 Successful swarming probability, function of defended territory occupancy

Three kinds of threat/targets, against which 10, 20 or 30 AMPs, with fulfilled aforementioned conditions, can accomplish successful swarming, have been considered in the model. The characterristic results of the simulation have been presented in the figures 7 and 8.

## 6. CONCLUSION

The research results achieved so far could be used for planning of armoured and mechanized units in the territory defense. Provided the knowledge of the characteristics of target/threaths, the achievement of appropriate density, i.e. number of AMPs per unit of the defended territory area, would result in certain level of the successful issue probability of combat activities using the swarming tactics.

The AMP group swarming simulator realized so far is limited to the case of a single threat/target unit simultaneous occurrence. That limitation is not a problem when relatively smaller AMP groups, which are often engaged in combat action against one serious threat/target, are considered.

However, even an armoured battalion, as the basic tactical unit, has opportunities to apply swarming simultaneously against 2, 3 and even more different threats/targets. Besides obvious quantitative increasing of the simulator capabilities, this imposes some new issues related to target selection, priorities, AMPs compatibilities with targets and the capabilities of the group for the self-organizing in such situations. Therefore, the algorithm should be enhanced in further research, so that the swarming of the AMP group against multiple threats/targets, being simultaneously in the defended territory, could be simulated.

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# New method for ranking chess Olympics teams

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## Abstract:

Ranking method in a team chess championship has long been an issue of great concern. In quest of the "golden standard," World Chess Federation officials applied many different ranking methods. In Torino 2006, final rankings were based on the sum of individual points won by each player as a member of a certain team. This method of rankings was changed for the 38th Chess Olympics in Dresden 2008 were each win, gained with a result of 2.5:1.5 or better, is awarded 2 match points. In this paper, we will examine the results of Chess Olympics held both in Torino 2006 and in Dresden 2008. Applying the I-distance ranking method, we came to some interesting results and we see them useful for the quality evaluation procedure of the abovementioned ranking methods. Results indicated that Torino 2006 ranking system proved to be more valid than its Dresden 2008 successor.

**KEYWORDS**

chess Olympics, I-distance method, ranking of countries

## 1. INTRODUCTION

For many centuries people regarded chess as a board game for rulers. The significant turning point and the rise in popularity of this sport happened in the early 1900's. In 1924 the World Chess Federation (FIDE) was founded and only three years later, the first Chess Olympic Games took place in London. There were 16 nations taking part in the first Chess Olympics. By the 37th Chess Olympics, there were 133 nations participating in this contest. For all these years, the world best chess players entered the Olympics as the representatives of their respective countries. Each national team was composed of four players. The first board is usually the best player in the team, whereas fourth is the weakest player. In a match between two teams the first boards play against each other; the same applies to boards two, three and four. Until Dresden 2008, the final rankings were based on the sum of individual points won by each player as a member of a certain team. This method of rankings was changed for the 38th Chess Olympics. Each win, gained with a result of 2.5:1.5 or better, is awarded 2 match points. For example, a team can win all four games on four boards or win with only one but draw on the rest of the three boards, to receive 2 match points. In the case of a draw match (2:2), a team wins one match point and gets no match points for losing a match (FIDE, 2011). The final standing is based on the sum of match points. This ranking method provoked some serious criticism regarding the fact whether it can provide objective final rankings or not. This paper is focusing on the examination of Torino 2006 and Dresden 2008 Olympics results using the statistical I-distance method.

## 2. THE I-DISTANCE METHOD

The ranking of specific issues is quite often carried out in a way that can seriously affect the process of taking exams, entering competitions, UN participation, medicine selection and many other subjects (Ivanovic, 1973; Ivanovic and Fanchette, 1973; Jeremic and Radojicic, 2010; Jeremic et al., 2011a; Al-Lagilli et al., 2011).

I-distance is a metric distance in a n-dimensional space. This method has been proposed and defined in various publications that have appeared since 1963 by B. Ivanovic (Ivanovic, 1977). Ivanovic created this method to rank countries according to their level of development on the basis of several indicators. Many socio-economic development indicators have been considered in the use of this method and the problem has been how to use all of them in order to calculate a single synthetic indicator which will represent the rank.

For a selected set of variables $X^T = (X_1, X_2, ..., X_k)$ chosen to characterize the entities, the I-distance between the two entities $e_m = (x_{1m}, x_{2m}, ..., x_{km})$ and $e_n = (x_{1n}, x_{2n}, ..., x_{kn})$ is defined as:

$$D(m,n)=\sum_{i=1}^{k}\frac{\left|d_i\left(m,n\right)\right|}{\sigma_i}\prod_{j=1}^{i-1}\left(1-r_{ji.12\ldots j-1}\right) \qquad (1)$$

where $d_i\left(m,n\right)$ is the distance between the values of the variable $X_i$ for $e_m$ and $e_n$, e.g. the discriminate effect,

$$d_i\left(m,n\right)=x_{im}-x_{in}\ , \quad i\in\left(1,\ldots,k\right) \qquad (2)$$

$\sigma_i$ the standard deviation of $X_i$, and $r_{ji.12\ldots j-1}$ is a partial coefficient of correlation between $X_i$ and $X_j$, $\left(j<i\right)$, (Ivanovic, 1973).

The construction of the I-distance is iterative. It is calculated through the following steps:

1. calculate the value of the discriminate effect of variable $X_1$ (the most significant variable, which provides the largest amount of information on the phenomena needed to be ranked);
2. add the value of the discriminate effect of $X_2$ which is not covered by $X_1$;
3. add the value of the discriminate effect of $X_3$ which is not covered by $X_1$ and $X_2$;
4. repeat the procedure for all variables (Mihailovic et al., 2009).

This I-distance fulfils all 13 conditions for defining the measures of distances and has proven useful in overcoming differences in measures. It is essential to point out that the I-distance method requires a standardization of all data.

Sometimes it is not possible to achieve the same sign mark for all variables in all sets, therefore a negative correlation coefficient and negative coefficient of a partial correlation may occur (Jeremic et al., 2011b; Jeremic et al., 2011c). Thus, the use of the square I-distance is even more desirable, which is given as:

$$D^2\left(m,n\right)=\sum_{i=1}^{k}\frac{d_i^2\left(m,n\right)}{\sigma_i^2}\prod_{j=1}^{i-1}\left(1-r_{ji.12\ldots j-1}^2\right). \qquad (3)$$

In order to rank the entities - countries in this case- in the observing set using the I-distance methodology, it is necessary to fix one entity as a referent: an entity with a minimal value for each indicator. The ranking of the entities in the set is based on the calculated distance from the referent entity.

## 3. RESULTS

In this paper, we examined the results (Chess Results, 2011) of the 2006 Torino and 2008 Dresden Olympics Women Section. In both Olympics, countries participated in the eleven-round tournament. In Torino 2006, final standings were based on the sum of individual points won by the team members, while in Dresden 2008 match points were crucial ranking criterion. Aim of our paper was to determine which of the previously mentioned ranking criterion provides the impartial results. For the purpose of this study we decided to implement all known variables that represent achievements of every national team. Only three variables were included in the final standings based on the FIDE regulations. On the other hand, in our work we examined fourteen variables. The first four variables were percentage values of points gained by each of the players from certain national team. We also incorporated the performance of each team member in the form of another four variables. Other variables expressed the (number of) team wins, draws and losses. Finally, we included all three variables proposed by the FIDE regulations (Match points, Sum of individual points, Tie-Break). Due to a high number of participating nations, round-robin tournament could not be an option for the Chess Olympics. As a logical consequence of this, draw plays significant role in final standings. Definitely, this very issue the current FIDE ranking system could not solve. As a way to come to an improvement, we suggested a much larger number of variables (fourteen compared to three) should be applied. Apart from this novelty, a crucial improvement should be to implement every team member performance as a variable. Elo defined performance as numerical representation of a chess player's strength in that particular tournament (Elo 1986). Performance is function of results achieved in a particular game and chess strength of its opponent. That is to say, chances for any team to

rise to the top of the ranking list by playing with weak opponents are minimal. According to our proposal, the influence of draw and luck on final standings will be significantly reduced.

### 3.1. Torino 2006 results

The results achieved through the use of the I-distance ranking method are presented in Table 1. As we can see from Table 1, the Olympics winner Ukraine topped the I-distance list as well. In every input variable we analyzed, it was one of the best national team and its players absolutely deserved the first place.

Table 1. The Results of the Square I-distance Method, I-distance Value and Rank

| Country | I-distance | Rank I-distance |
|---|---|---|
| Ukraine | 102.15 | 1 |
| Russia | 89.52 | 2 |
| China | 76.31 | 3 |
| USA | 72.11 | 4 |
| Hungary | 61.96 | 5 |
| Georgia | 61.17 | 6 |
| Armenia | 59.99 | 7 |
| Bulgaria | 56.81 | 8 |
| Netherlands | 56.22 | 9 |
| Slovenia | 55.97 | 10 |
| Germany | 55.57 | 11 |
| Greece | 55.06 | 12 |
| India | 54.58 | 13 |
| Czech Republic | 54.52 | 14 |
| Romania | 53.74 | 15 |
| Poland | 52.86 | 16 |
| Cuba | 52.27 | 17 |
| Slovakia | 52.02 | 18 |
| France | 51.46 | 19 |
| Latvia | 51.29 | 20 |
| ---------- | ---------- | ---------- |
| Nigeria | 16.68 | 90 |
| Costa Rica | 15.16 | 91 |
| Trinidad & Tobago | 15.05 | 92 |
| Fiji | 14.39 | 93 |
| Namibia | 12.36 | 94 |
| Kenya | 12.31 | 95 |
| Libya | 8.92 | 96 |

Results achieved with the I-distance method are quite similar with the official Torino 2006 results. As the matter of fact, correlation of ranks is very high with $r_s$=.977, p<.01. Nonetheless, crucial issue is to provide information which indicator (input variable) is the most important one for determining final rankings. Thus, data set was further examined and a correlation coefficient of each indicator with the I-distance value was determined, the results of which are presented in Table 2.

Table 2. The Correlation between I-distance and Input Indicators

| Indicators | r |
|---|---|
| Sum of individual points | .961** |
| Match points | .918** |
| Tie-Break | .870** |
| Performance Board 2 | .852** |
| Performance Board 3 | .837** |
| Number of Wins | .792** |
| Number of Losses | .791** |
| Performance Board 1 | .735** |
| Performance Board 4 | .605* |
| Percentage Board 2 | .573* |
| Percentage Board 4 | .508* |
| Percentage Board 1 | .489 |

| | |
|---|---|
| Percentage Board 3 | .395 |
| Number of Draws | .236 |

**p < .01
*p < .05

The results accomplished by means of this analysis are far from the unexpected. Sum of individual points won by each player as a member of a certain team is assumed as the first ranking criteria in the Torino 2006 Olympics and it has the largest correlation with the I-distance value. In a line with this, its significance is undisputed. The correlation coefficient for the other two official FIDE criteria (Match points and Tie-Break) also has large effect on final rankings, but decision to promote Match points into crucial indicator for ranking in Dresden 2008 doesn't seem to be statistically justified. Also, the performance of every single team member has a strong correlation with the I-distance value. However, the real curiosity is that the first board is not the most significant one, although it is the board with the strongest player. This finding provides us with a new perspective on team chess tournaments. So far we have anticipated that the first board players are the best to be the team leaders. It appears that the crucial team members are boards two and three.

### 3.2. Dresden 2008 results

After conducting Torino 2006 analysis, we applied I-distance method on Dresden 2008 Olympics. The results achieved through the use of the I-distance ranking method are presented in Table 3. As we can see from Table 3, the Olympics silver medallist Ukraine topped the I-distance list. On the other hand, winners Georgia deserved second place according to I-distance method.

Table 3. The Results of the Square I-distance Method, I-distance Value and Rank

| Country | I-distance | Rank I-distance |
|---|---|---|
| Ukraine | 79.63 | 1 |
| Georgia | 59.28 | 2 |
| Russia | 50.41 | 3 |
| USA | 50.18 | 4 |
| Poland | 48.06 | 5 |
| China | 46.31 | 6 |
| Armenia | 43.13 | 7 |
| France | 40.94 | 8 |
| Serbia | 39.47 | 9 |
| Israel | 38.82 | 10 |
| Belarus | 37.33 | 11 |
| Hungary | 37.29 | 12 |
| Greece | 35.31 | 13 |
| Cuba | 35.28 | 14 |
| Romania | 34.85 | 15 |
| India | 34.72 | 16 |
| Netherlands | 33.7 | 17 |
| Croatia | 32.85 | 18 |
| Italy | 32.82 | 19 |
| Slovakia | 32.79 | 20 |
| ---------- | ---------- | ---------- |
| Malta | 8.1 | 94 |
| Honduras | 7.87 | 95 |
| Yemen | 7.26 | 96 |
| Nigeria | 6.98 | 97 |
| Iraq | 6.88 | 98 |
| Ireland | 5.97 | 99 |
| Macau | 5.94 | 100 |

Results achieved with the I-distance method are similar with the official Dresden 2008 results, with $r_s$=.929, p<.01. Although correlation is very high, one must notice the decline comparing to Torino 2006. Further on, a correlation coefficient of each indicator with the I-distance value was determined, the results of which are presented in Table 4.

Table 4. The Correlation between I-distance and Input Indicators

| Indicators | r |
|---|---|
| Sum of individual points | .965** |
| Match points | .928** |
| Tie-Break | .884** |
| Performance Board 2 | .851** |
| Number of Wins | .846** |
| Performance Board 3 | .837** |
| Performance Board 4 | .809** |
| Performance Board 1 | .806** |
| Number of Losses | .787** |
| Percentage Board 4 | .600* |
| Percentage Board 2 | .594* |
| Percentage Board 3 | .548* |
| Percentage Board 1 | .520* |
| Number of Draws | .222 |

**p < .01
*p < .05

Results indicated that once again sum of individual points won by each player as a member of a certain team is the most important variable, with r=.965, p<.01. This finding challenges the call made by FIDE officials by implementing the Match points as crucial criterion for ranking in Dresden 2008 Olympics.
Particularly interesting finding is that Board 2 and 3 are the most important boards. This provides us with completely different perspective on team chess competitions. It is obvious that coaches and team captains must acknowledge this analysis and incorporate it into their strategies.

## 4. CONCLUSION

Ranking method in Chess Olympics has long been an issue of great concern. In a quest for "golden standard" FIDE officials tried a number of different options. The last change for 2008 Dresden Olympics raised many issues. In this paper, we analyzed the proposed FIDE ranking by employing statistical I-distance method. The FIDE ranking provides opportunity for teams to achieve far better results than expected. Calculative play, defining various strategies, significantly diminished the quality of the games. I-distance method adds a new sense of objectivity to rankings and points out the potential weaknesses of every team. It can provide us with a unique opportunity to further analyze every team and focus on certain issues. I-distance clearly demonstrates the influence of each team member's play on the result. Particular contribution of this paper is finding that second and third board are crucial for good team results. So fare we took for granted "fact" that first board is the place were medals are decided. Analysis pointed out that excellent 2nd and 3rd board are path to success in team chess championships. We hope that this study will add to the growing body of studies in sport sciences (Duch et al. 2010; Lucifora and Simmons 2003; Brillinger 2007; Van Tuyckom and Jöreskog 2010) and its results could be conductive for a better understanding of individual player performance in team activities.

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# Towards an Evaluation of Sustainable Development: A Statistical Approach

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## Abstract:

Sustainable development represents a constant commitment to advancing well-being. Despite numerous attempts, sustainable development is a difficult issue to measure and there is no one method agreed upon. As a possible remedy to this issue, a statistical method which synthesizes many sustainable development indicators into one quantitative indicator is herein proposed. Issues raised in this paper concerning sustainable development are of huge significance, since the growing demand of resource consumption is continually affecting the world economy and is a determining factor in many countries' strategic planning. In this paper, the sustainable development of the 27 member states of the European Union has been analyzed. The results have demonstrated that Sweden and Denmark are the two top EU countries concerning sustainable development. On the other hand, the newest EU members such as Latvia, Hungary, Estonia, Lithuania and Slovakia remain at the bottom of the ranking. In conclusion, the need for EU countries to take lead on sustainable development must be emphasized. Only through their groundbreaking work will other countries follow the path of an individual country's well-being and sustainable development. Moreover, the proposed method presented can be applied to other regions and can provide useful information as to whether the world is actually moving towards its goals of sustainable development.

**KEYWORDS**

sustainable development, I-distance method, ranking of countries

## 1. INTRODUCTION

The most often quoted definition of sustainability comes from the 1987 report published by the World Commission on Environment and Development (WCED), also known as the Brundtland Commission (Our Common Future, 1987). Environmental sustainability and development are defined as a single, indivisible issue, which, consequently, led to the following definition: "Sustainable development is development that meets the needs of the present without compromising the ability of future generations to meet their own needs" (Our Common Future, 1987, p. 43). Based on this widely accepted definition of sustainable development, the philosophy of welfare through generation progress can be observed, studied and measured in the same manner as a country's social, economic and environmental development. Also, the Earth Charter shows the evolution of the sustainability concept, particularly in respect to the inclusion of social and economic justice components (The Earth Charter, 2008), and possesses the power to challenge the dominant paradigm of economic development as being the yardstick for individual, community, national, and global progress, since it "is founded on the principle that caring for the earth and caring for people are two dimensions of the same task" (Greenwood, 2004, p. 96).

Since sustainability is a multidimensional concept, economic social and environmental aspects must be considered and integrated (Pope et al., 2004). The appropriate instrument for multidimensional representation is a suitable set of indicators that must be an integral part of an assessment methodology to be used for the purposes of measuring sustainability (Ness et al., 2007; Moffat et al., 2001). Improvements in the way the indicators are constructed and used are very important research issues (Munda and Nardo, 2009). Precisely, this paper shall evaluate sustainability by implementing many variables and create one synthesized indicator. The methodology used here is very similar to propositions made by Ivanovic in his work (Ivanovic, 1973); according to whom, the choice of social and economic development indicators is certainly one of the most important problems when evaluating countries: the more complete the information about national development levels contained in the selected indicators is, the better the results will be. There are a certain number of indicators, which are always thought to provide the most reliable information on national social and economic development (Ivanovic, 1977). They are most frequently used in evaluating a country's development level; for example, its per capita GDP, its literacy rate or the percentage of its population not employed in agriculture. However, the lists of development indicators used in various national or international research institutions are not always identical, and there is constant controversy in regard to the value and importance of one or another of these well-

established indicators. In fact, it may be speculated upon as to whether there are actually any objective reasons as to why they should all be considered as social and economic development indicators and if others, so far overlooked or neglected, might not be more suitable than certain, traditionally used ones (Ivanovic, 1973).

In accordance with these ideas, sustainable development shall be evaluated though the headline indicators of European Union Sustainable Development Indicators. This paper is organized as follows: section 2 focuses on the methodology used to perform the analysis, section 3 features the results of the analysis and the final section of the paper highlights conclusions of the analysis.

## 2. THE I-DISTANCE METHOD

The ranking of specific issues is quite often carried out in a way that can seriously affect the process of taking exams, entering competitions, UN participation, medicine selection and many other subjects (Ivanovic, 1973; Ivanovic and Fanchette, 1973; Jeremic and Radojicic, 2010; Jeremic et al., 2011a; Al-Lagilli et al., 2011).

I-distance is a metric distance in an n-dimensional space. This method has been proposed and defined in various publications that have appeared since 1963 by B. Ivanovic (Ivanovic, 1977), who created this method to rank countries according to their level of development on the basis of several indicators. Many socio-economic development indicators have been considered in the use of this method and the exact problem has been how to use all of them in order to calculate a single synthetic indicator which will represent a ranking.

For a selected set of variables $X^T = (X_1, X_2, ..., X_k)$ chosen to characterize the entities, the I-distance between the two entities $e_r = (x_{1r}, x_{2r}, ..., x_{kr})$ and $e_s = (x_{1s}, x_{2s}, ..., x_{ks})$ is defined as:

$$D(r,s) = \sum_{i=1}^{k} \frac{\left| d_i(r,s) \right|}{\sigma_i} \prod_{j=1}^{i-1} \left(1 - r_{ji.12...j-1}\right) \qquad (1)$$

where $d_i(r,s)$ is the distance between the values of the variable $X_i$ for $e_r$ and $e_s$, e.g. the discriminate effect,

$$d_i(r,s) = x_{ir} - x_{is}, \qquad i \in (1,...,k) \qquad (2)$$

$\sigma_i$ the standard deviation of $X_i$, and $r_{ji.12...j-1}$ is a partial coefficient of correlation between $X_i$ and $X_j$, $(j < i)$, (Ivanovic, 1973).

The construction of the I-distance is iterative and is calculated through the following steps:

1. calculate the value of the discriminate effect of variable $X_1$ (the most significant variable, which provides the largest amount of information on the phenomena needed to be ranked);
2. add the value of the discriminate effect of $X_2$ which is not covered by $X_1$;
3. add the value of the discriminate effect of $X_3$ which is not covered by $X_1$ and $X_2$;
4. repeat the procedure for all variables (Mihailovic et al., 2009).

This I-distance fulfils all 13 conditions for defining the measures of distances and has proven useful in overcoming differences in measures. It is essential to note that the I-distance method requires a standardization of all data.

Sometimes it is not possible to achieve the same sign mark for all variables in all sets, therefore a negative correlation coefficient and negative coefficient of a partial correlation may occur (Jeremic et al., 2011b; Jeremic et al., 2011c). Thus, the use of the square I-distance is even more desirable, which is given as:

$$D^2(r,s) = \sum_{i=1}^{k} \frac{d_i^2(r,s)}{\sigma_i^2} \prod_{j=1}^{i-1} \left(1 - r_{ji.12...j-1}^2\right). \qquad (3)$$

In order to rank the entities - countries in this case- in the observing set using the I-distance methodology, it is necessary to fix one entity as a referent: an entity with a minimal value for each indicator. The ranking of the entities in the set is then based on the calculated distance from the referent entity.

## 3. RESULTS

In this paper, the sustainable development of the 27 member states of the European Union has been analyzed. As input variables, the titles of Sustainable Development Indicators of EU Sustainable Development Strategy have been used (Eurostat 2010). In order to rank the countries of the European Union, the following 11 variables have been used:

Table 1. Sustainable Indicators for Determining a Country's Welfare

| ***Theme*** | ***Headline Indicators*** |
|---|---|
| *Socio-economic development* | *1. Growth rate of real GDP per capita* |
| *Sustainable consumption and production* | *2. Resource productivity* |
| *Social inclusion* | *3. Population at-risk-of-poverty or exclusion* |
| *Public health* | *4. Healthy life years - male* |
| | *5. Healthy life years - female* |
| *Climate change and energy* | *6. Greenhouse gas emissions* |
| | *7. Share of renewable energy in gross final energy consumption* |
| *Demographic changes* | *8. Employment rate of older workers - male* |
| | *9. Employment rate of older workers - female* |
| *Sustainable transport* | *10. Energy consumption of transport relative to GDP* |
| *Global partnership* | *11. Official development assistance as a share of the gross national income* |

The results achieved through the use of the I-distance ranking method for the countries' welfare are presented in Table 2. Sweden and Denmark topped the list according to the I-distance method; while the newest EU members such as Latvia, Hungary, Estonia, Lithuania and Slovakia came in at the bottom of the ranking.

Table 2. The Results of the Square I-distance Method, I-distance Value and Rank

| **Country** | **I-distance** | **Rank I-distance** |
|---|---|---|
| Sweden | 44.645 | 1 |
| Denmark | 30.5 | 2 |
| Luxembourg | 29.453 | 3 |
| The Netherlands | 26.512 | 4 |
| The United Kingdom | 24.629 | 5 |
| Cyprus | 24.422 | 6 |
| Spain | 19.084 | 7 |
| Malta | 18.726 | 8 |
| Ireland | 17.693 | 9 |
| Germany | 17.676 | 10 |
| Poland | 17.097 | 11 |
| Bulgaria | 16.103 | 12 |
| Austria | 15.724 | 13 |
| Greece | 15.581 | 14 |
| Italy | 15.411 | 15 |
| The Czech Republic | 13.935 | 16 |
| France | 13.739 | 17 |
| Portugal | 13.281 | 18 |
| Finland | 13.152 | 19 |
| Belgium | 11.383 | 20 |
| Romania | 9.426 | 21 |
| Slovenia | 7.685 | 22 |
| Latvia | 6.708 | 23 |
| Hungary | 6.694 | 24 |
| Estonia | 5.736 | 25 |
| Lithuania | 5.66 | 26 |
| Slovakia | 3.838 | 27 |

This data set was further examined and a correlation coefficient of each indicator with its I-distance value was determined, the results of which are presented in Table 3 (the Pearson correlation test has been used here).

Table 3. The Correlation between I-distance and Input Indicators

| Indicators | r |
|---|---|
| *Official development assistance as share of gross national income* | .803** |
| *Healthy life years - male* | .706** |
| *Growth rate of real GDP per capita* | .670** |
| *Resource productivity* | .600** |
| *Healthy life years - female* | .628** |
| *Employment rate of older workers - male* | .535** |
| *Employment rate of older workers - female* | .429* |
| *Greenhouse gas emissions* | .341 |
| *Share of renewable energy in gross final energy consumption* | .112 |
| *Energy consumption of transport relative to GDP* | .024 |
| *Population at-risk-of-poverty or exclusion* | .022 |

** p<.01.
*p<.05.

As the results show, Official development assistance as share of gross national income is the most significant variable, with r=.803, p<.01. Additionally, a very interesting finding is that "resource productivity" has a significant correlation with I-distance ranking, with r=.600 (p<.01).

On the other hand, the growth rate of real GDP per capita was ranked as being the third most significant indicator. Although GDP has very often been used as the most significant indicator in the past (even more so, as only indicator (Davidson, 2000)), it must be now acknowledged that it is essential to implement components of sustainable development into research.

## 4. CONCLUSION

The environment provides numerous goods and services to humanity. Keeping with this, the integration of ecology into general development, according to the principles of sustainable development, is an obvious necessity. Therein, the approach presented in this paper has demonstrated that EU countries have to dramatically improve their policies concerning sustainable development.

Policies had improved since their first "start up" stage until the 1970s, when the first significant efforts for environmental protection were originally undertaken. At the national level, governments started to approve general legislative framework, to establish environmental ministries and to create protected areas; while on a global and regional scale, multilateral treaties for the conservation of species (CITES) and ecosystems (Ramsar Convention), as well as the protection of human rights and cultural heritage (WHC) and the control of pollution (CLRTAP) were agreed upon. The Stockholm Conference in 1972 was a recognizable milestone, which resulted in the creation of UNEP.

A second stage characterized by stagnation in sustainable development policy followed and continued until around 1986. Although the creation of protected areas continued to grow at the national level, multilateral efforts in environmental areas refraineas a result from a shift to economic prosperity as being the solution for development problems. In addition to this less favorable atmosphere, the WCS (in 1980) gave visibility to the concept of sustainable development. However, a concept similar in meaning to that coined in 1987 by the Brundtland Commission appeared for the first time in 1976 (in the text of the Agreement Establishing the South Pacific Regional Environment Programme, according to a search in Mitchell, 2008). The approximate decade from 1987 until 1995 represented another period of significant effort by the international community to push for sustainable development. The Brundtland Commission set the stage in their report "Our Common Future". Sustainable development thereafter definitely entered the lexicon and became a popular expression, receiving widespread agreement. The 1992 UNCED was another obvious breakthrough, as it is supported by peaks in all indicators of Figure 3 around this date and by an impressive number of political milestones that followed: the environmental agreements CBD/UNFCCC, institutional arrangements, the financial mechanisms CSD/GEF and acknowledged "soft laws" (Rio Declaration, Agenda 21). The balanced position offered by WCED was

neglected during UNCED, whose agenda was significantly "greener". During the 1990s, governance issues such as transboundary cooperation and public participation were finally addressed by multilateral agreements.

The fourth period, which started around 1996, was marked by a decline in global sustainable development policy. The fear of terrorism and the globalization of the world economy are the most probable reasons, but a short peak in political indicators around 2000 – certainly related to the coming of WSSD – makes conclusions less straightforward. Yet even WSSD as a milestone was not as influential as UNCED, if judgment is based on their outcomes (Table 3). For instance, no agreements, institutional arrangements or financial mechanisms followed WSSD; and although several targets had been approved, Röpke (2005) remarked on a notable 'implementation deficit" that the international community seems unable to counteract. As already proposed by UNEP (2007, p. 376), perhaps a new stage of sustainable development policy is now emerging that shifts its attention to the implementation of existing norms and policies, including the growing number of national sustainable development strategies, which already amount to 82 (UNDSD, 2008a).

The topics and goals addressed at the global level by multilateral agreements and by other political milestones are in ever constant evolution, depending on the perception of what problems require the most primary attention, on disasters or specific events that trigger immediate responses and on influences from the scientific realm. An analysis of Table 3 reveals a transition from single issues to more complex and integrated framework. Ecosystem concerns succeeded species conservation initiatives, and it was not until the 1990s that governance issues were directly addressed by MEA. Since 2000, this interest in the institutional dimension of sustainable development seems to have been continuing, at least in the discourse of the Millennium Declaration and of WSSD, although it has not been accompanied by implementation mechanisms. It can also be seen that treaties have been approved as of late to regulate new topics that pose specific threats; such as genetically modified organisms and persistent organic pollutants. Several targets have been agreed on by the international community, namely through the Millennium Development Goals. These commitments are meritorious, yet are also biased towards human development, as they are significantly more detailed and in a greater number than environmental protection goals. Sustainability concerns have shifted from an emphasis on pollution and the availability of natural resources to a more balanced position that puts human and social development – particularly freedom and the expansion of individual capacities – in the spotlight.

As a final synthesis regarding which goals and topics are most relevant in the context of sustainable development, six main policy pillars are here proposed to encompass a variety of sustainability issues:
• sustaining natural capital – biodiversity, water, air;
• sustaining life support systems – ecosystems, ecosystem services, resources;
• minimizing human impact – climate change, pollution, waste, desertification, population growth;
• developing human capital – human rights, political liberties, learning, equity, health, wealth;
• developing social capital – solidarity, community, culture;
• developing economies – economy, agriculture, consumption, employment, technology;
• developing institutions – proper governance, democracy, transparency, public participation, international cooperation (Quental et al. 2011).

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# Evaluation of Educational and Scientific Quality of Serbian Universities from the Employees' Aspect

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**Abstract**

According to ISO standards, in order to achieve sustainable success of the organization, quality management system (QMS) should be based on the principles that refer to: efficient resource utilization, fact-based decision making, focusing on needs, expectations and satisfaction of consumers and all interested parties. „Interested party is person or group that has a stake in the success or performance of an organization, or that is interested for organization's activities in any other way".

Hence, it is not wonder that the representatives of all interested parties are included in the committees for quality at the universities which follow the claims and criteria of quality prizes or ISO standard all over the world. The procedure of evaluation of educational processes from the standpoint of students has been already done in Serbia for several years, so it is now turn to evaluate the educational processes and recent reforms from the standpoint of employees. The survey is done among the professors and assistants at the Serbian universities, and the results are presented in this paper. First part represents the results of the scientific quality of the examinees, while the second part represents their answers regarding quality of educational processes at the universities.

The survey results keep us in the belief that hypothetically noticed problem exists, and they also encourage us to further elaboration of the possible and necessary actions to establish standards for measuring educational and scientific processes at the universities in Serbia.

**KEYWORDS**

Quality management, higher education, reform

## 1. INTRODUCTION

ISO 9000 family of standards dates from 1987 year. The first revision of the standards was in 1997 year. The main aim of these standards was to examine and assess the capabilities of manufacturers to produce the required product. The standards gave unique criteria for organizing and evaluating the quality systems of suppliers. Companies were able to certify their quality by the following standards: ISO 9001, ISO 9002, ISO 9003. There was also the standard 9004 - Quality management and quality system elements that consist of five parts. [1].

Family of standards ISO 9000:1987 and 1996 was referred to the Quality System (QS) of organization. They didn't include the quality of people and products. Since the main aim was the establishment of QS according to the requirements of ISO 9000 and its confirmation by the third-party – certification, the quality of products was neglected. The opinion was that the achievement of planned firm's performances also provided a product quality and ensured customer satisfaction. Disadvantages of 1996 year standards were the following: by introducing quality systems into the organization and its production functions, quality of the product would be provided. But the organization often used to introduce the QS and have documentation for it, while the product didn't meet the strict requirements defined by customers. Requirements were defined in 20 points and referred only to the key activities - the production functions. Since other activities were not defined, it didn't ensure the functioning of the process and obtain high quality output from the process - the

results. The organization is a set of subsystems. Therefore, in order for organizations to be managed, the system and all subsystems must be managed. Also, family of standards is defined for manufacturing companies. If the service company wanted to certify their QS, it also had to meet the requirements of ISO 9001 and 9004-2. Companies had to meet numerous requirements, and create and follow a forest of documentation. For these reasons, in 2000 the following standards have been revised.
The new concept of the revised standard – process approach, clearly defined principles and requirements which are based on principles that enabled the elimination of these shortcomings [2].
The new revision of ISO 9000 and ISO 9001 was created in 2008 year. The name, concept, principles and standard structure have remained the same. The changes are not large. The main changes relate to: the inclusion of risk management from the environment for the design and management of QMS and defining competencies of employees. [5, pp. 6]. Further, text version from 2008 says that the design and implementation of QMS in an organization depend on: environment of the organization, changes in the environment and the environment risks, the various needs of the organization, the specific objectives of the organization, products / services delivered, a process applied, organization size and its organizational structure [6]. Also, standard 9004 were revised in 2009 year, and it is quite different from the 2000 standard. Standard 9004:2009 provides the introduction to organization for achievement sustainable success in a complex, demanding and changing environment by using the approach of continuous improvement QMS [7]. It also provides guidance for improving the overall performance of the organization. From existing programs of sustainable development, which were prepared by the international or national organizations or large international business systems, it follows that an organization can achieve sustainable development by applying the eight quality management principles in managing their systems in the process that: constantly monitor and analyse the environment of the organization including all external and internal factors and risks that may affect the achievement of goals and behaviour as well as relationships with customers and stakeholders; should determine the needs and expectations of users and stakeholders; create and maintain the mission, vision and strategic value in order to meet the demands and expectations of users and stakeholders; define and implement strategies and policies in order to fulfil the mission, vision and strategic goals that support organizational values, identify, provide and manage internal and external resources needed for achieving goals in short as well as in long term, should provide outputs that meet the requirements and continually exceed customer expectations and other interested parties; regularly monitor, measure, analyze and question the performance of your organization / process / output / staff and customer satisfaction, and all interested parties in order to lead to continuous improvement, innovation and [7, picture 1, pp. 6].
The standard particularly highlights the importance of all stakeholders. “Model of stakeholders quality is therefore fundamentally different from the previous “[4, pp. 29]. It is so different that it can represent the beginning of the third generation of quality model that will gradually replace the previous [3]. In order to achieve the basic goals of the organization as well as its sustainable success, organisation by continuously fulfilling the needs and expectations of its stakeholders should adopt a procedure that could follow the following steps: identifying stakeholders and evaluating their potential impact on the organization performance and balanced the fulfilment of their needs and expectations, identification of values for each, i.e. establishing programs and processes that have added value to each stakeholder, determination of the transaction which is required for achieving these values, determination of the conditions and expectations of stakeholders as well as determination how to meet the needs and expectations of all stakeholders and balanced them, specifying and developing access to mutually beneficial relationships with stakeholders, definition of forms of ongoing involvement of stakeholders and continuous communication with them, specifying the different approaches and consensus in order to achieve a balanced relationship between, often conflicting, goals of individual stakeholders [ 7, pp. 5]. This is an iterative process, because the relationships, stakeholder’s goals are dynamic categories. The procedure of each of following iteration has to follow the Deming cycle of improvement.
The users of university and college services are students and all stakeholders. Stakeholders are: society, economy, government institutions, organizations, professionals, other universities, alumni, etc... It passed four years since Bologna reform in Serbia. First students graduated, so the conditions are to assess the quality of teaching and educational process and the reform of higher education in terms of teaching staff of universities. The importance of interviewing of all stakeholders was also emphasized by Turajlić. In her article Turajlić describes “tuning project” and analyzes some of the results. “Tuning Project”, among other things, had a task to raise the question which competences should have received priority from the standpoint of employment, and to what extent it could be considered that they had been acquired during their studies. [8]

The survey involved teachers of universities, employers who employ students from these universities and students who were hired by these employers after graduation. Therefore, there is recommendation that these surveys in future should be conducted annually as well as that next year should begin with the application of survey economy and job market. [8]

## 2. STATISTICAL HYPOTHESES

The following hypotheses are accepted in the paper:

- It is assumed that the sample is homogenous, i.e. there are no statistically significant differences in the examinees' responses by scientific field, gender and academic degree. The null hypothesis is the positive answer to the question.

Specific hypotheses are:

- Test for the significance of differences between empirical and theoretical frequencies – particular empirical frequencies are equal to the theoretical and the total difference between empirical and theoretical frequencies is in the range of stochastic variations.
- Test for the significances of differences of relative proportions of two subsets in the population – the proportions of the answers of two different subsets are equal, so the differences in observed characteristics in the population are stochastic.

## 3. RESEARCH METHOD

***Research type.*** The questionnaire research was done in the July and August 2010. The target population was all teaching employees at the Serbian state-owned and private-owned universities and faculties. The e-mail with the appeal to take part in the research and to answer on the on-line questionnaire was delivered to every employee which e-mail was available at the university's internet site. The mail was sent personally to the all employees on the state-owned universities and on the Megatrend University, and to the contact persons of all other private-owned universities and faculties. The questionnaire was made by the team of professionals, including the sociologist and the pedagogue. The questionnaire was anonymous. Thanks to the Austrian centre for social researches, the questionnaire was installed on its internet site. The questionnaire consists of two parts. The first part (from the 4th to the 19th question) consists of the questions about the competent and the quality of scientific work of the examinees. The second part (from the 20th to the 32nd question) refers to the examinees' opinion about the reforms' success.
***Sample.*** The research was done on the simple, random sample of 604 examinees, of which the 53% are males and 47% are females; 36% belongs to the technical sciences, 31% to the social sciences, 26% to the natural sciences and 7% to the medical sciences; 69% of the examinees have PhD degree, 28% Master degree and 3% bachelor degree. The sample is representative (5% of the total population is 500 examinees). There are approximately 8200 researches and 1100 associates in the higher education institutions in Serbia.
***Measurement scales.*** Different scales are used for the questions 5 to 19, while the qualitative scale is used in the second part of the questionnaire. Available answers were: yes, no, and without answer. At the beginning of the questionnaire, examinees quote the personal data: gender, scientific field and scientific degree. The high percent of correctly filled questionnaire is achieved of nearly 100%, although the examinees often did not answer to the all question (see Table 1 for some questions).
***Statistical methodology.*** Data are saved in Microsoft Excel 2003 and SPSS (Statistical Package for the Social Science for Windows, version 13.0) formats. The following statistical methods are used for the analyze:

- **Statistical description**: relative frequencies and response structure are calculated for all examinees by gender, scientific field and academic degree;
- **Statistical analyses**: Test for the significance of differences between empirical and theoretical frequencies ($\chi^2$) and test for the significance of differences in relative proportions of two subsets of population are used; i.e. hypotheses of statistical significance of differences between the number of particular responses in subsets by the gender, scientific field and academic degree are tested. The

hypothesis of the proportions equality is also tested. The analysis is done by using the SPSS and Microsoft Excel. $\chi^2$ and *z*-test are used, according to the statistical rules and preconditions. Confidence level is set at $\alpha=0.05$.

## 4. RESULTS

The presumptions of the $\chi^2$ –test application ar4e tested in the first step of the analysis. The condition of the frequencies of the numerical distribution is fulfilled. The second condition is that the sum of the empirical frequencies has to be equal to the sum of the theoretical frequencies, i.e. $\sum_{i=1}^{k} f_{iE} = \sum_{i=1}^{k} f_i^*$ , *i=1, 2... k*. This condition is also fulfilled for all responses. The third condition is that all theoretical frequencies have to be higher then 5. If any frequency is less then 5, than it is added to the first next frequency higher then 5, so the condition is fulfilled in every case.

Null hypothesis is: particular empirical frequencies are equal to the theoretical and the total difference between empirical and theoretical frequencies is in the range of stochastic variations, i.e.:

*Ho:* $f_{1E}= f_1^*$; $f_{2E}= f_2^*$;.....; $f_{kE}= f_k^*$, where $f_E$ denotes empirical frequency, $f^*$ denotes theoretical frequency and $k$ is the number of the subsets.

Alternative hypothesis is:

$H_1$: at least two frequencies are not equal and the total difference between empirical and theoretical frequencies is in the range of stochastic variations.

Test statistics is: $\chi^2 = \sum_{i=1}^{k} \frac{\left(f_i - f_i^*\right)^2}{f_i^*}$.

The decision of acceptance of the null hypothesis has to be made on the basis of following condition. If $\chi^2 \leq \chi^2_{\alpha,\nu}$ and $p \succ \alpha$ then the $H_0$ is accepted with the confidence level of 1-α. $H_0$ is not accepted with the risk level of α if $\chi^2 \succ \chi^2_{\alpha,\nu}$ and $p \leq \alpha$.

The test for the second hypothesis of equal proportions of two subsets is also started with the check of the fulfillment of assumed preconditions, which are:
Normal distribution of X:

$X_1 : N(P_1, \sigma^2_{P_1})$; $X_2 : N(P_2, \sigma^2_{P_2})$;

Two large simple samples are chosen from the population. Test statistics in the I sample is $\hat{p}_1$ and $\hat{p}_2$ in the II sample. These statistics follow Bernoulli's distribution, but the distribution are approximately normal for the $n_1, n_2 \succ 50$.

Proportions of the population are unknown, but normally distributed, so are their differences, with the variance:

$$\sigma^2_{P_1-P_2} = \sigma^2_{P_1} + \sigma^2_{P_2} = \frac{P_1Q_1}{n_1} + \frac{P_2Q_2}{n}.$$

$\sigma^2_{(P_1-P_2)}$ has to be calculated from:

$s_{\hat{p}_1-\hat{p}_2} = \sqrt{p_w q_w (\frac{1}{n_1} + \frac{1}{n_2})}$, where $p_\omega$ denotes weighted arithmetical mean which is calculated as:

$$p_w = \frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2}, q_\omega = 1 - p_\omega.$$

Samples' proportions are: $\hat{p}_1 = a_1 / n_1$, $\hat{p}_2 = a_2 / n_2$.

Null hypothesis: $H_o$: $P_1 = P_2$.
Alternative hypothesis: $H_1$: $P_1 \neq P_2$.

Test statistics is $|z| = \left| \frac{\hat{p}_1 - \hat{p}_2}{s_{(\hat{p}_1 - \hat{p}_2)}} \right|$, and it follows standardized normal distribution *N (0, 1)*.

The samples are independent and the following conditions have to be fulfilled:

$n_1 \succ 50; \quad n_1\hat{p}_1 \geq 5; \; n_1\hat{q}_1 \geq 5$

$n_2 \succ 50; \quad n_2\hat{p}_2 \geq 5; \; n_2\hat{q}_2 \geq 5.$

The null hypothesis is accepted if the empirical value of test statistics |*z*| is less or equal to the theoretical value $z_{\alpha/2}$ and $p \geq \alpha$. The null hypothesis is not accepted if the empirical value of test statistics |*z*| is higher then the theoretical value $z_{\alpha/2}$ and $p \prec \alpha$.

As the number of the examinees in the field of medical sciences is 42 (less than 50), the hypothesis cannot be tested for this subset. All other subsets fulfilled the sample size condition. However, for some questions the product of the sample size and relative frequency is not adequate. In those cases the hypothesis of the equality of relative frequencies is not tested, too.

In some cases different hypotheses for the same responses are accepted by $\chi^2$ test and z-test. Namely, $\chi^2$ test shows that there is no statistically significant difference between empirical and theoretical frequencies for some questions, while z-test shows that there is statistically significant difference. Because of lower sensitiveness of un-parametric tests compared with parametric tests, both methods are used in the analysis.

## 5. DISCUSSION OF THE STATISTICAL ANALYSIS RESULTS

Beginning with research of the subject we didn't have any kind of previous exploration of the theme as a model, neither in Serbia nor in surrounding countries.

We have been guided with necessity which appeared due to observing the problem of lack of teaching and educational processes quality measurement by the subjects themselves, i. e. teaching staff, as a key factor in maintaining systems quality in higher education organizations (HEO).

Authors of this study themselves – as a part of the teaching-learning community, based on personal experience, intuition, and professional support from the colleagues in the field of science and technology research development, formulated a questionnaire for 32 issues. The first three survey questions were related to the gender, the scientific field and the title of respondents.

With following issues, from 5$^{th}$ to 19$^{th}$, we tried to get a picture of the average, scientific, research and professional competence of the sample of teaching staff, bearing in mind that in Serbian universities, beside teaching most of the staff are working on science and research (S&R) projects.

In the second part of the questionnaire from 20$^{th}$ to 32$^{nd}$ question, we tried to get an aggregate score of current transitional processes - resulting from the implementation of the Bologna Declaration in Serbian institutions of higher education - from the personal attitudes of employees.

Questions: 4 (*Do you try to continually upgrade your teaching skills and introduce new methods of teaching?*), 5 (*When the last time you attended a seminar in order to improve your knowledge*?) and 19 (*Have you been abroad for professional training?)* are related to the professional training of survey participants. According to the *Frascati* Manual [9] these activities cannot be characterized as research and development, but as part of general professional training, which in the long run, as a cumulative process can be translated into a research activity and measured as a research and development. On the one hand, 92% of all respondents considered that they were very devoted to improving their skills and teaching methods,

however, only 65% of them attended a seminar in the current or previous year, while 10% of respondents had never attended such a seminar. When it comes to training abroad, 43% were in vocational training less than three times, while over a third of respondents had never attended this kind of events abroad (37%). These answers correspond to answers to questions 6 (*How many times do you have in the past two years had a presentation at scientific conferences in the country?)* and 7 (*How many times do you have in the past two years had a presentation at scientific conferences abroad?).* Almost half of respondents (49%) said they had 1 to 3 paper presentation in the country and almost the same percentage abroad, while 15% of respondents had no presentation in the country, and there was twice as many (30%) of those who never had this kind of presentation abroad. These results indicate a general slow vertical mobility of employees in the SEO, e.g. senior positions in the organizational hierarchy are not subject to (fast) changes.

Further, concerning the science and research results of faculties staff - question 8 (*How many published papers do you have in leading national magazines (national type category M51) in the last 5 years?*) 9 (*How many published papers do you have in journals with SCI?*) 10 (*How many professional books you have published as author or co-author?)* 11 (*Do you have published a textbook?).* The vast majority of respondents (37%) have published three papers (cat. M51) in a given period, while, on the other hand, a quarter of the sample (25%) has no published work in one of the leading national professional journals. The highest frequency on the ninth question also, have has an answer "from 1 to 3" (24%), while almost 1/3 of respondents do not have any published papers in journals from SCI list. Most published papers are in the field of medicine and natural sciences, and they authors are mostly PhD holders, and men. Research in social sciences have more local character, which can be a problem for editing of results in journals from SCI list, also there are significant differences in gender and/or scientific degree that show already mentioned slow vertical mobility on gender basis. About 37% of respondents have up to three published books, while 44% did not publish any book. 11$^{th}$ question has response similar to the previous question: 40% have up to three published books, and 45% have published their textbook. Of course, this does not necessarily mean that teachers and teaching staff are not able to write a good textbook, but may mean that in their research fields, there are good books and also that they do not think they can write better. Earlier legislation, with obligatory election to the position of full professor was required from teachers to have published books that created hyper-production of textbooks in certain scientific fields, but not necessarily of certain quality.

Published papers in the aforementioned categories, as a result of S&R projects are crucial for assessment in the current methods of evaluation of project proposals of the Ministry of Science and Technology Development (MSTD). In correlation with the answers to previous questions is current involvement of the sample in MSTD's projects (questions 15 and 16), which still exceeds the assumed level of evaluation assessment, and indicates the need to increase S&R projects output at universities. We also notice a difference in engagement of respondents from medical sciences (90%) and social sciences (65%), which may be just the result of strategic priorities of MSTD, if they really are important during evaluation and projects selection for funding.

Questions 12, 13 and 14 are related to the membership in professional and/or scientific associations and editorial boards at home and abroad. Over 70% of respondents said that they are members of a professional association, while 23% are not members of any such organization. In editorial boards of national professional journals is the fourth of the sample, while twelve percent of respondents are in the editorial boards of international journals, that could be very flattering score, if the most of these journals is not published in Serbia.

Questions 15 and 16 refer to the involvement of participants in the projects of the Ministry of Science and Technology Development or other state institutions. Over 4 / 5 samples (82%) were engaged in projects in the country, and currently 70% of total respondents. 15% have never had this type of engagement, while a quarter is not currently engaged (25%).

Over half of respondents (52%) had previously been involved in the project was financed by foreign institutions or a foreign country government (question 17), currently are engaged 24% of respondents (question 18), while the vast majority - 71% were not engaged. And here comes to the fore the differences in gender, where 77% of women currently is not engaged in international projects, vs. 65% of male respondents.

The next group of questions from 20 to 26 is in connection with the impact of the higher education system reform processes in the HEO, provides us mainly with answers based on which we can conclude that the reform so far, according to the vision of the teaching staff, made more negative than positive results. The lowest appreciation of reforms gives the respondents in the field of technical sciences. On the other hand, respondents were more positive in the evaluation of teaching and curriculum at their schools, concerning that

they are in accordance with contemporary trends in science and profession. They also state that students are interested in modern teaching methods and actively participate in them. Although positive responses to these questions in general, are about 50%, however they can be regarded as a sign that shows some progress in adoption of new processes at universities.
If we would attempt from above mentioned to generalize, competence of typical profile of the employee at the university could include the following:
• Very committed to improving their skills and teaching methods, but fails to realize its maximum commitment for example in participating in seminars or trainings,
• Generally do not have published papers in journals with SCI, by gender is female, in the field of social sciences, or, he has a 4-9 published papers in journals with SCI, is a male and in the field of medical or natural sciences,
• Member of project team financed by MSTD, sometimes working on projects funded by foreign institutions or governments,
• Member of the professional association, sometimes member of the editorial board of the international journal,
• Less than 1 - 3 times was on the professional training abroad.
The survey results keep us in the belief that hypothetically noticed problem exists, and they also encourage us to further elaboration of the possible and necessary actions to establish standards for measuring educational and scientific processes at the universities in Serbia. Survey that we conducted in order to obtain answers regarding the hypothesis of the research, was only a small step, but it keep us in the belief and intention to define issues in the next survey research in way that would result in a set of indicators as a basis for future periodical measurements of teaching and educational processes quality at universities in Serbia from the employees' aspect.

## 6. CONCLUSIONS AND RECOMMENDATIONS

Based upon the research and appropriate analysis of each question, we could draw the following general conclusions:
• It is necessary to define and adopt a strategy of higher education;
• It is necessary to define and adopt the strategic objectives in higher education. All goals, including strategic, need to be measurable. It should be borne in mind that average of exam passing may be and it is, the most conflicted with the level of system / process / output / teachers quality;
• It is necessary to organize a public debate in universities and colleges before the above be accepted;
• It is necessary to defined processes;
• It is necessary to standardize performance measures for the system / processes / services rendered / output / staff and controls their fulfillment;
• The procedure for the survey of teaching staff at universities and other stakeholders should be defined by the team of competent experts. The survey should be conducted annually. During 2011 year should begin surveying industrial organization and state institutions about their satisfaction, and analysis of the employment rate of newly graduated students. (In 2010 first "Bologna students" began to graduate);
• The formation of the pyramidal structure of the Commission for the QMS (Quality Management System). Commission for the QMS must be an independent body (competent staff, independent of political parties and state bodies). Also, at all levels should be employed one person, who would be responsible for implementing decisions of the Commission and informed the Commission of the implemented measures and activities (the feedback link);
• It is necessary to carry out internal and external evaluation, analysis of results and definition of preventive and corrective measures in specified intervals;
• To be prepared for international comparative analysis of students functional knowledge assessment;
• Create teams of experts who will train teachers who receive a to low or to high ratings in the polls;
• To bring decision that students members of the Students Parliament can not be students who have lost a year and have an average rating below 7;
• To train students to fill in surveys about their satisfaction and introduce them to possible consequences arising from the survey results;

• Initiate projects that will investigate whether there is mobbing and corruption at universities.

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# Large scale data mining models via parallel regularized RBF Neural Networks

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**Abstract**

This paper investigates the development of non-linear data mining tools by means of different parallelization methods for regularized Radial Basis Function Neural Networks. Three different parallel approaches are studied, namely data parallel, neuron parallel as well as hybrid session parallel. A typical master-worker programming model is used for the implementation of all parallel structuring approaches discussed. The parallel neural networks are analyzed experimentally using the Message Passing Interface (MPI) library standard and measured for performance on a homogeneous Linux Cluster. For all cases, results are provided for speedup and scaling of the algorithms and discussed further in the text.

**KEYWORDS**

Parallel processing, Neural networks, data mining, RBFs.

## 1. INTRODUCTION

With the availability of inexpensive storage and the progress in data capture technology, many organizations have created ultra-large databases of business and scientific data, and this trend is expected to grow, thus giving rise to Parallel Processing in Data Mining techniques in recent years. High performance in data mining literally means to take advantage of parallel data systems and additional CPUs in order to gain performance benefits. By adding additional processing elements, more data can be processed, more models can be built and the accuracy of these models can be improved.

Artificial Neural Networks (ANNs) are not as common among a part of the data mining community yet, as they are considered to be time consuming. Their general abilities when compared with other data mining techniques to deal with data features show that they are very good in handling noisy data, poor in process large data sets, have excellent predictive accuracy, poor explanation capability, good integration abilities and difficult operation capabilities. However as a rule of thumb, ANNs are more accurate than many data mining techniques and the choice decision of the appropriate data mining tool is usually a cost-benefit analysis when it comes to real life applications. One percent increase in the accuracy may reveal most important improvements for many domains in the mining process.

This work presents an effort to combine Parallel Processing for Radial Basis Functions (RBF) Neural Networks and Data Mining models. The training phase of RBF Neural Networks can be considered as an optimization procedure towards searching for the best network parameters where several training methods can be applied. Popular algorithms for optimized training in RBF NNs are [Karayiannis and Randolph-Gips 2003] strategies selecting the RBFs randomly, strategies employing supervised procedures for RBF center selection and weights estimation, or unsupervised procedures for RBF center selection. But unlike the global optimization in Multilayer Perceptron (MLP) Neural Networks, the training in RBF NNs resembles more the local search in Facility Location Problems [Yingwei et al 1998] or mixed integer problems [Peng et al 2006]. This is due to the local nature of the RBF learning units. In general ANNs with computation-intensive training times can be well benefited by parallel processing as many studies and simulations of MLP Neural Networks and the Back-Propagation Algorithm on parallel computers [Šerbedžija 1996] [Suresh et al. 2005] have shown. Here by studying parallelism in one of the fastest existing RBF Neural Network training algorithms, more suitable for data mining applications, with time complexity almost O(N), we also set a starting point, to be further extended in the future, by employing training methods with higher complexities.

# 2. PARALLEL RBF NEURAL NETWORK IMPLEMENTATIONS

The Radial Basis Function Neural Network [Wang and Fu 2005] represents a special ANN model that uses radial basis functions as activation units, suitable for function approximation, classification [Oyang et al 2005] or system identification tasks, which has certain advantages over other types of ANNs, such as better approximation capabilities, simpler network structures and faster learning algorithms. On the contrary to classical MLP model, it is a network with local RBF units motivated by the presence of many local response units in human brain. Because of their universal approximation, more compact topology and faster learning speed, RBF neural network algorithms have been widely studied and applied in many science and engineering fields [Du and Zhai 2008] while recently attract considerable interest as data mining tools [Wang and Fu 2005] in moderate size datasets for rule extraction, classification and feature selection tasks.

The first parallel implementation of RBF Neural Networks was the Orthogonal Least Squares learning algorithm [Demian et al 1996] due to its higher time complexity, $O(N^4)$, and memory requirement $O(N^2)$, where N the potential train data centers. This complexity however is probably forbidden for large scale data processing. Thus we concentrate our study on the fastest sequential RBF algorithms, having complexity less than $O(N^2)$, which are the three step learning and the gradient learning. The later case of the gradient descent training algorithm with RBF neuron parallelism has been well studied before [Strey 2004]. The former case, on which we focus here, is divided into three steps [Wang and Fu 2005] [Sarimveis et al 2002] corresponding to the three distinct sets of network parameters, which are analyzed in the next paragraphs.

## 2.1. RBF Neural Network Topology

A typical RBF neural network has three layers, consist of input neurons, hidden neurons and output neurons respectively, with two synaptic weight matrices C and W in between. The hidden RBF layer is nonlinear, whereas the output layer is linear. Gaussian functions at the hidden layer with appropriate center means c and covariance matrices σ are popular choice of RBFs.

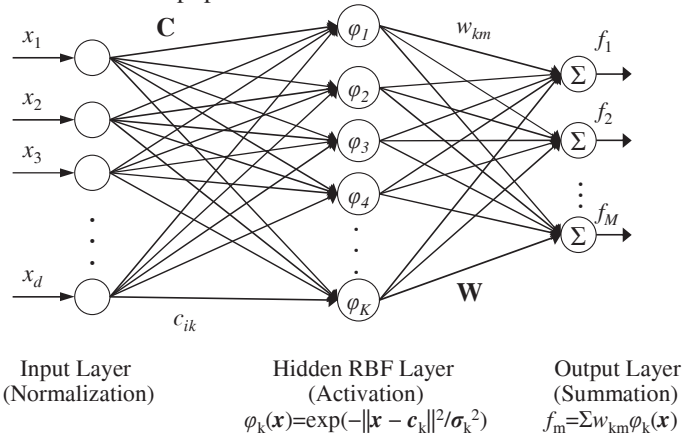


Figure 1. A three layer RBF Neural Network with d inputs, K hidden and M outputs.

Input neurons usually normalize the incoming input vectors x. The hidden RBF neurons in Fig. 1 are fully connected to all input neurons, with their synaptic weight matrix C to be formed by the centers of the radial basis functions. The hidden units form a layer of K receptive fields, which have localized response radial basis functions in the input space. The hidden layer performs a nonlinear transformation and maps the input space onto a new space. Here neuron k computes the Euclidean distance $d_k= \|x-c_k\|$ between an input vector x and the weight vector $c_k$ (the $k^{th}$ center of the weight matrix C). A radial symmetric function, typically a Gaussian, is applied to $d_k$ to form the response $\varphi_k(d_k)= \exp(-d_k^2/(2\sigma_k^2))$. The resulting outputs $\varphi_k$ of the hidden neurons are communicated via weighted links $w_{km}$ to the M neurons of the output layer where the sums $f_m$ are calculated. Thus the output layer implements a linear combiner on the transformed space, where the adjustable parameters are the synaptic weights W. Using exponentially decaying localized nonlinearities, RBFs construct local approximations to nonlinear input-output mappings.

## 2.2. RBF Neural Network Training

With a given training set $\{\vec{x}_i, \vec{y}_i\}_{i=1}^{N}$, the three step training algorithm of the network search the parameter space to find the best of all centers $c_{ik}$, sigmas $\sigma_k$ and weights $w_{km}$ (the center coordinates, widths and heights of the Gaussian bells). The first step consists of determining the centers by clustering. In the second step the additional hidden unit's parameters (widths, weighted norm matrices) are estimated. During the third step the output weights are determined and optimized by regularization.

To determine the parameters of the weight matrix C, which form the centers of the Gaussian Radial Basis Functions, the training data are clustered using fast unsupervised clustering methods. Some experimentation will be needed to find an adequate number of clusters K. Algorithms such as k-means, fuzzy c-means, or substractive clustering are all well documented procedures for this RBF training stage. Once the clustering is completed, the mean of each cluster C can be used as the center. Sequential k-means algorithm randomly choose k number of cluster centers, then assign data to each cluster, then recalculate the centers and repeat the processes. The parallel k-means uses data parallelism with Master/Worker message passing. Data points are evenly partitioned among all processes while the cluster centers are replicated. Global-sum reduction for all cluster centers is performed at the end of each iteration in order to generate the new cluster centers.

To find the spreads (widths) of every radial basis function one can simply set them equal to the variance of each cluster C, or else determine them by using the k-nearest neighbour rule. No matter which method is being used, the effect of the widths on the RBF hidden unit outputs will be tuned and regularized by the synaptic weights W and the regularization parameter λ, as it can be shown in the following.

To find the weights W between the hidden layer and the output layer, assuming that all the other parameters (the number of hidden units, centers, and widths of RBFs) have been fixed, one must minimize the Sum of Squared Errors (SSE) function commonly used as the objective function to measure network performance. The SSE for the RBF Neural Network is [Wang and Fu 2005]:

$$SSE = \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{M}\{\sum_{k=0}^{K} w_{km}\varphi_k^n - y_m^n\}^2$$

where N is the number of patterns, M is the number of outputs, and K is the number of hidden units. $W_{km}$ is the weight connecting the $k^{th}$ hidden unit with the $m^{th}$ output unit. $\varphi_k^n$ represents the output of the $k^{th}$ kernel function for the $n^{th}$ input pattern. $y_k^n$ represents the target output of the $m^{th}$ output unit when inputting the $n^{th}$ pattern. Minimization can be achieved by using gradient descend or directly solving the linear system: GW= Y where the matrix G={ $\varphi_{ij}(\| x_i - c_j \|^2, \sigma_j^2)$ , $i$=1,..,$N$, $j$=1,..,$K$ } have all the outputs of the RBF units, the matrix W={ $w_{ij}$, $i$=1,..,$K$, $j$=1,..,$M$ } contain all weights and the matrix Y={ $y_{ij}$ , $i$=1,..,$N$, $j$=1,..,$M$ } have elements all the desired outputs vectors. As we typically have more training samples N than hidden units K, the system is overdetermined. The weights can be computed as $W = G^{+}Y$ where $G^{+}= (G^{T}G)^{-1} G^{T}$ is the pseudo-inverse. It is noted that, if the matrix $(G^{T}G)$ is singular there exists no unique solution and methods such as Singular Value Decomposition must be used to find the pseudo-inverse. Following regularization theory of Tikhonov one can redefine the pseudo-inverse as $G^{+} = \lim_{\lambda->0} (G^{T}G + \lambda I)^{-1} G^{T}$ where $I$ is the unit matrix and λ a small positive regularization parameter. As $\lambda \rightarrow 0$, the $G^{+}$ solution becomes the least square solution. Instead of using $I$, the more general choice is using the sum of squared weight centers [Hu 2002] $G_0$={ $\varphi_{ij}(\| c_i - c_j \|^2, \sigma_j^2)$ , $i$=1,..,$K$, $j$=1,..,$K$ } which is a symmetric square matrix with elements the outputs of RBF units for inputs the same centers.

Thus the approximation RBF problem is to find the weight matrix W by minimizing the Sum of Squared Errors (SSE) function subject to the constraint $W^{T}G_0W$ = a constant. The solution to this constrained optimization problem become [Hu 2002]: $W = (G^{T}G + \lambda G_0)^{-1} G^{T} Y$ .

This regularization method is an easy way to avoid over-fitting the training set by tuning the architecture based on a loss function. The method simply penalizes 'wiggly' functions, often for neural networks, by adding λ times the sum of the squared weight centers. An iterative procedure can be used to compute the optimal regularization λ parameter as well as the number K of hidden neurons that minimizes the generalized cross-validation Error. Cross-validation techniques are based on the idea that we split the data into two parts, a training set and a validation set. The network is trained on the train set and then the error on the validation set is evaluated. The learning algorithm is then run on the train set with different setups (metaparameters).

## 2.3. Parallel RBF structuring approaches in MPI

Although RBF Neural Network training is faster than that in MLP neural networks, it takes a significant amount of time to train when it comes to a massive number of data. However the RBF hidden layer is much easier to interpret, by following simple locality principals, when compared to the black box solution in its MLP counterpart. This is a must in data mining [Wang and Fu 2005] and diagnosis models where explanation is important. When RBF training is finished their operation is usually slower than an MLP. This is due to their local nature which needs more hidden neurons to cover larger areas as opposed to the global approximation nature of the MLPs.

To improve the RBF Neural Networks performance, it is necessary to use the hardware platform as much as possible via parallelism in learning and testing. Implementation of parallel Neural Networks is not a straightforward task. In general it requires complete control on the ANN software architecture [Suresh et al. 2005] in order to reproduce specific parallel structures and control mechanisms. Since the architecture of each ANN model defines its possible parallel solutions, and as they differ very much from model to model, it is difficult to have a generic solution.

Exploitation of parallelism in parallel systems can be achieved in many different levels such as bit-level parallelism, instruction-level parallelism, data parallelism, task parallelism and system-level parallelism.

In the same spirit, structuring approaches for implementing parallelism in ANNs propose general guidelines (see taxonomy review in [Šerbedžija 1996]) on how to break an ANN structure from coarse grained to fine grained pieces in order to run them in parallel [Beckenkamp and Pree 1999][Pethick 2003]. The first approach in the parallel neuro-simulations taxonomy, the Session parallelism, places different training model sessions on different processors. A given ANN is trained simultaneously with different learning parameters in the same data patterns, allowing the implementation of parallelism by distributing the different ANN models on different processors for learning and testing processes. The second, the Data parallelism (or pattern-parallel), is implemented by simultaneous learning in different training data examples within the same session. A data set is split among processors and a corresponding neural network model, same for all processors, is trained simultaneously. The third, the Layer parallelism, provides concurrent computation for layers. Different layers are distributed on different processors and are pipelined so that learning examples are fed through the network in a way that each layer works in a different training example simultaneously in the pipeline. The fourth, the Neuron parallelism, splits and distributes the ANNs among the different processors. The distributed neurons perform weighed input summation and other computations in parallel. This typical biologically realistic structured parallelism also called network partitioning, is working well for moderate communication/computation ratios. The fifth, the Weight parallelism refines Neuron parallelism allowing the simultaneous calculation of each weighed input. This form of parallelism which is at the level of each synapsis can be implemented in most of ANN models. However, depending on the ANN type and train method, this fine-grained implementation can be many times proved to lead to performance slowing down, due to thread overload and the bottleneck effect. The sixth approach, the Bit parallelism, where each bit of the calculations is processed in parallel, is a hardware dependent solution.

In the work presented herein we implement parallel structuring approaches for the RBF neural network topology and training algorithms including RBF Session Parallel, RBF Data Parallel, RBF Neuron Parallel as well as weight matrices parallelism, and measured all cases for performance analysis and speedups. Exploiting loosely-coupled distributed systems all codes were implemented in C using Message Passing Interface (MPI) standard [Pacheco 1997] and tested for compatibility on two Linux Clusters. Message passing is a programming paradigm used widely on parallel computer, especially scalable parallel computers with distributed memory, and on networks of workstations. Parallel Programming environments like MPI are based on libraries running in user space, and thus require no alterations to the underlying operating system or the hardware that supports it. Therefore, the same program can also be used in heterogeneous cluster architectures, consist of processors with different speed and memory, which offer very powerful environment for parallel computing. Each MPI library provides a set of functions or primitives, allowing a programmer to create distributed programs.

The next paragraphs present performance results. For each resulting network we evaluate the error on the validation set. The network with the lowest validation error is then picked as the one with the best generalization ability. The usual 10-fold cross-validation process consists of 10 trials is used. In each trial, a subset is retained as the validation set, and the remaining 10−1 subsets are used as the train set.

# 3. ALGORITHMS AND RESULTS

Each RBF neuron executes a very simple calculation, and sends an output signal to the neurons of the next layer. The fact that the basic RBF neuron units execute local calculations, along with their batch training methods makes them more suitable for parallel processing.

The performance tests are conducted on a Linux Cluster of 12 Intel Pentium E5200 dual core machines with a clock speed of 2.5 GHz and 2GB of memory. The machines are interconnected with a 1000 Mbps Ethernet LAN. Results are determined for 1..2..4..6..8.. and 10 processors in the cluster, to demonstrate the scalability of the programs. All RBF Neural Network implementations are written in C using MPI library.

First we measure the classification performance. The benchmark datasets Iris, Wine, Wisconsin, Yeast and Diabetes came from the UCI Repository of Machine Learning Databases from the category 'Most Popular Datasets'. All the experiments measure also the classification rate on the test set (Classification Rate) = Correctly Classified Samples/(Total Samples).

Table 1. Classification rates on the cross-validated test set for several benchmark datasets

| Database | 1-Nearest Neighbor rule | RBF Neural Network |
|---|---|---|
| Iris | 93% | 98% |
| Wine | 93% | 98% |
| Wisconsin | 95% | 98% |
| Yeast | 51% | 60% |
| Diabetes | 70% | 75% |

The RBF column in table 1 shows the test errors from the RBF networks which are trained with 10-fold cross-validation and optimized for the number of RBF hidden units. The final models selected are those with the smallest cross-validated sum of squared errors, while we also observe and indicate the classification rate. The 1-NN column displays the results from the simple 1-Nearest Neighbor classifier in order to compare reasonably the linearity and difficulty of each problem.

## 3.1. RBF Session Parallelism

Typically, a classifier is parametric and the optimized parameters a(i) must be found for the final model. This is the procedure of model selection, using cross - validation. Several different models are produced from the train set, validated on the test set and the parameters with the best cross-validated error are chosen. The final model is built by training the classifier in all the available data using the optimal parameters. The execution times for training RBF Neural Nets are measured for kfolds= 10, max lamdas = 10, max neurons = 9 for the same well known benchmark datasets Iris, Wine, Wisconsin, Yeast, Diabetes.

The time measurements are taken in a Cluster using 1..2..4..6..8..10 processors. A well known measure of parallel programming efficiency is the Speedup = S / P , where P the time taken by the program to run in parallel machine, and S the time taken by the program to run in sequential machine. A near-linear speedup indicates the efficient exploitation of parallelism.

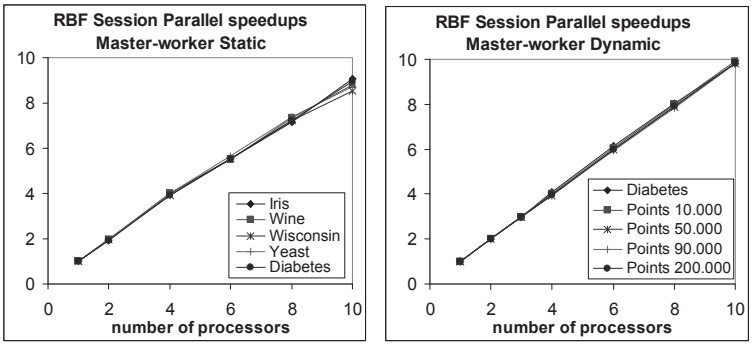


Figure 2. (A) RBF neural network Session Parallel with Master-Worker Static where one can observe almost linear (9/10) Speedups. (B) RBF neural network Session Parallel with Master-Worker Dynamic where linear speedups (10/10) are measured and this result is even more favoured by increasing the problem size and algorithm complexity.

For this RBF NN session parallelism we tried two parallel architectural approaches that of Master-Worker Static in Fig. 2A and Master-Worker Dynamic in Fig. 2B. In usual Master-Worker architecture the Master node can use the static partition and allocation of work or the dynamic. In the first, the static, load balancing is calculated and all works are distributed in the beginning of computations from Master node. This allows the Master to participate in the computations after finishing the distribution of work parts in each Worker. In Master-Worker dynamic the Master node doesn't equal-distribute from beginning the same amount of work in Workers, but the distribution is conducted dynamically. The Master dispatches small pieces of work in each Worker, while iteratively receiving results from a Worker that finished calculations and sending the next piece of work to this Worker. This send-receive process continuous until the work list is completed.

The time saving with increasing processors and the linear speedups are clearly shown in Fig. 2A. Although the sub-problems, the sequential RBF training for every fixed parameter set, and the construction of several models, have linear time complexity in a single processor, the speedups via parallelism of the total problem are almost linear even for small sized datasets. Note here that, each worker acts in a different given dataset each time, because the entire process Session Parallel is hybrid task and data parallelism. The tasks are the different parameters and the data are the different each time folds. In experiments with small number of RBF neurons the problem has linear time complexity on tasks and linear complexity on data size. In the hybrid parallelism the closer we approach linearity for these two quantities, tasks - data, it becomes more difficult to efficiently parallelise the problem. However the total number of calculations in a problem is large enough and thus the results of parallelism are most satisfactory. The Master / Worker Dynamic programming model can achieve very good performance, as shown in Fig. 2B. For a large number of processors the central control of Master may cause some communication bottleneck, which in turn can reduce the performance of the parallel program. It is possible to improve the scalability of the model by replacing the single Master with a group of Masters, each one of which would control a different team of Workers.

## 3.2 RBF Data Parallelism

For complex multidimensional problems, the number of training points is usually large, while the number of model parameters (weights) remains comparatively small. To tackle the problem, the Data Parallelism scheme can be adopted to distribute data across different processors. Each processor k of the p total processor nodes contains a copy of the entire RBF neural network architecture and the data set to be processed is then split up equally among the nodes. This approach could be easily extended to a grid environment with heterogeneous nodes. A function calculating the number of distributed points per node could send different number of points to nodes that is commensurate with its computational capability.

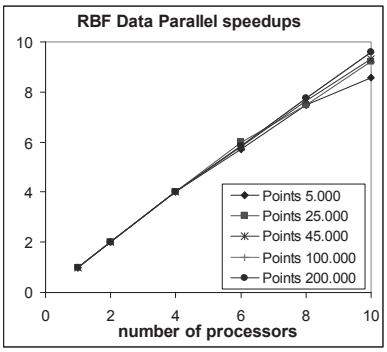


Figure 3. RBF Data parallelism speedups for points 5.000 to 200.000 as indicated.

In RBF Data parallelism with Master/Worker Architecture, Master node sends to all Workers the same copy of Neural Network. Then Master partition the data set and send a different portion to every worker. For each epoch (passes through the entire data set), in parallel the worker nodes independently process their local portion of the data set, exchanges its updates with other nodes, computes local error rates and send the results back to the Master. This approach ensures that all values needed during training phase, like output and error calculation, are locally available, reducing the communication among nodes and the need for algorithm synchronization. In distributed memory machines this RBF data parallelism approach seems to be very efficient, leading to great gain of operation performance, which clearly observed in Fig. 3.

Fig. 3 illustrates almost linear speedups (9.6/10) and there is clearly an improvement for larger problem sizes, a typical situation in parallel systems. The data parallelism on cluster systems reduces the amount of communication and can be an effective solution to all the causes of long training and execution times in sequential RBF neural networks. This type of data parallelism in a batch learning network should offer similar speedups for moderate and large sizes of layered neurons provided that the data set is large enough.

## 3.3 RBF Neuron Parallelism

In the Neuron Parallelism approach, Neurons are initially distributed among the processor nodes. While we mostly consider hidden RBF Neuron partitioning, similar to MLPs studied in [Suresh et al 2005], input or output neurons can be distributed as well. With large numbers of hidden neurons, a common phenomenon in locally response neural networks, this method can speed up the computations. In RBF hidden Neuron parallelism with Master/Worker programming model, the Master partition and distribute RBF hidden neurons and the corresponding weight matrices C and W across Workers. The input and output neurons are common for all processors. Each Worker uses its local neurons and spread parameters. Then, in a loop cycle, Master Broadcast in batches data points to Workers. Workers compute distances, kernels and send back partial sums for each output. Master node reduces these results to find labels for each data point, and compute errors. This testing cycle can be repeated for several parameters, until a stopping criterion. The batch size can be adjusted to modify the system's latency and to accommodate the availability of data.

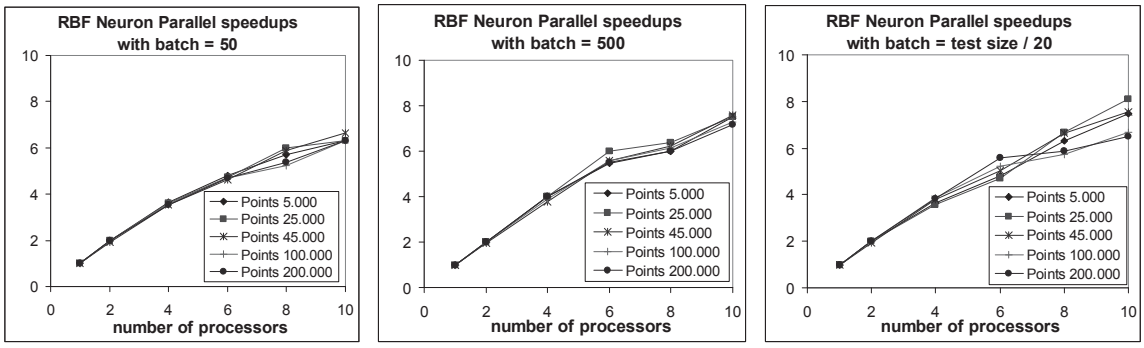


Figure 4. Illustrates from left to right: (A) Neuron parallel speedups for fixed batch number 50, (B) same parallel speedups for fixed batch number 500, and (C) same speedups for variable batch number.

A neural network belongs to the synchronous class of problems. The same operation is carried out for each pattern in the training set. Neuron parallelism attacks the training time problem by improving the time to process a single pattern. However the communication-intensive neuron-parallel RBF neural network implementation is much more likely to slow down on a cluster of workstations due to the high communication/computation ratio. The network latency and bandwidth on most cluster systems will significantly limit the degree of this type of parallelism. Often all the nodes of the cluster connected by a single network bus will need to send data at the same time, for instance at a synchronization point, which will partially saturate the network. Thus sub-linear speedups like those in Fig. 4 are commonly expected. As far as the batch size effect is concern, typical choices are shown in Fig. 4. For small batch size 50 in Fig. 4A, the cost of communication between neurons on different cluster nodes eventually overwhelms the potential speedup from parallelization and sub-linear speedups are emerging by increasing processors. For large batch size like 500 in Fig. 4B the situation is clearly improved, while speedups remain sub-linear. For a variable batch size as in Fig. 4C, equal to a portion of each dataset, the performance results depend on the total problem size. In the case of relatively small problem size (that is 5.000 to 45.000 points) the actual batch size lies between 250 and 2250 points, achieving the best results of the three experiments. In the case of lager problems (100.000 and 200.000 points) the actual batch size is 5000 or 10000 points, yielding lower speedups. It is evident that both very small and very large batch sizes result in lower performance. The optimal batch size for each problem is determined not only by the actual problem but also by the parallel system used. Systems with high communication bandwidth or even with shared memory architecture can accommodate larger batch sizes.

# 4. CONCLUSION

This work explores three different approaches for implementing parallel regularized RBF neural networks models in speeding up data mining tasks. Mappings of the neural network structures and modifications of their algorithms are tested here in order to achieve better parallel performance. The simulation results, on the session parallel neural network with master-worker static or dynamic as well as on the data parallel one, display almost linear speedups in a variety of benchmarks and tests running in multiple processors. As a consequence a reasonable linear scaling of RBF models on very large sized data sets can be achieved. This is an attractive result when considering the RBF connectionist model as an alternative to the Multi Layer Perceptrons for several tasks. Data parallelism should offer similar speedups for moderate and large neural network sizes of layered neurons, provided that the data set is large enough. For very small sized data sets, the overhead of any of the parallel algorithms will overshadow all the benefits. Experiments on the third approach, the communication intensive RBF neuron parallel, with respect to the batch size reveals that the overhead produced as a result of frequent synchronizations after every computation step may limit the degree of parallelism, producing sub-linear speedups and decelerating the simulations. It is commonly expected that both very small and very large batch sizes result in lower performance. Further, an optimal batch size can be calculated, which depends not only on the problem at hand but on the parallel system specifications as well. In the future we will try to analyze other types of training algorithms on a more extensive set of databases, as well as different neuron distribution strategies for performance comparisons. Finally, we will try to combine the different parallel implementations in a multilevel parallel hybrid scheme.

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# Measuring inflation dynamics in Central and Southeastern Europe within the Markov switching model approach

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**Abstract**

The correlation pattern of inflation dynamics is considered in the following countries from the Central and Southeastern Europe: Slovakia, Czech Republic, Poland, Hungary, Romania and Serbia. Sample covers monthly data from January, 1995 to May, 2010 for Poland, Hungary and Slovakia, and a previous year, 1994, in the case of Czech Republic. Inflation in Romania is considered for period January, 2002 - June 2010. The shortest sample is used for Serbia, given a late start of transition process (January, 2003 - September, 2010). The inflation dynamics is captured by the Markov switching autoregressive model approach that allows for random changes in economic regimes. It is found that inflation persistence is still of moderate to high magnitude in Hungary, Poland, Romania and Serbia. In Slovakia and Czech Republic inflation persistence is estimated to be of smaller order. It is also documented that changes in inflation persistence often correspond to changes in variability and mean of inflation.

**KEYWORDS**

inflation dynamics, inflation targeting, persistence, Markov switching models.

## 1. INTRODUCTION

The measure of inflation dynamics provides valuable information for central banks to conduct such a monetary policy that would meet announced target level. The correlation structure we refer to is also known as inflation persistence. Understanding the path of inflation persistence is *per se* relevant for those Central and Southeastern European countries (CSEECs) that have adopted inflation targeting. More specific, difference between inflation persistence in current euro area and in CSEECs that are EU members could suggest asymmetric impact of common shocks between two groups of countries and indicate extent to which their convergence towards the current euro area might decline upon the euro is adopted. In addition, the successfulness of recently introduced inflation targeting in some CSEECs can be assessed by taking into account experience of other economies from the same region.

It is widely noted that the adoption of inflation targeting should be associated with the sharp decrease in persistence. The identification of this fall in persistence has been extensively investigated for major industrial countries (the U.S. and euro area). In recent years performances of inflation targeting in emerging market economies have also been analyzed. Survey based on descriptive and econometric evidence (Siklos, 2008) confirms prevailing empirical findings that inflation targeting has diminished inflation persistence in industrial countries, but only in a handful of emerging market economies.

In this paper we consider the following CSEECs: Czech Republic, Poland, Hungary, Romania and Serbia. Czech Republic, Poland and Hungary have relatively long experience in inflation targeting, while Romania and Serbia introduced inflation targeting recently. The objective of the paper is to measure the magnitude of inflation persistence in these five economies. In addition, inflation dynamics is also considered in Slovakia, being the only country from this region currently in the euro zone. Estimation for Slovakian inflation will be used as a benchmark case.

Our analysis is based on monthly data which provide accurate insight into the inflation dynamics without inducing false autocorrelation (cf. Paya et al., 2007). The Markov switching regime time-varying model is used as a baseline framework (Hamilton, 1989, 1990). It is a non-linear extention of ARMA specification that induces more flexibility in capturing inflation dynamics over time.

Sample covers monthly data from January, 1995 to May, 2010 for Poland, Hungary and Slovakia, and a previous year, 1994, in the case of Czech Republic. Inflation in Romania is considered for period January, 2002 - June 2010. The shortest sample is used for Serbia, given a late start of transition process (January, 2003 - September, 2010). Our results contribute to the existing empirical results for CSEECs in the following

way: sample period is extended by including recent years of relatively high inflation rates, and Romania and Serbia, not being previously considered, are included.

Paper is structured as follows. Section 2 represents a short overview of empirical literature on inflation persistence in CSEECs and methodology applied in this paper. Findings derived are presented in Section 3. Section 4 concludes.

## 2. PREVIOUS LITERATURE ON INFLATION PERSISTENCE IN CSEECS AND SHORTH OVERVIEW OF METHODOLOGY APPLIED

The issue of inflation persistence in CSEECs has been considered in a number of papers. Early researches on this topic are based on micro data analyses for Czech Republic (Babetski et al., 2008), Poland (Konieczny and Skrzypacz, 2005) and Slovakia (Coricelli and Horvath, 2006). Monthly data are used for the sample that does not exceed year 2005. Persistence among different price indices is measured with the only finding for the overall CPI inflation in Czech Republic suggesting that inflation seems to be less persistent after adoption of inflation targeting. Magnitude of inflation persistence in Hungary is estimated by univariate and structural time series methods for the sample that ends in 2005 and mid 2006 (Darvas and Varga, 2007 and Menyhert, 2008). It is found that for that period inflation persistence in Hungary was higher than in US and euro area and that inflation dynamics was determined equally by past inflation and forward-looking expectations. Contrary to individual case studies, Franta et al. (2007) and Hondroyiannis et al. (2008) cover several countries from CSEE along with euro area based on sample that ends in the first quarter of 2006 and the third quarter of 2005 respectively. While estimation of different univariate and structural time series methods is performed in Franta et al. (2007), only structural models are considered in Hondroyiannis et al. (2008). The main results indicate that inflation persistence was comparable to that in the euro area with dominant role of backward behaviour in inflation dynamics.

Our results are based on seasonally adjusted monthly CPI index, denoted as $P_t$, based on which monthly inflation rate is calculated in percent as: $\pi_t = 100(\log P_t - \log P_{t-1})$. Baseline method in time series analysis to measure the persistence is the sum of autoregressive coefficients, $\sum_{i=1}^{p} \phi_i$, from the autoregressive model of order p, $\pi_t = \alpha_0 + \sum_{i=1}^{p} \phi_i \pi_{t-i} + e_t$, which can be rewritten as: $\pi_t = \alpha_0 + \rho \pi_{t-1} + \sum_{i=1}^{(p-1)} \delta_i \Delta\pi_{t-i} + e_t$, such that parameter $\rho = \sum_{i=1}^{p} \phi_i$ contains information about the sum of autoregressive parameters and thus provides measure of inflation persistence. Error term, $e_t$, is Gaussian white noise.

This specification can be modified in a number of different ways to take account of possible regime changes and nonlinearity in inflation rate. In fact, several empirical papers on inflation persistence in transition economies (for example, Franta et al., 2007 and Darvas and Varga, 2007) argued that linear specification is not rich enough to capture true dynamics in inflation rate.

To allow for changes in some parameters we employ the Markov-switching autoregressive model assuming that mean, variability and persistence differ among two regimes. The relevant specification is of the following form (Hamilton, 1989, 1990):

$$\pi_t = \left(\alpha_0 + \alpha_1 S_t\right) + \left(\rho + \rho_1 S_t\right)\pi_{t-1} + \delta_1 \Delta\pi_{t-1} + \ldots + \delta_{p-1}\Delta\pi_{t-p+1} + \left(h_0 + h_1 S_t\right)e_t \quad (2.1)$$

$S_t$ is the unobserved random variable that follows a Markov chain defined by transition probabilities between two states. The full matrix of transition probabilities for two states reads as follows:

| State at *t+1* | Condition at *t* | |
|---|---|---|
| | $S_t$=0 | $S_t$=1 |
| $S_{t+1}$=0 | q=$p_{0/0}$ | f=$p_{0/1}$ |
| $S_{t+1}$=1 | $p_{1/0}$ | $P_{1/1}$ |

Switches of economy from state 0 to state 1 is governed by introduced random variable $S_t$. Under this specification we have two different regimes: regime 0 (i.e. $S_t$ =0) and regime 1 (i.e. $S_t$ =1). The parameters $\alpha_1, \rho_1, h_1$ capture the changes in the mean of inflation, persistence of a shock to inflation and the variance during regime 1 relative to regime 0.

Parameters of the Markov-switching model can be estimated by the method of maximum likelihood. The likelihood of the model can be evaluated efficiently using the filtering procedure of Hamilton (1990).

## 3. EMPIRICAL RESULTS

This section contains summary of empirical results reached. Data are taken from WIIW Monthly Database for Eastern Europe and Central Bank of Serbia. Oxmetrics6.1 package is used (Doornik, 2009). Due to lack of space detailed results are presented only for some economies including two Balkan countries considered (Romania and Serbia).

*Hungary*

Two-state Markov switching model fits well dynamics of monthly inflation rate in Hungary for period January, 1995-May, 2010. Two different inflation persistence regimes have been detected. Regime 0 has lower inflation persistence characterized by estimated magnitude $\hat{\rho} = 0.38$. This is also a regime of lower mean inflation rate and lower variability, 0.07. Regime 1 is found to have higher inflation persistence: estimate is 0.82. During regime 1 inflation rate exhibited higher mean value and higher variability, 0.30. Statistically, there is a significant difference between two parameters of inflation persistence.

The probability *q* of remaining in the regime of lower persistence, while being in that regime is 0.55. The probability of remaining in regime of higher persistence, as already being in that regime is 0.92, implying that the probability *f* of switching from the regime of higher to regime of lower persistence is small and is equal to 0.08. Most of the time, 84.62%, economy is in the regime 1 of relatively high inflation persistence. This also holds for the last years of relatively high inflation rate.

Simple linear autoregressive model suggested that inflation persistence is of moderate size, 0.78. Since this specification is improved significantly by estimates obtained, our conclusion is that inflation persistence is relatively high in Hungary as prevailing estimate is 0.82.

*Czech Republic*

Two-state Markov switching model is also used to estimate monthly inflation rate in Czech Republic for period January, 1994-May, 2010. Results are reported in Table 3.1. Regime 0 has higher inflation persistence estimated to be 0.57. During this regime inflation rate exhibits higher mean, 0.135/(1-0.570)=0.31, and variability, 0.39. Regime 1 is described as regime of lower inflation persistence with estimate 0.46. During regime 1 inflation rate has lower mean value, 0.138/(1-0.460)=0.26, and variability, 0.15. Although two estimates of persistence do not differ significantly, we proceed with two regimes analysis given significant difference in inflation mean and variability.

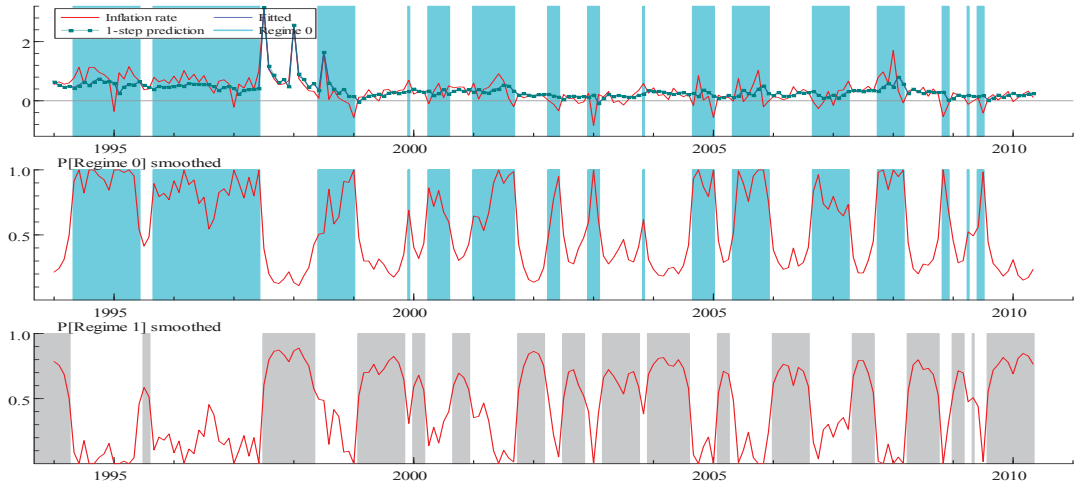
The probability *q* of remaining in the regime of higher persistence, while being in that regime is 0.80. The probability of staying in regime of lower persistence, as already being in that regime is 0.74, so that probability *f* of switching from the regime of lower to regime of higher persistence is 0.26. Visual inspection of regimes, from Graph 3.1., indicates that two regimes are equally split with 49.75% months in regime 0 and 50.25% months in regime 1. When we take into consideration last two years of data (25 months) it appears that this economy was most of the time (20 months) in the regime of lower inflation persistence, mean and variability.

Table 3.1. Estimated inflation model for Czech Republic

| Parameter | Estimate | t-ratio |
|---|---|---|
| $\alpha_0$ | 0.135 | 2.22 |
| $(\alpha_0 + \alpha_1)$ | 0.138 | 4.03 |
| $\rho$ | 0.570 | 5.54 |

| | | |
|---|---|---|
| $(\rho+\rho_1)$ | 0.460 | 6.73 |
| $h_0$ | 0.39 | 9.30 |
| $(h_0+h_1)$ | 0.15 | 4.26 |
| q | 0.80 | 6.28 |
| f | 0.26 | 2.09 |
| $\delta_1$ | -0.259 | -4.33 |
| $\delta_2$ | -0.123 | -2.48 |
| $\delta_3$ | -0.110 | -2.54 |
| Linearity test : $\chi^2_5=17.21(0.003)$; Box – Pierce Q(36) $\chi^2_{36}=47.54(0.10)$, ARCH1 F(1,181) $=0.212(0.64)$; Normality $\chi^2_2=1.15(0.56)$ Three impulse dummy variables are included that have non-zero value 1 for the following months: 1997:7, 1998:1 and 1998:7. | | |

Graph 3.1. Two regimes of inflation persistence in Czech Republic




*Poland*

Based on sample January, 1995 - May, 2010 monthly inflation rate is modeled within two-state Markov switching specification. Regime 0 is detected to have higher inflation persistence with estimate 0.89. Regime 1 is found to be of lower inflation persistence (estimate is 0.61). Parameters differ significantly ($\chi^2_1$ =3.70(0.04)). The probability *q* of staying in the regime of higher persistence, while being in that regime is 0.85. The probability *f* of switching from the regime of lower to regime of higher persistence given the regime of lower persistence is 0.40.

Almost 80% of the time economy was in regime 0 of higher persistence. The rest of 20% is described by regime 1 that is detected for blocks of one up to four months, mostly characterized by non-standard values. Due to these transitory shocks inflation uncertainty is estimated to be higher in this regime of lower persistence. Relatively high degree of persistence is associated with last two years of estimation which, as in case of Hungary, emphasizes the potential problem with ongoing economic instability on the level of inflation persistence.

*Romania*

Two-state Markov switching model performs well for Romanian monthly inflation using period January, 2002-June, 2010 (Table 3.2.). Regime 0 has higher inflation persistence estimated to be 0.73. Regime 1 is

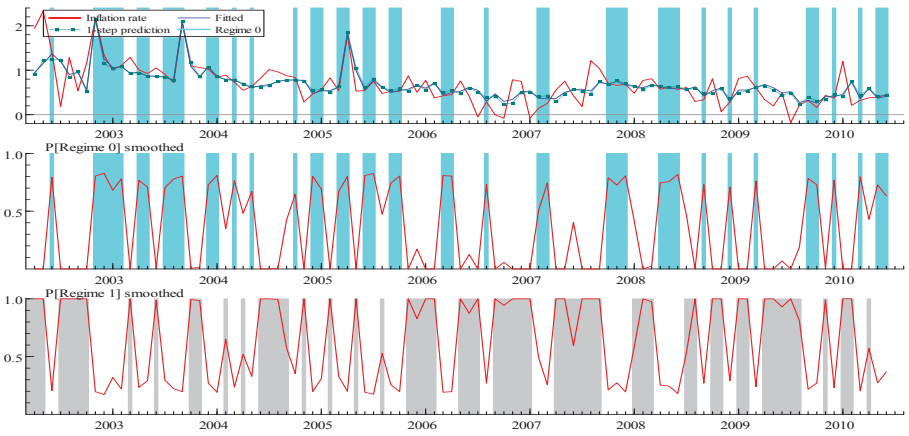
described to have lower inflation persistence (estimate is 0.52). These measures of persistence differ significantly. Average duration of regime 0 is 1.79 months and it takes 43.43% of the sample. The rest of 56.57% belongs to regime 1 that lasts on average 2.33 months (Graph 3.2.). Economy moves from one to another regime frequently. In this case higher inflation persistence episodes are associated with lower mean and less variable inflation rate.

The probability of staying in regime of lower persistence, as already being in that regime is 1-f=0.66, while the probability of switching into another regime is 0.34.

Table 3.2. Estimated inflation model for Romania

| Parameter | Estimate | t-ratio |
|---|---|---|
| $\alpha_0$ | 0.092 | 2.49 |
| $(\alpha_0+\alpha_1)$ | 0.308 | 3.22 |
| $\rho$ | 0.733 | 18.9 |
| $(\rho+\rho_1)$ | 0.522 | 4.46 |
| $h_0$ | 0.06 | 4.44 |
| $(h_0+h_1)$ | 0.40 | 10.2 |
| q | 0.41 | 2.80 |
| f | 0.34 | 3.69 |
| $\delta_1$ | -0.253 | -5.66 |
| $\delta_2$ | -0.204 | -7.43 |
| Linearity test : $\chi^2_5 = 31.93(0.00)$; Box – Pierce Q(36) $\chi^2_{36} = 26.04(0.89)$, ARCH1 $F(1,84) = 3.22(0.08)$; Normality $\chi^2_2 = 1.30(0.53)$ Three impulse dummy variables are included that have non-zero value 1 for the following months: 2002:11, 2003:9 and 2005:4. | | |

Graph 3.2. Two regimes of inflation persistence in Romania




*Serbia*

Inflation dynamics in Serbia is examined for the sample that covers shorter time interval than used for the rest of the countries. Serbia entered transition process at the end of 2000. To avoid shocks due to price liberalization and adjustment of several price indices undertaken at the beginning of this process, our sample starts in January 2003. It ends in September 2010.

Models with three regimes fits Serbian inflation rate satisfactory well. Estimated model is presented in Table 3.3., while corresponding transition probabilities are reported separately in Table 3.3a. Regime 0 has high persistence estimated to be 0.81. Average duration of regime 0 is 1 month taking 22.47% of the sample. Regime 1 is also associated with relatively high inflation persistence (estimate is 0.72). Similarly to regime 0, this one also lasts 22.47% of the sample with average duration of 1.25 months. Smallest persistence is found in regime 2, 0.48, that lasted on average 1.58 months. Among three regimes detected highest inflation variability is estimated for regime 2. Visual inspection of regimes is provided in Graph 3.3.

Adequacy of estimated model is confirmed by several specification tests. While simple AR model suggests that inflation persistence is just the value found for regime 2, 0.48, within this approach we are able to make distinction among episodes of different inflation behaviour. Similarly to results found for Romania, Serbian monthly inflation rate exhibits frequent changes of regimes that occur almost at monthly level. This could suggest extremely high level of sensitivity to unexpected random shocks. This is confirmed by inflation uncertainty being higher when the persistence is lower. In addition, given regime 2 of lower persistence there is almost equal probability (about 0.40) that economy will stay in that regime and switch to regime 0 of higher persistence.

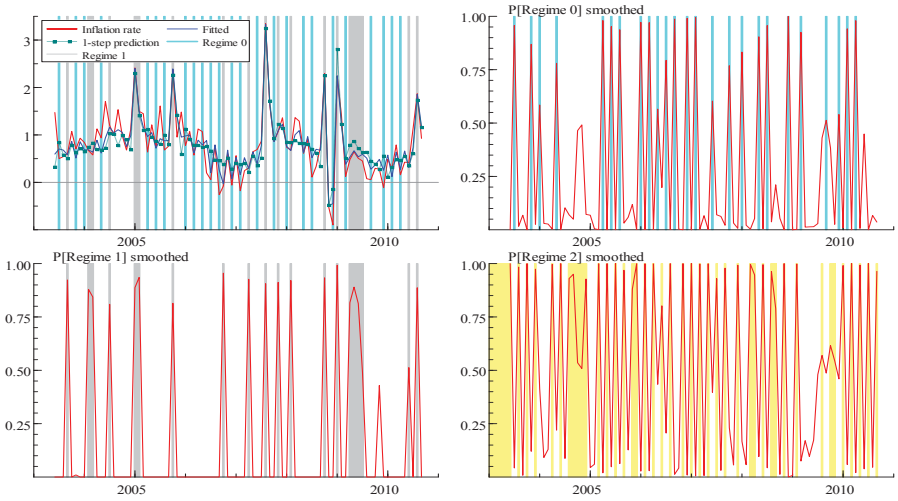
Table 3.3. Estimated inflation model for Serbia

| Parameter | Estimate | t-ratio |
|---|---|---|
| $\alpha_{0\ \text{regime 0}}$ | -0.448 | -1.67 |
| $\alpha_{0\ \text{regime 1}}$ | 0.081 | 7.96 |
| $\alpha_{0\ \text{regime 2}}$ | 0.533 | 5.27 |
| $\rho_{\ \text{regime 0}}$ | 0.815 | 5.46 |
| $\rho_{\ \text{regime 1}}$ | 0.719 | 63.5 |
| $\rho_{\ \text{regime 2}}$ | 0.479 | 6.15 |
| $h_{0\ \text{regime 0}}$ | 0.221 | 1.93 |
| $h_{0\ \text{regime 1}}$ | 0.014 | 5.10 |
| $h_{0\ \text{regime 2}}$ | 0.350 | 9.20 |
| $\delta_1$ | -0.179 | -14.1 |
| $\delta_2$ | -0.191 | -20.6 |
| $\delta_3$ | -0.032 | -4.48 |
| Linearity test : $\chi^2_{10} = 43.63(0.00)$; Box – Pierce Q(36) $\chi^2_{36} = 26.50(0.88)$, ARCH1 F(1,165) = 0.02(0.90); Normality $\chi^2_2 = 4.12(0.13)$ Five impulse dummy variables are included to have non-zero value 1 for the following months: 2005:1, 2005:10, 2007:8, 2009:1 and 2010:8. Transitory dummy with only non-zero values 1 and -1 for 2008:11 and 2008:10 respectively is also introduced. | | |

Table 3.3a. Estimated transition probabilities

| State at $t+1$ | Condition at $t$ | | |
|---|---|---|---|
| | $S_t=0$ | $S_t=1$ | $S_t=2$ |
| $S_{t+1}=0$ | 0 | 0 | 0.40 |
| $S_{t+1}=1$ | 0.19 | 0.18 | 0.22 |
| $S_{t+1}=2$ | 0.81 | 0.82 | 0.38 |

Graph 3.3. Three regimes of inflation persistence in Serbia




*Slovakia*

At the end of our modeling Slovakian monthly inflation is considered for period January, 1995-May 2010. Estimated model for Slovakia implies one regime of inflation persistence, but two regimes of inflation mean and variability (Table 3.4.). Estimated inflation persistence is 0.28. Regime of lower inflation mean covers 96% of the sample and has average duration of 21.75 months. The rest of 4% is described by higher mean inflation rate that lasted on average 1 month. Slovakian economy has been permanently in the regime of lower mean inflation since November 2005.

Table 3.4. Estimated inflation model for Slovakia

| Parameter | Estimate | t-ratio |
|---|---|---|
| $\alpha_0$ | 0.206 | 7.11 |
| $(\alpha_0+\alpha_1)$ | 0.914 | 14.9 |
| $\rho$ | 0.283 | 5.81 |
| $h_0$ | 0.20 | 16.5 |
| $(h_0+h_1)$ | 0.10 | 2.40 |
| $\delta_1$ | -0.174 | -3.87 |
| $\delta_2$ | -0.130 | -3.30 |
| $\delta_3$ | -0.109 | -3.21 |
| $\delta_4$ | -0.064 | -2.41 |
| Linearity test: $\chi^2_3 = 16.70(0.0)$; Box – Pierce Q(36) $\chi^2_{35} = 42.590(0.18)$, ARCH1 F(1,163) = 0.37(0.54); Normality $\chi^2_2 = 0.85(0.65)$ | | |
| Four impulse dummy variables are included to have non-zero value 1 for the following months: 1999:1, 1999:7, 2000:2 and 2003:1. Also, two composite dummies are introduced such that the first one takes non-zero value 1 for 2001:2 and 2000:2, while the second one has non-zero value 1 for 2004:1 and 2003:1. | | |

Findings reported are summarized in Table 3.5. Inflation in Slovakia exhibits by far the lowest level of persistence. Relatively modest magnitude of persistence was determined for Czech Republic. Only in these two economies there was no significant difference between inflation persistence across regimes.

Inflation persistence remains at relatively high level in Hungary and Poland, such that regimes of higher inflation persistence dominate over sample considered. The same was concluded for Romania and Serbia. Additional finding for later two economies is that they switch from one to another regime frequently in a way that relatively lower inflation persistence is characterized by higher inflation uncertainty. This could also be a sign of high sensitivity of inflation rate to unexpected random shocks. Reaction of inflation to these shocks may be described by lower persistence in one of detected regimes, but at the cost of more variability in inflation rate.

Table 3.5. Estimated inflation persistence across countries

| Country | Estimated inflation persistence | |
|---|---|---|
| | Regime of lower persistence | Regime of higher persistence |
| Hungary | 0.38 | 0.82 |
| Czech Republic | 0.46 | 0.57 |
| Poland | 0.61 | 0.89 |
| Romania | 0.52 | 0.73 |
| Serbia | 0.48 | 0.72/0.81 |
| Slovakia | 0.28 | |

## 4. CONCLUSION

It is shown that inflation persistence is still of moderate to high magnitude in Hungary, Poland, Romania and Serbia. In Slovakia and Czech Republic inflation persistence is estimated to be of smaller order. It is documented that changes in inflation persistence often correspond to changes in variability and mean of inflation. Contrary to countries from Central Europe, in two Balkan countries, Romania and Serbia, inflation dynamics exhibit more frequent switches across different inflation persistence regimes.

Comparison among current euro zone countries and economies covered here is relevant as enables assessment of extent to which inflation dynamics in CSEECs might converge to that of euro zone. We can take the case of Slovakia as a benchmark, for which we have an evidence of the lowest persistence level. There is no enough evidence to conclude that inflation dynamics in the countries analyzed is likely to converge to the inflation dynamics of the current euro zone members. Similar conclusion is reached when structural econometric approach is implemented (Mladenović and Nojković, 2011).

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# Analysis of the urban air pollution in Bucharest

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**Abstract**

We studied the possibility to predict the air pollution analyzing the time series of measurements for NO, CO and PM 2.5 at the center of the city of Bucharest using the techniques of the dynamics of chaos. We found that the individual prediction of the pollutants give better results that the simultaneous prediction. This is due perhaps to fact that some pollutants do not follow the model of evolution chosen by us. The predictions are more accurate for short time (one day) but sometimes we got surprisingly good results for a week.

**KEYWORDS**
Air pollution, dynamics of chaos, time series.

## 1. INTRODUCTION

The use of the methods of dynamics to analyze the time series of air pollution is not new (see [7]-[9]). The evolution of the air pollution was also analyzed using ARIMA models or neural networks ([10]-[13]). Depending on the case one method of prediction or another were found to give more accurate predictions. For the present study we had at our disposal the hourly measurements of NO, CO, PM 2.5 at the center city of Bucharest between 1/1/2007 01:00 - 12/30/2008 24:00. Effectively we used the data between 1/1/2007 01:00 - 6/30/2008 24:00 for calculations and the predictions of the pollution for the next few days were compared with the measured data. One finds a review of the theory used here in [1]-[2], and practical applications in [3]. For computations we used the TISEAN package, version 3.0.1 ([4], [6]).

## 2. METHODS AND RESULTS

### 2.1 Mathematical aspects

Assuming the data follow a deterministic evolution

$$\overline{x}' = \overline{X}(x) \tag{1}$$

with $\overline{x}$ in some finite or infinite dimensional normed space E, after a certain amount of time the trajectory of the dynamical system (1) approaches an attractor A. The dynamics in A can be captured virtually by any function $s : E \to R$. Let $\Phi(t, x)$ the flow of (1) and $F(\overline{x}) = \Phi(\tau, \overline{x})$, $F^{-1}(\overline{x}) = \Phi(-\tau, \overline{x})$, for a fixed value of $\tau$, and let $s : U \to R$ be defined in a neighborhood of A. Then according to a theorem of Takens ([1], [2], [5]), under general conditions for s and for enough large m the application

$$\begin{aligned} &S : A \to R^m \\ &S(\overline{x}) = \left(s\left(F^{-(m-1)}(\overline{x})\right), s\left(F^{-(m-2)}(\overline{x})\right), \ldots s\left(F^{-1}(\overline{x})\right), \overline{x}\right) \end{aligned} \tag{2}$$

is an embedding of A in $R^m$. In practice we have a time series $(s_n)_{n\in N}$, the observed values of $s: U \to R, U \supset A$ with

$$s_n = s(\Phi(n\tau, \bar{x}_0)) = s(F^n(\bar{x}_0)) \tag{3}$$

On the embedded attractor $S(A) \subset R^m$ we have

$$S(\Phi(n\tau, \bar{x}_0)) = \bar{s}_n = (s_{n-(m-1)}, s_{n-(m-2)} \ldots s_{n-1}, s_n) \tag{4}$$

The dynamics F on the attractor A is expressed on its image $S(A)$ by

$$\bar{s}_n \to G(\bar{s}_n) = \bar{s}_{n+1} \tag{5}$$

A slight modification of (4) is

$$\bar{s}_n = (s_{n-(m-1)d}, s_{n-(m-2)d} \ldots s_{n-d}, s_n) \tag{6}$$

with $d \in N^*$ some delay. For functions s with values in $R^p$ the embedding (2) is in $R^{mp}$.
The optimum delay for (6) is recommended (see [1], [2]) the first minimum of the empirical mutual information

$$H = \sum_{i,j} p_{i,j}(d) \ln\left(\frac{p_{i,j}(d)}{p_i\, p_j}\right) \tag{7}$$

The probabilities $p_i$ and $p_{i,j}(d)$ in (7) are obtained by partitioning the data $(s_n)_{1\le n\le T}$ in a number of boxes: $p_i$ is the probability to get s in the box i and $p_{i,j}$ is the probability of transition from the box i to the box j after d units.
To determine the embedding dimension m we can use the method of the false nearest neighbor (see [1], [2]). The idea is that if $\bar{s}_j$ is the nearest neighbor of $\bar{s}_i$ in $R^m$ then $\bar{s}_{i+1} = G(\bar{s}_i)$ and $\bar{s}_{j+1} = G(\bar{s}_j)$ are also close and the ratio $R_i = \| \bar{s}_{i+1} - \bar{s}_{j+1} \| / \| \bar{s}_i - \bar{s}_j \|$ is not too large. For an $R_i$ greater than an heuristic threshold the point $\bar{s}_j$ is marked as false neighbor of $\bar{s}_i$ and m is considered too small if the fraction of points having a false neighbor is too large (for example greater than a given threshold).
To predict further values of s we used the zero order prediction and the local linear prediction. The zero order prediction is given by

$$s_{n+k} = \frac{1}{\mathrm{card}(U_n)} \sum_{\bar{s}_j \in U_n} s_{j+k} \tag{8}$$

where $U_n$ is a small neighborhood of $\bar{s}_n$ in $R^m$. The predictions are done for $n + k > T$, the last measurement.
The local linear prediction is given by the formula

$$s_{n+1} = \bar{a}_n \bar{s}_n + b_n \tag{9}$$

where the coefficients $\bar{a}_n \in R^m$, $b_n \in R$ are determined such that

$$\sigma^2 = \sum_{\bar{s}_j \in U_n} (s_{j+1} - \bar{a}_n \bar{s}_j - b_n)^2 \tag{10}$$

is minimum. The sum is performed over those $\bar{s}_j$ in a small neighborhood $U_n$ of $\bar{s}_n$ but not too small to contain enough points to ensure the least squares problem (10) is nonsingular.
For measurements s taking values in $R^p$ the above formulas have to be modified accordingly.

## 2.2 Results

Using the TISEAN package for NO data we found the optimum delay d=8 and for this d the fraction of points with false nearest neighbor is under 6.8% if $m \geq 30$. For CO data the optimum d is 8 and same m as for NO. For PM 2.5 data the optimum delay is reported by the procedure"mutual" of TISEAN as d=21. For zero order predictions (see (8)) we used d=1 and d=optimal. For the dimension m of the embedding (6) we used $m = 48 \geq 30$ (two weeks of data) and we used also m=100. Experimentally we found that the neighborhood $U_n$ has to be enough small such that in (8) the average should be performed with $k \approx 10$ points. Too many points make the graph of the predicted values too smooth and far from the measured values. In the Table 1 we summarized the results of individual zero order predictions (8) for the week 7/1/2008-7/7/2008 (168 hours) and for one day, 7/1/2008. As the measure of fit of the predicted values with the measured values we used the correlation ratio

$$\eta_{yy_x} = \sqrt{1 - \frac{\sum (y - y_x^*)^2}{\sum (y - \bar{y})^2}} \tag{11}$$

where y are the measured values and $y_x$ are the predicted values. This ratio takes values between 0 and 1 and its significance is checked with Fisher statistics

$$F = \frac{n-k}{k-1} \frac{\eta^2}{1-\eta^2} \tag{12}$$

If $F < F_{\alpha=5\%}(k-1, n-k)$ the dependence between $y_x$ and y is insignificant.

Table 1. Individual zero order prediction of the pollutants

| Predicted poluttant | $\Sigma(y-y_x^*)^2$ | Correlation ratio $\eta_{yy_x}$ | $\Sigma(y-y_x^*)^2$ | Correlation ratio $\eta_{yy_x}$ |
|---|---|---|---|---|
| | 168 hours | | 24 hours | |
| NO m=48, d=1, k=10 | 28884.41 | insignificant | 4805.636 | insignificant |
| NO m=48, d=8, k=10 | 19239.02 | insignificant | 2389.156 | insignificant |
| NO m=100, d=1, k=10 | 22175.74 | insignificant | 1268.734 | insignificant |
| NO m=100, d=8, k=10 | 13218.82 | 0.2858*** | 1208.371 | insignificant |
| CO m=48, d=1, k=10 | 0.65150 | 0.9429*** | 0.00158 | 0.9975*** |
| CO m=48, d=8, k=10 | 0.56106 | 0.9510*** | 0.03147 | 0.9494*** |
| CO m=100, d=8, k=10 | 8.87495 | insignificant | 0.61507 | insignificant |
| CO m=100, d=1, k=10 | 0.58149 | 0.9492*** | 0.00141 | 0.9978*** |
| PM 2.5 m=48, d=1, k=10 | 13057.64 | insignificant | 1370.578 | insignificant |
| PM 2.5 m=48, d=21, k=10 | 11202.69 | insignificant | 1977.665 | insignificant |
| PM 2.5 m=100, d=1, k=10 | 20684.21 | insignificant | 1096.740 | 0.4793* |
| PM 2.5 m=100, d=21, k=10 | 24997.12 | insignificant | 2340.423 | insignificant |

The NO prediction is bad according to the Table 1. In the Fig. 1 we see the best prediction of NO which corresponds to m=100, d=8.

In Fig. 2 we see that CO prediction is well predicted for $m \geq 48$, for d=1 but for the optimal delay d=8 the prediction fails for large m (Fig. 2, d). Perhaps this is due to insufficient number amount of measured data.

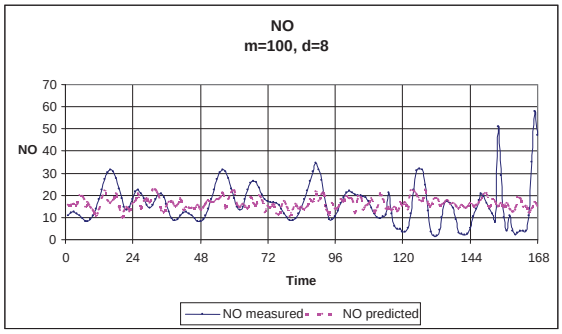



Fig. 1. The NO prediction is bad for each choice of m and d.
Here we have the best result we get for m=100, d=8

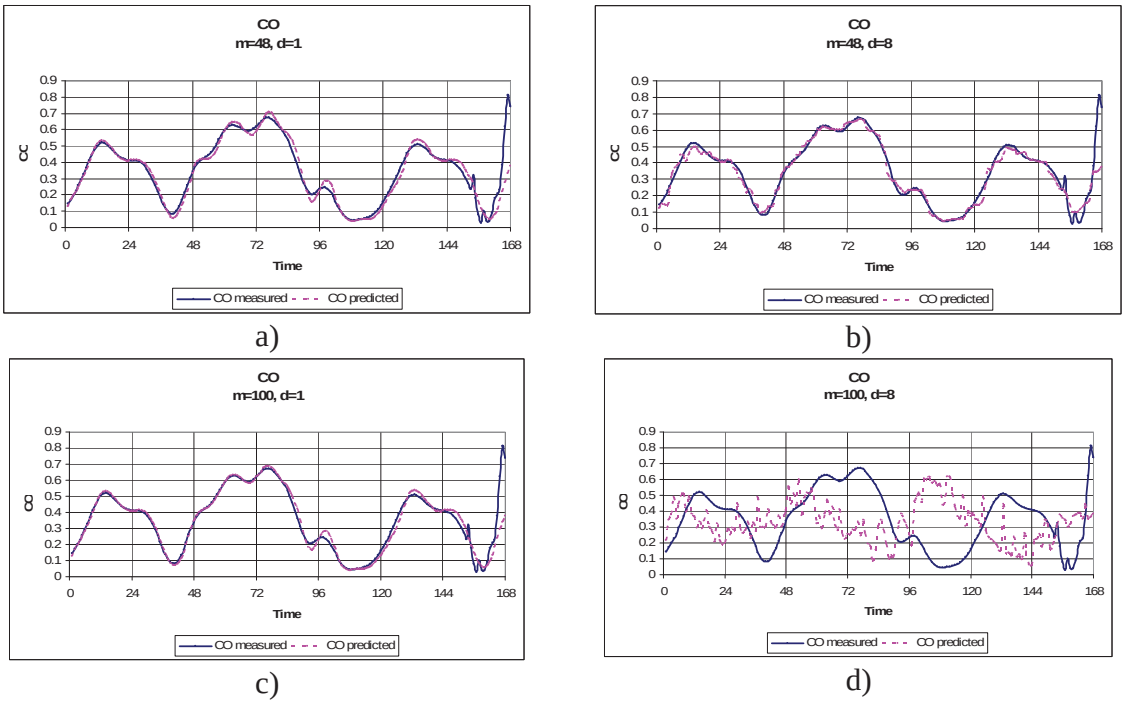
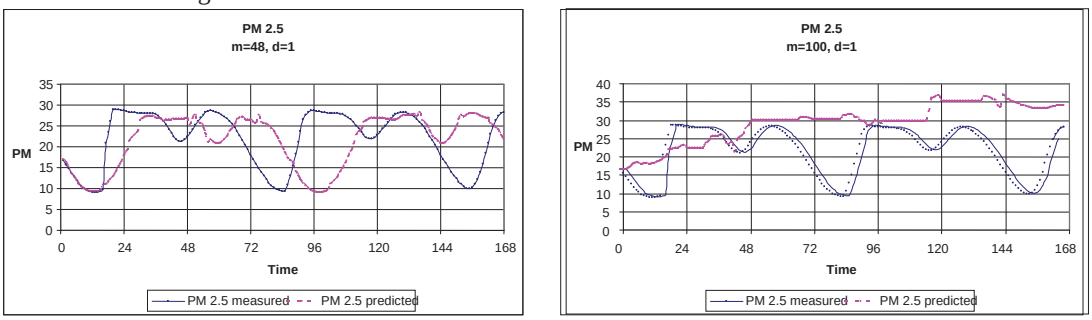



Fig. 2

As we can see in the Table 1 the 168 hours prediction for PM 2.5 is bad for all the cases of m and d, but for m=48, d=1 the prediction is good for the first 18 hours. The results indicate that the choice of a delay equal to the first minimum of the mutual information is not always a good choice. It is better to choose the delay d=1.

We tried also a simultaneous prediction of these 3 pollutants. The embedding dimension in this case is $3 \cdot m$. Using m=48 and d=1 we have the results in the Table 2. The best predictions are obtained at d=1 and are significant only for CO and PM 2.5 for a time interval less than 20 hours as we see in the Fig. 4. One sees that the simultaneous prediction is worse that individual prediction. Perhaps NO is not following an evolution according to the section 2.1 and this has influence on the simultaneous prediction of CO and PM 2.5.
The above results make apparent that m=48 and d=1 are good choices for the prediction in the chosen location. In the Fig. 5 we see the prediction of PM 2.5 for the week 7/1/2009-7/7/2009 using measured data between

1/1/2007-7/1/2009. The prediction is almost identical to the measured date. Unfortunately the predictions for NO and CO are not so good.

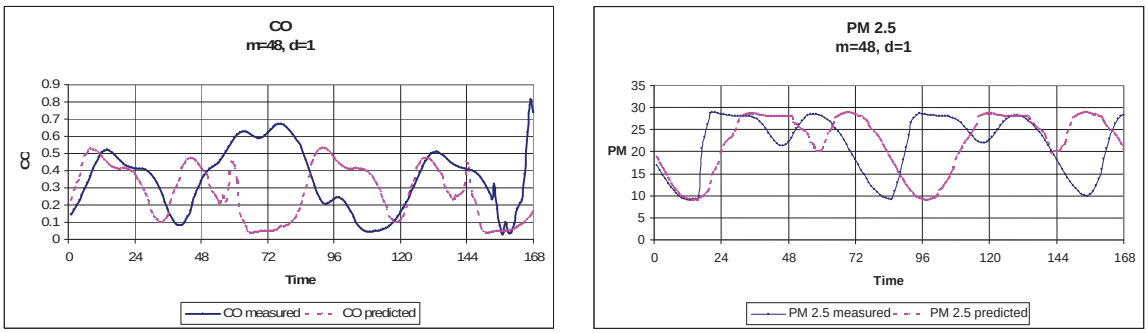



a) b)

Fig.3

Table 2. Simultaneous prediction of the pollutants for the zero order prediction

| Predicted poluttant | $\Sigma(y-y_x^*)^2$ | Correlation ratio $\eta_{yy_x}$ |
|---|---|---|
| | 24 hours | |
| NO m=48, d=1, k=10 | 3607.249 | insignificant |
| CO m=48, d=1, k=10 | 0.234383 | 0.7178*** |
| PM 2.5 m=48, d=1, k=10 | 1396.125 | 0.3738*** |
| NO m=100, d=1, k=10 | 1572.624 | insignificant |
| CO m=100, d=1, k=10 | 4.281166 | insignificant |
| PM 2.5 m=100, d=1, k=10 | 1734.327 | insignificant |
| NO m=48, d=8, k=10 | 1918.217 | insignificant |
| CO m=48, d=8, k=10 | 0.69648 | insignificant |
| PM 2.5 m=48, d=8, k=10 | 1976.788 | insignificant |
| NO m=100, d=8, k=10 | 2687.823 | insignificant |
| CO m=100, d=8, k=10 | 2.843159 | insignificant |
| PM 2.5 m=100, d=8, k=10 | 1291.255 | 0.5525*** |




a) b)

Fig. 4

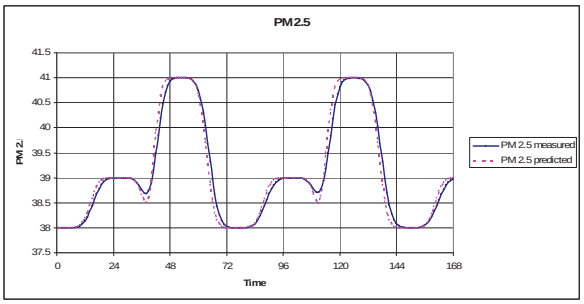



Fig. 5

We found the linear predictions (formulas (9), (10)) give less accurate results as zero order predictions. In the Table 3 we summarized the results we get by running the linear prediction procedure with the indicated parameters. The prediction is compared with the measured results. It is worthily to note that for linear prediction it is necessary to adjust the minimal number of points in the neighborhood $U_n$ such that the minimization in (10) be not singular. We found experimentally that a number of points four times the number of parameters to estimate is enough. For long term the zero order prediction is more accurate but for one short term the linear prediction works too.

Table 3. Individual prediction of the pollutants for the local linear prediction

| Predicted poluttant | $\Sigma(y-y_x^*)^2$ | Correlation ratio $\eta_{yy_x}$ |
|---|---|---|
| | 24 hours | |
| NO m=48, d=1, k=200 | 2163.91 | insignificant |
| NO m=48, d=8, k=200 | 11548.26 | insignificant |
| NO m=100, d=1, k=400 | 1915.36 | insignificant |
| NO m=100, d=8, k=400 | 2624.02 | insignificant |
| CO m=48, d=1, k=200 | 0.07 | 0.8795*** |
| CO m=48, d=8, k=200 | 0.05 | 0.9229*** |
| CO m=100, d=1, k=400 | 0.04 | 0.9393*** |
| CO m=100, d=8, k=400 | 0.64 | insignificant |
| PM 2.5 m=48, d=1, k=200 | 989.65 | 0.5816*** |
| PM 2.5 m=48, d=8, k=200 | 1338.94 | insignificant |
| PM 2.5 m=100, d=1, k=400 | 1044.06 | 0.5494*** |
| PM 2.5 m=100, d=8, k=400 | 4653.74 | insignificant |

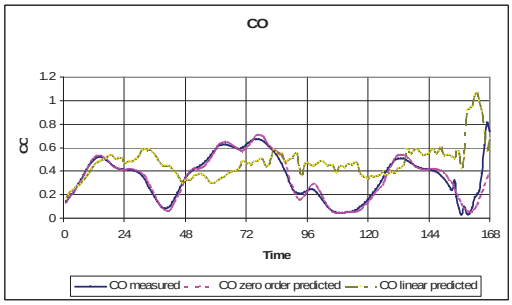



Fig. 6

## 3. CONCLUSIONS

1. Using a delay d=1 is a good choice, sometimes better that the delay at which the mutual information reach a local minimum.
2. A good choice for the number k of points in the formula (8) is to be found experimantally. For linear prediction a good choice of the number of points in the formula (10) is at least four times the number of parameters to estimate.
3. The choice of the embedding dimention such that the fraction of points having the nearest neighbor false is less than 6% give good results. It is also indicated to choose m such that the vector $\bar{s}_n$ encompasses the data of a period if it is suspected to exist one.
4. It is possible the data do not follow a deterministic law $s_{n+1} = F(\bar{s}_n)$. In this case any choice of m, d is a bad choice.
5. Short term predictions are generally more accurate. The good prediction of CO for a week (Fig.2, a, b, c) is rather surprisingly.

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# Analysing volatility of Belgrade stock market using multifractal analysis

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**Abstract**

Nowadays, it is obvious that the linear paradigm cannot explain widespread critical events on financial markets. There are some general ways of explaining market dynamics. Besides variety of common stands, devoted to classical econometric standpoint, there are a lot of nonlinear techniques, looking for deterministic nonlinear and chaotic dynamics in different economical systems, and in particular in the markets. In that way, different recent multifractal techniques proved to be useful in giving a lot of information about volatility of financial time-series.
Analysis of cumulative distribution for stocks, have led to conclusion that stocks have power law behavior on tails of distribution. This was an evidence of presence of scaling in investigated data. This important step gave rise to applying the methodology of multifractal theory to various financial instruments.
In this paper, four firms contributing the Belgrade Stock Exchange Index – BELEX15, have been studied. The first two are found among the 5 most developed and strong firms from BELEX15. They are the most "efficient" in academic sense. The third one is one of the 5 worst firms, and the fourth one is among the 5 medium firms. In our work for numerical study we used the *Fraclab* Matlab application. In the study we have shown that the proposed methodology of the multifractal singularity spectrum is applicable and works well for classification of firms contributing in BELEX15, based on similarity of fluctuation of their stock exchange prices and BELEX15, in Belgrade stock market. The multifractal methodology is applicable to firms with essentially different dynamics.

**KEYWORDS**

Multifractal analysis, stock market time series, volatility.

## 1. INTRODUCTION

Financial markets are systems where a large number of participants interact with one another. They react to some information in order to determine the best price for a given item. In financial stock markets, the dynamical change of price over time shows a highly complex behavior because it is generated by the nonlinear interactions among heterogeneous agents [1]. When someone inspects a time series of the time evolution of the price, volume, and number of transactions of a financial product, he recognizes that the time evolution is unpredictable. At first sight, one might sense a curious paradox. An important time series, such as the price of a financial good, is essentially indistinguishable from a stochastic process.

Markets are complex systems that incorporate information about a given asset in the time series of its price. The most accepted paradigm among researchers in finance is that the market is highly efficient in the determination of the most rational price of the traded asset.

The efficient market hypothesis was originally formulated in the 1960s [2]. A market is said to be efficient if all the available information is instantly processed when it reaches the market and it is immediately reflected in a new value of prices of the assets traded. The theoretical motivation for the efficient market hypothesis has its roots in the pioneering work of Bachelier [3], who at the beginning of the twentieth century proposed that the price of assets in a speculative market be described as a stochastic process. This work remained almost unknown until the 1950s, when empirical results [4] about the serial correlation of the rate of return showed that correlations on a short time scale are negligible and that the approximate behavior of return time series is indeed similar to uncorrelated random walks.

The efficient market hypothesis was formulated explicitly in 1965 by Samuelson [5], who showed mathematically that properly anticipated prices fluctuate randomly.

Financial time series look unpredictable, and their future values are essentially impossible to predict. This property of the financial time series is not a manifestation of the fact that the time series of price of financial assets does not reflect any valuable and important economic information. Indeed, the opposite is true. The time series of the prices in a financial market carries a large amount of nonredundant information. Because the quantity of this information is so large, it is difficult to extract a subset of economic information associated with some specific aspect. The difficulty in making predictions is thus related to an abundance of information in the financial data, not to a lack of it. The efficient market is an idealized system. Real markets are only approximately efficient. The concept of the efficient market is useful in any attempt to model financial markets. After accepting this paradigm, an important step is to fully characterize the statistical properties of the random processes observed in financial markets.

In the works studying cumulative distribution for stocks [6], the authors concluded that stocks have power law behavior on tails of distribution. This was an evidence of presence of scaling in investigated data. This important step gave rise to applying the methodology of multifractal theory to various financial instruments. The empirical results show that the multifractal models can obtain the forecasting accuracy in stock markets[7].

In our paper, for numerical study we used the *Fraclab* Matlab application. We have shown that fluctuations of returns in stock market time series show multifractal properties. This multifractal characteristic is reflected in a definite geometry for the series, arranged around fractal components of characteristic power law behavior under changes in scale. In this paper, taking 2 years high-frequency data (16.01.2009.- 21.01.2011.) consisted of daily prices of the Belgrade Stock Exchange Index BELEX15 (closing prices for 510 working days), as an example, we propose a classification measure based on the multifractal spectrum of the high-frequency price variability within a trading day.

## 2. MULTIFRACTAL ANALYSIS

Multifractals were first introduced by Benoit B. Mandelbrot who conceived and developed a new geometry of nature and implemented its use in a number of diverse fields [8]. Multifractal formalism relies on the fact that the highly nonuniform probability distributions arising from the nonuniformity of the system often posses rich scaling properties including that of self-similarity [9]. The study of long-term dynamical behavior of the physical system can then be attempted by the characterization of the fractal properties of a measure that can be associated with the nonuniform distribution.

In general, two different types of multifractality in time series can be distinguished [10]:

- Multifractality due to a broad probability density function for the values of the time series, In this case the multifractality cannot be removed by shuffling the series.
- Multifractality due to different long range (time-) correlations of the small and large fluctuations.

Due to the nature of the stock exchange time series, they can be treated as a long range process. Also, the stock exchange time series exhibit self-similarity and hence can be subject of multifractal analysis. Multifractal analysis (MA) is concerned with the study of local irregularities of measures or functions in a geometrical and statistical fashion [11]. The main feature of MA is the multifractal spectrum, a function which quantifies the number of points having the same singularity.

The multifractal formalism describes statistical properties of these singular measures in terms of their singularity spectrum, or their generalized dimensions [9]. In particular, if we cover the support of the measure with boxes of size $l$ and define $Pi(l)$ to be the probability in the $i$-th box, then we can define an exponent (singularity strength) $\alpha_l$ by

$$Pi(l) \sim l^{\alpha} \tag{1}$$

If we count the number of boxes $N(\alpha)$ where the probability $Pi$ has singularity strength between $\alpha$ and $\alpha + d\alpha$, then $f(\alpha)$ can be loosely defined as the fractal dimension of the set of boxes with singularity strength $\alpha$ by

$$N(\alpha) \sim l^{f(\alpha)} \tag{2}$$

This formalism leads to description of multifractal measure in terms of interwoven sets of Hausdorff dimension $f(\alpha)$ possessing singularity strength $\alpha$ [9].

Before showing the results obtained in this paper we give an intuitive interpretation of the actual values ($\alpha$, $f(\alpha)$) seen in a spectrum. A typical example of a multifractal spectrum is illustrated in Fig. 1. The

multifractal spectrum provides a global description of the singularities of the observed measure. The parameter $\alpha$ quantifies the degree of regularity in a point x. Values $\_(x)<1$ indicate, thus, a burst of events around $x$ 'on all levels' (bursts of bursts), while $\alpha(x)>1$ is found in regions where events occur sparsely. The spectrum $f(\alpha)$ captures how 'frequently' a value $\alpha(x)=\alpha$ is found [12].

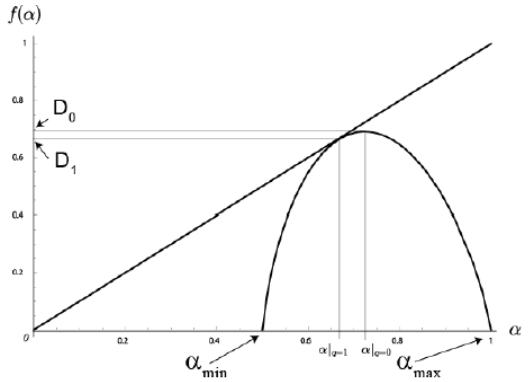


Figure 1. Example of multifractal spectrum.

Now days there are different methods for determination of singularity spectrum. Multifractal detrained fluctuation analysis (MDFA) has been applied in a multitude of diverse scientific areas, and has proven successful in calculation of multifractal spectrum in analysis of stock exchange time series [1]. In this work Legendre Multifractal Spectrum is used for characterization of multifractal properties of stock market time series.

## 3. Methodology and results

In this paper, four firms contributing the Belgrade Stock Exchange Index – Belex15, have been studied. First firm is one of the 5 worst firms (firm A), second one is among the 5 medium firms (firm B). Third firm and fourth firm are treated as the most developed and strong firms (firm C and firm D). Firm D is the best firm, one in the 5 most developed firms. It is the most "efficient" in academic sense. We followed prices in 2 years time (16.01.2009.- 21.01.2011.) for firms A,B,C and D. In our work for numerical study we used the Fraclab Matlab application.

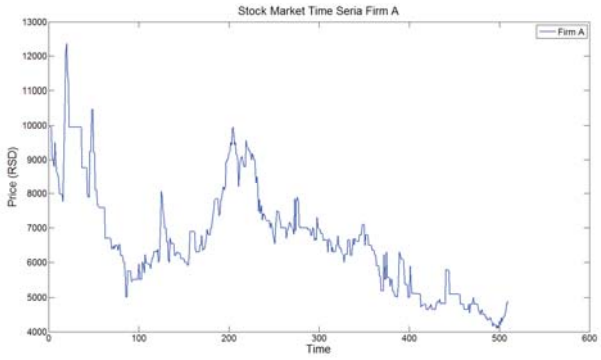


Figure 2. Stock market time series for firm A.


Figure 3. Stock market time series for firm B.

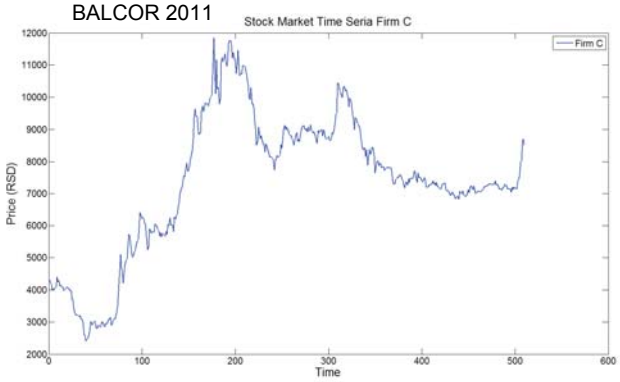


Figure 4. Stock market time series for firm C.

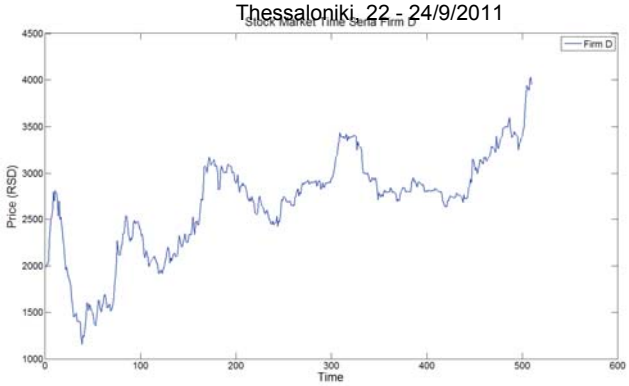


Figure 5. Stock market time series for firm D.


Figure 6. Stock market Belex15 Index time series.

Figures 2-5 represent the time series of prices of companies A, B, C and D, concerning the closing trading price of their stocks, while Figure 6 shows the time series of the Belgrade Stock Exchange index BELEX15 affecting the companies A, B, C and D with appropriate weights, for 2 years time period (510 working days). Looking at time series of companies, we cannot assign clear classification of companies in terms of impact on the value of the index BELEX15. Knowing the pure weights influence of the individual companies on the index, cannot give a clear answer to the question of which companies actually dictate the movement of the index BELEX15 of the stock market, because the weights below 0.25 (range of values 0-1) for each company, which is insufficient in order to change the value of shares of any company at any time, have a significant impact on the change of the index.

If we observe the correlation of whole time series of share prices of companies that determine the movement of the stock market index and the index itself, we can see that certain companies have a greater similarity to the trend of the index than the actual value of the percentage of participation in the formation of the index (weight). The aim of this paper is to attempt to classify companies by separating a group of companies whose share price fluctuation has a similar trend of fluctuation as the stock exchange index, which can greatly assist in planning and investing on the stock exchange.

By analyzing weights, we noticed that certain figures have significant impact on BELEX15. Nevertheless, this does not mean that BELEX15 will follow the movement of prices of thoose firms. This phenomenon cannot be detected simply by observing correlation of time series of firms prices and BELEX15. For that reason, we applied MA, to verify similarity based on weights.

The correlation between firms A, B, C and D and BELEX15, respectively, are shown in figures 7-10.

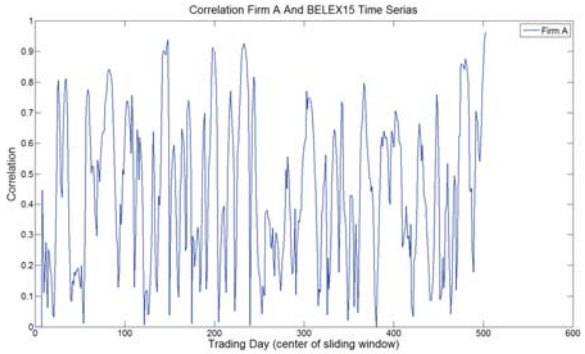


Figure 7. Correlation of time series for Firm A and Belex15.

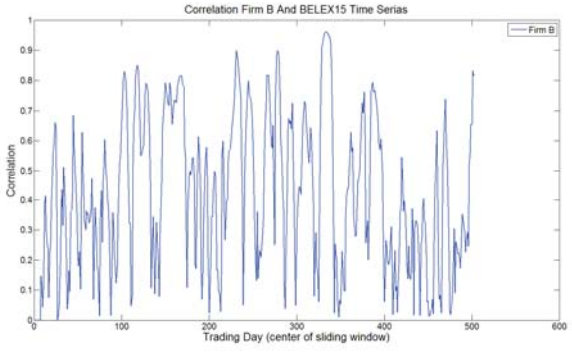


Figure 8. Correlation of time series for Firm B and Belex15.

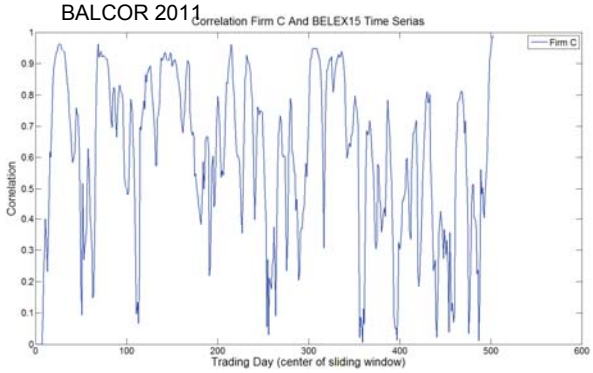


Figure 9. Correlation of time series for Firm C and Belex15.


Figure 10. Correlation of time series for Firm D and Belex15.

As already mentioned, the correlation on the level of the whole time series is not sufficiently accurate to determine the changes of trends in the index. For this reason, we introduced the sliding window, which represents the time interval of 15 trading days within which the correlation was analyzed. In Figures 6,7,8 and 9 the results of the comparison are shown. The company A doesn't have a high correlation at the local level with an index of the stock exchange, company B has certain periods of time when it followed the trend of the index. Companies C and D have longer periods of time when the share prices trend followed the trend of the index.

Furthermore, the observation of the signal in the local time intervals is not suitable for classification, and for that reason it is necessary to describe the time series in a more appropriate manner, both locally and globally. The most suitable tool for this purpose is the application of multifractal analysis (MA) on the time series. In this case we used the MA Fraclab software, running on Matlab platform. Specifically, to calculate the singularity spectrum of MA we used *Legendre Multifractal spectrum*, based on Box Counting method of calculation. Multifractal spectrums of time series are shown in figures 11-17.

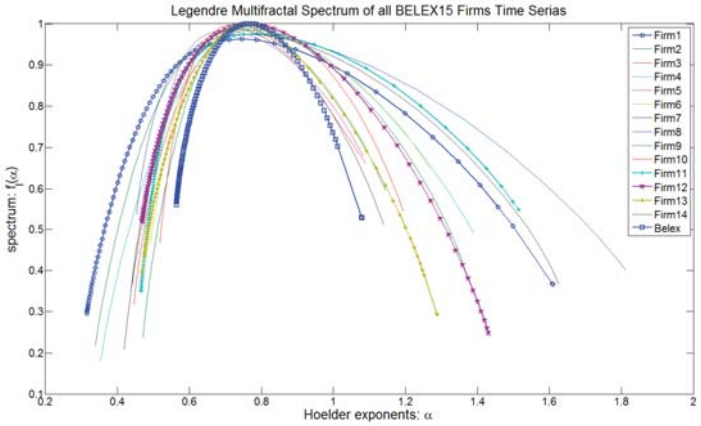


Figure 11. Legendre Multifractal Spectrum of all time series for Belex15 firms.

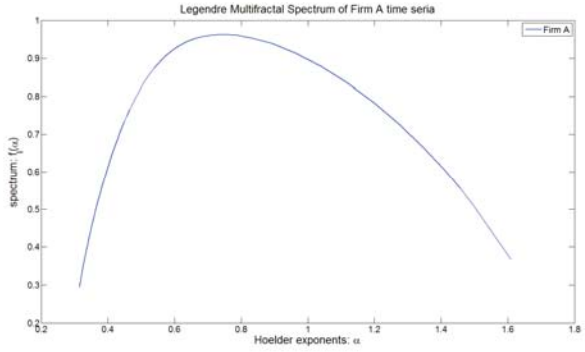


Figure 12. Legendre Multifractal Spectrum of time series of firm A.

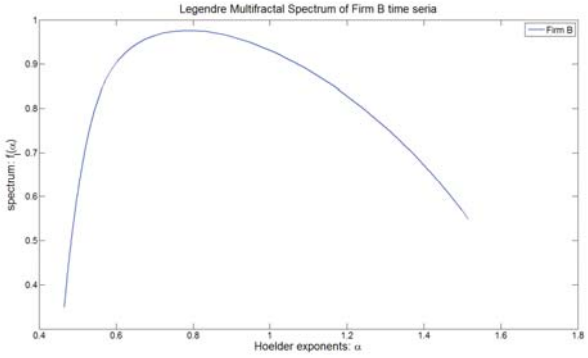


Figure 13. Legendre Multifractal Spectrum of time series of firm B.

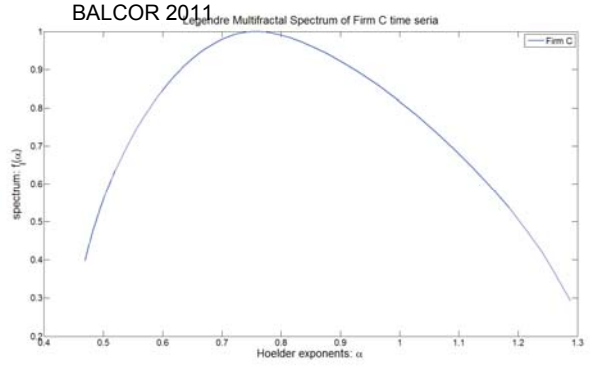


Figure 14. Legendre Multifractal Spectrum of time series of firm C.

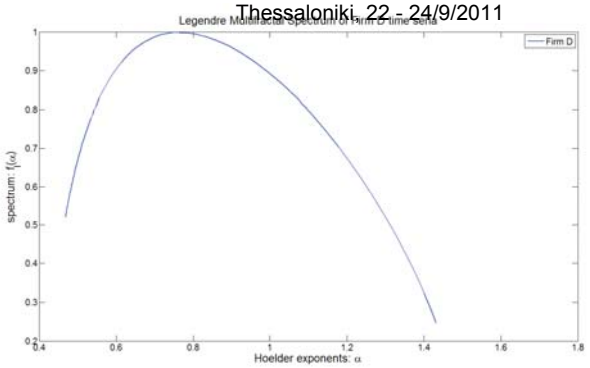


Figure 15. Legendre Multifractal Spectrum of time series of firm D.

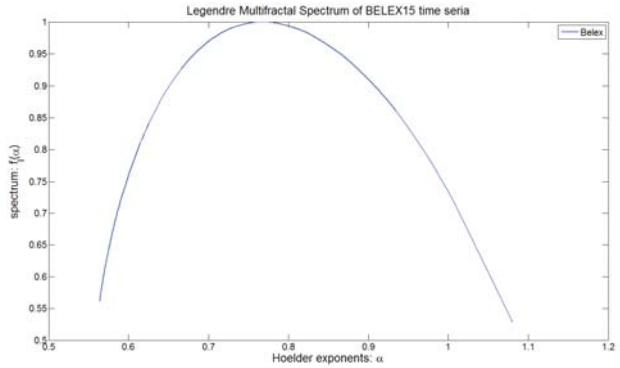


Figure 16. Legendre Multifractal Spectrum of time series of Belex15.

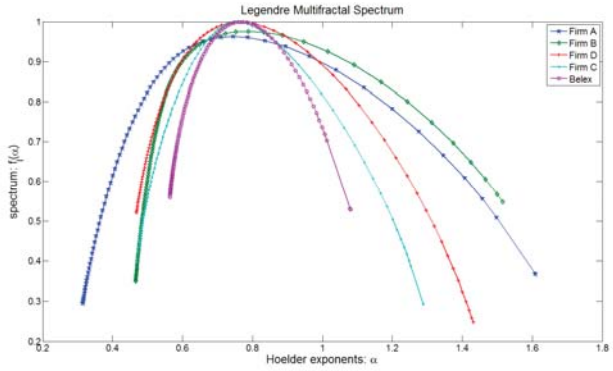


Figure 17. Legendre Multifractal Spectrum of time series of firms A, B, C, D and Belex15.

In Figures 11-16, the multifractal spectra of time series of stock prices of companies that influence BELEX15 are shown. Figure 11 gives a summarised multifractal spectrum of all companies within BELEX15 index. Figures 11-15 show individual spectrums of the selected companies and the stock market index BELEX15. In Figure 17 we grouped spectra of 4 selected companies. In Figure 11, we can clearly see the group of spectra that have a similar shape as the spectrum of BELEX15 (Company 2, Company 3, Company 10, Company 12, Company 13). The correlation of prices of time series of firms (Index basket constituents) and BELEX15 is given in Table 1. The resemblance is there, but not emphasized. This result is in accordance with the results of application of correlations on the whole time series.

On figure 17, we can see that the firm C is closest (most similar) to BELEX15, which is in accordance to global correlation of total time series. But, we consider that situation on the local level (individual firm) is not in accordance with the global correlation. For that reason, in order to make clear classification at the local (individual) level, it is necessary to apply MA on sliding window correlation results. Sliding window referes to observing sequence of 15 working days with one day step during 2 years time (510 working days).

Table 1. The correlation of prices of time series of firms (Index basket constituents) and BELEX15.

| Firm | Correlation (Firm & BELEX15) |
|---|---|
| Firm 1 | 0.1715 |
| Firm 2 | 0.8853 |
| Firm 3 | 0.8109 |
| Firm 4 | 0.5255 |
| Firm 5 | 0.7478 |
| Firm 6 | 0.8667 |
| Firm 7 | 0.6679 |
| Firm 8 | 0.7329 |
| Firm 9 | 0.5048 |
| Firm 10 | 0.8911 |
| Firm 11 | 0.709 |
| Firm 12 | 0.8002 |
| Firm 13 | 0.9655 |

| | |
|---|---|
| Firm 14 | 0.7945 |

Figures 18-23 show results obtained by applying multifractal analysis on the results of correlation within a sliding window with time interval of 15 working days. In Figure 18, it can be seen that companies that follow the trend of the index BELEX15 are quite evident and easy to observe and classify. In this case, the Company B is not in that group, although it has a high correlation with the index, at the level of the whole time series. When applying correlation to the time interval, a different form of time series is obtained. This is very specific when it comes to time series, which each have the same trend. Since we reduced the interval of the calculation of correlation, the local similarity of time series is emphasized.
Multifractal spectrum of sliding window correlation is shown in Figures 18-23.

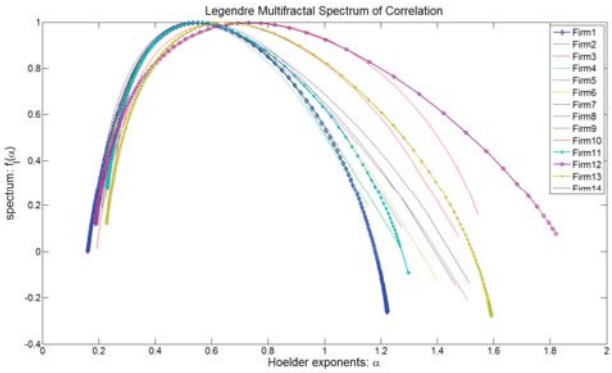


Figure 18. Legendre Multifractal Spectrum of Correlation for all Belex15 firms.

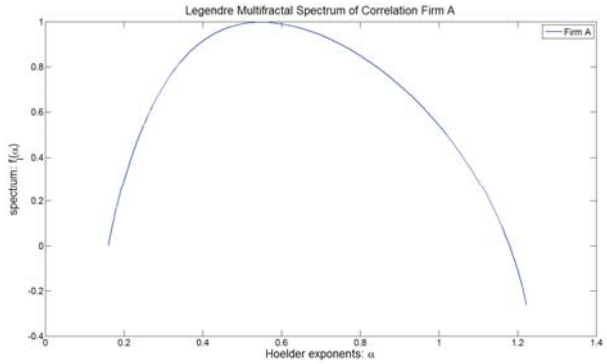


Figure 19. Legendre Multifractal Spectrum of Correlation for firm A.

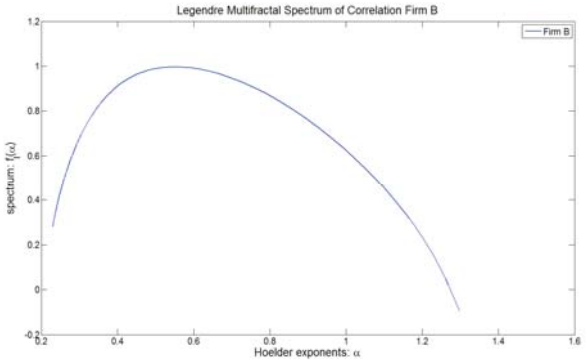


Figure 20. Legendre Multifractal Spectrum of Correlation for firm B.

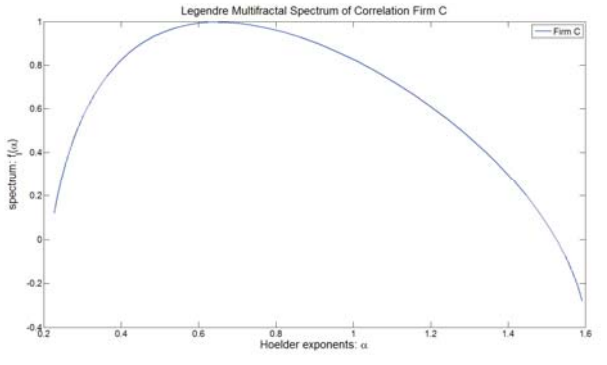


Figure 21. Legendre Multifractal Spectrum of Correlation for firm C.

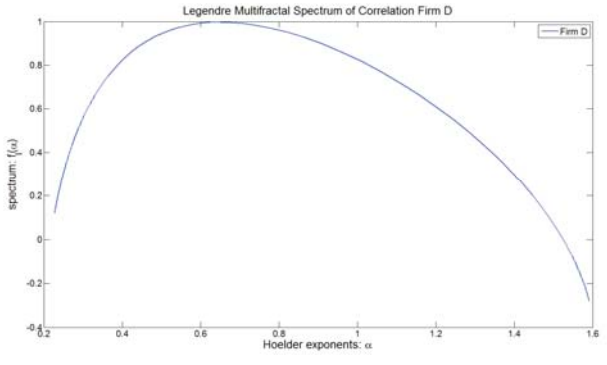


Figure 22. Legendre Multifractal Spectrum of Correlation for firm D.

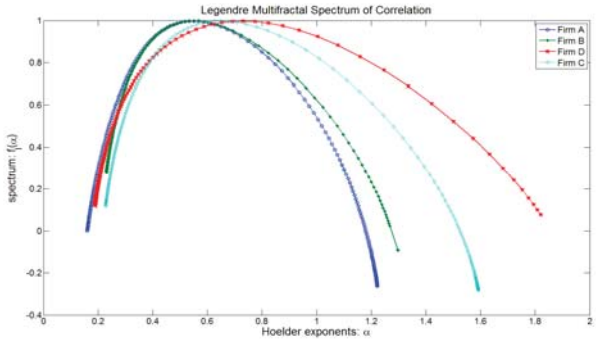


Figure 23. Legendre Multifractal Spectrum of Correlation for firms A,B,C,D and Belex15.

The value of correlation has increased, so the new series produced by such a calculation has different characteristics compared to the basic time series. The series locally very similar to BELEX15 have wider intervals of high correlation, which is reflected in the increased width of multifractal spectrum, and marked sudden changes that are reflected in higher values of Hoelder exponent. In that way, we can easily detect time series with fluctuations very similar to the fluctuation of stock market index.

In our case, when analized lokaly, for 15 working days period, we can see that firms C and D have very similar fluctuations as stock market index BELEX15.

We shall now compare figures 17 and 23. On figure 17 we can clearly mark similarities, so *Legendre Multifractal spectrum* tels us that firm C is more similar to BELEX15 than firm D. This is not the case on the local level. On figure 23, we showed local similarities with *Legendre Multifractal spectrum of sliding window correlation.* Large similarity on local level is shown trough wide curve, which showed that there are areas with high similarity of firm D prices and Belex15 (figure 23). It is characteristic for two high similarity areas that they include sharp descent between them.This is reflected by widness of curve. Besides, firms C and D have shifted peaks on the right, compared to firms with low local similarity. On this way, we obtained a different result and it is obvious that firm D is the most similar to index BELEX15.

## 4. CONCLUSION

Classification method presented in this paper combines multifractal analysis and statistiacl tools in order to extract main features of the stock exchange time series in Belgrade stock exchange. Characterization of the stock exchange time series as signals with multifractal properties provides new opportunities for analysis. The main purpose of this research was to demonstrate the feasibility of local classification of the stock exchange time series based on their multifractal properties, and to check possibility of application of MA on stock exchange time series data.

Obtained results indicate the feasibility of the method, which is able to classify the stock exchange time series.

By applying multifractal analysis on the results of correlation within a time interval of 15 working days, when trading on the Belgrade Stock Exchange, we selected companies whose trends of share price fluctuations are in line with the trend of fluctuations in stock market indices. Such a classification proves that when using combined multifractal analysis and correlation, we can classify the companies that can be considered as reflection of stock exchange movements in the market, based on monitoring index BELEX15.
Future directions concern the stability and econometric interpretation of this multifractal caracteristics of stock exchange time series, towards description of the actual time series, even prediction.

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# ECONOMETRIC MODELS AND FORECASTING

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**Abstract**

In the contemporary environment characterized by the dynamic structure of factors and the unpredictability of the relations existing between them, the central problem is the selection of strategic goals.
Forecasting is the necessary precursor to the planning process and includes research into the future course of events. Numerous methods and techniques of forecasting are used nowadays. Econometric models can be used successfully for predicting the future development of a phenomenon.

**KEYWORDS:** forecasting, econometric models, decision making

## 1. RESEARCH RESULTS

This paper uses data published by the Republic Office of Statistics of Serbia, obtained through Household Consumption Surveys.

The Household Consumption Survey is one of the oldest surveys conducted by the Republic Office of Statistics. The methodology of this survey has been revised several times since 1954, in order to obtain as reliable results as possible.

Since 2003, this research has been conducted in accordance with international standards and recommendations of EUROSTAT, ILO and UN, thus providing international data comparability. The survey gathers data on income, expenditure and household consumption, i.e. data on the basic elements of individual consumption.

A survey unit is taken to be every single or multi-member household, selected according to the sampling plan. A household is defined as: (a) a community of persons, whose members live, eat and jointly spend the earned income; (b) a single person living, eating and spending the earned income on his/her own.

200 households are surveyed every 15 days, i.e. 1200 households quarterly.

The gathered data refer to total disposable income, expenditure on food and non-alcoholic beverages, and the share of this expenditure in quarterly total individual consumption as well, and constitute the monthly averages per household (expressed in RSD and percentages). The data for all households were recorded separately for Vojvodina and the Republic of Serbia, for all households (Table 1).

Table 1. Disposable income and expenditure per household in Vojvodina and the Republic of Serbia by quarters

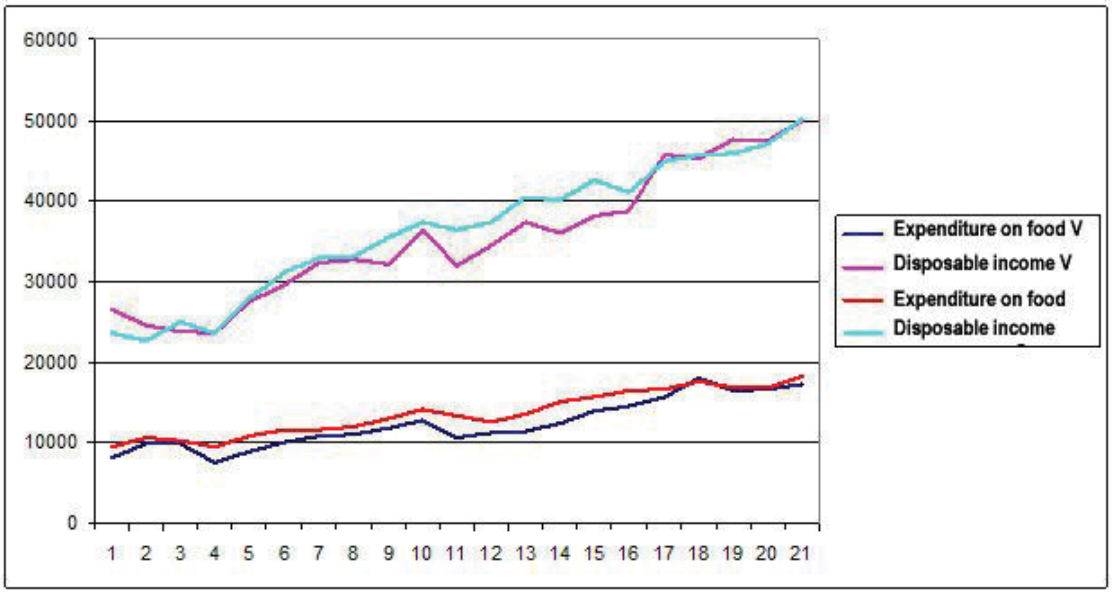
| **VOJVODINA, SERBIA – all households (MONTHLY AVERAGE PER HOUSEHOLD IN RSD)** | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | **Vojvodina** | | | | **Serbia** | | | | |
| **Year** | Quarter | Expenditure on food (Y) | **Disposable income (X)** | **Chain indexes** (Y) | **Chain indexes** (X) | **Share (%)** | Expenditure on food | **Disposable income (X)** | **Chain indexes** (Y) | **Chain indexes (X)** | **Share (%)** |
| 2004 | III | 8195 | 26480 | | | 36.70 | **9452** | **23569** | | | **40.50** |
| | IV | 9937 | 24657 | 1.2126 | 0.9312 | 34.20 | **10578** | **22726** | **1.1191** | **0.9642** | **38.30** |
| 2005 | I | 9898 | 23839 | 0.9961 | 0.9668 | 37.70 | **10282** | **25057** | **0.9720** | **1.1026** | **38.10** |
| | II | 7462 | 23522 | 0.7539 | 0.9867 | 33.80 | **9488** | **23587** | **0.9228** | **0.9413** | **37.80** |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | III | 8861 | 27567 | 1.1875 | 1.1720 | 33.30 | **10763** | **27816** | **1.1344** | **1.1793** | **37.10** |
| | IV | 10111 | 29588 | 1.1411 | 1.0733 | 36.90 | **11654** | **31160** | **1.0828** | **1.1202** | **37.10** |
| 2006 | I | 10806 | 32344 | 1.0687 | 1.0931 | 32.90 | **11644** | **32895** | **0.9991** | **1.0557** | **35.10** |
| | II | 10999 | 32641 | 1.0179 | 1.0092 | 35.30 | **12010** | **33027** | **1.0314** | **1.004** | **38.00** |
| | III | 11845 | 32129 | 1.0769 | 0.9843 | 38.40 | **13005** | **35337** | **1.0828** | **1.0699** | **39.90** |
| | IV | 12772 | 36422 | 1.0783 | 1.1336 | 36.30 | **14162** | **37358** | **1.0890** | **1.0572** | **39.50** |
| 2007 | I | 10690 | 32002 | 0.8370 | 0.8786 | 34.70 | **13310** | **36322** | **0.9398** | **0.9723** | **38.30** |
| | II | 11317 | 34501 | 1.0587 | 1.0781 | 36.50 | **12567** | **37396** | **0.9442** | **1.0296** | **39.30** |
| | III | ***11475*** | ***37276*** | 1.0140 | 1.0804 | 35.50 | ***13532*** | **40404** | **1.0768** | **1.0804** | **39.70** |
| | IV | 12433 | 35981 | 1.0835 | 0.9653 | 36.50 | **15083** | **40083** | **1.1146** | **0.9921** | **39.90** |
| 2008 | I | 13935 | 38209 | 1.1208 | 1.0619 | 37.90 | **15651** | **42672** | **1.0377** | **1.0646** | **39.90** |
| | II | 14462 | 38648 | 1.0378 | 1.0115 | 41.00 | **16365** | **41044** | **1.0456** | **0.9618** | **43.50** |
| | III | 15594 | 45642 | 1.0783 | 1.1810 | 37.30 | **16593** | **44835** | **1.0139** | **1.0924** | **41.40** |
| | IV | 18023 | 45361 | 1.1558 | 0.9938 | 39.60 | **17655** | **45637** | **1.0640** | **1.0179** | **40.90** |
| 2009 | I | 16531 | 47663 | 0.9172 | 1.0507 | 37.70 | **16929** | **45853** | **0.9589** | **1.0047** | **40.90** |
| | II | 16722 | 47434 | 1.0116 | 0.9952 | 39.10 | **16900** | **47068** | **0.9983** | **1.0265** | **41.90** |
| | III | 17200 | 49887 | 1.0286 | 1.0517 | 38.20 | **18177** | **50194** | **1.0756** | **1.0664** | **41.80** |
| | IV | 17194 | 45639 | 1.0000 | 0.9148 | 37.90 | **18249** | **47582** | **1.0040** | **0.9480** | **40.70** |

Source: **www.stat.gov.rs** (Communication LP-12 – Household Consumption Survey, 16 June 2010)

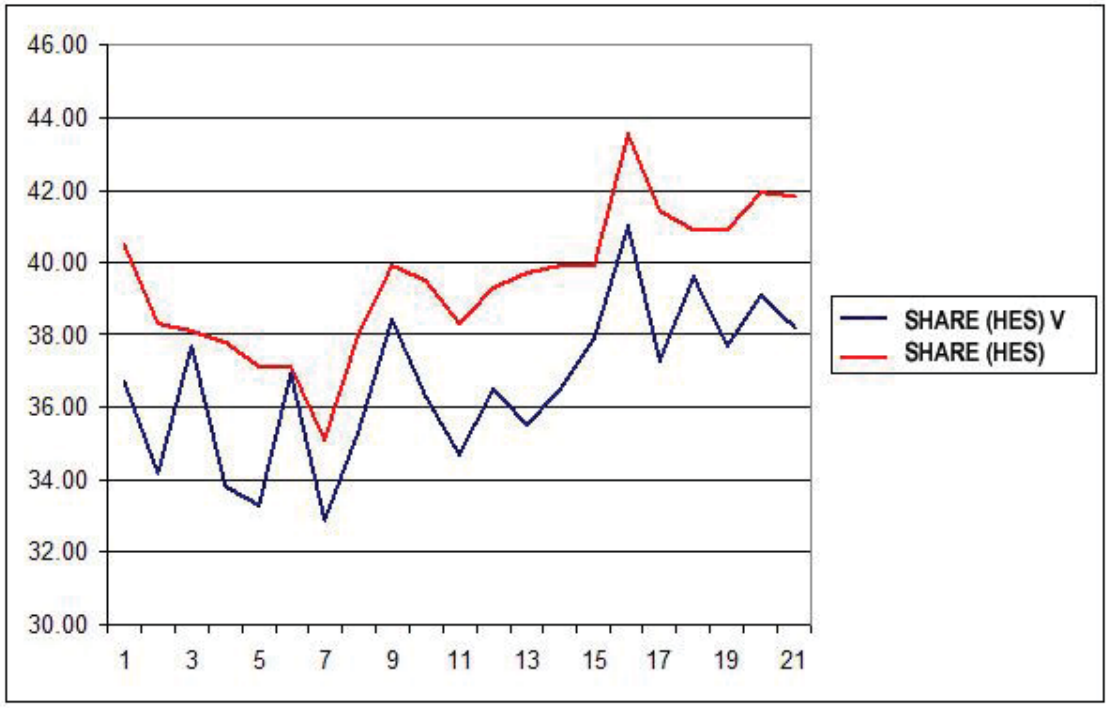
Viewing the trends in the values of disposable income and expenditure on food, as well as the Share of expenditure on food in total individual consumption over time, one can follow the trends in the population's living standard.

Graph 1. Trends in original data over time (disposable income and expenditure on food, monthly average per household in RSD – Vojvodina and the Republic of Serbia)




Source: Based on Table 1

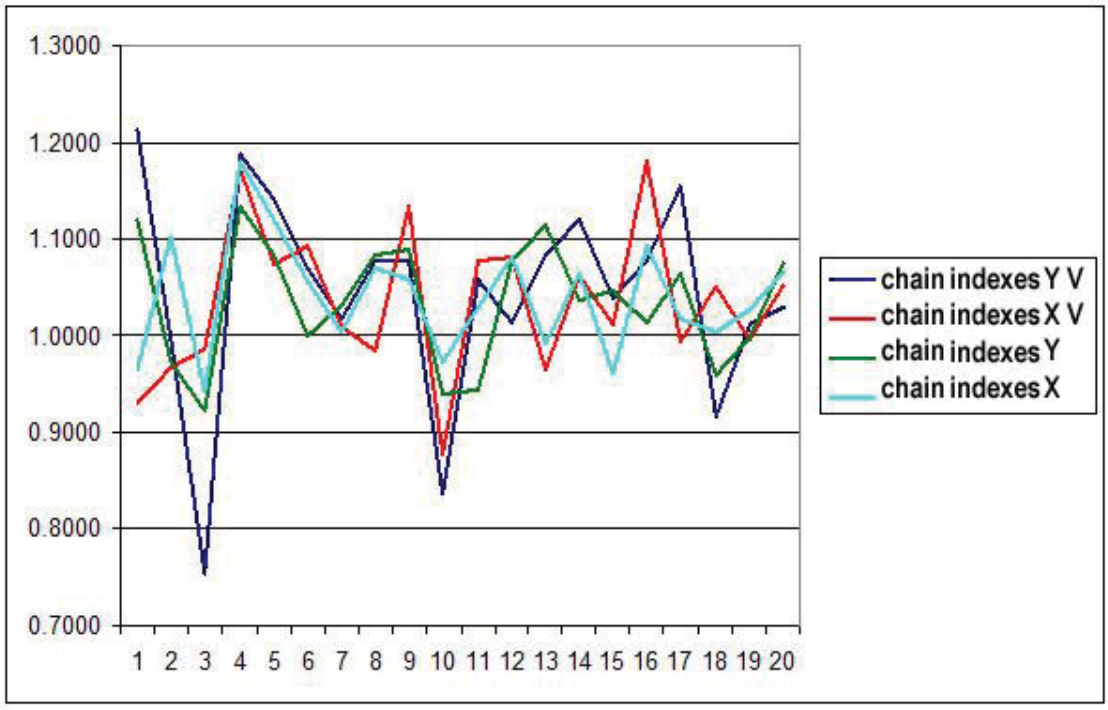
Graph 2. Trends in the share of expenditure on food and non-alcoholic beverages in total individual consumption (%).




Source: Based on Table 1

One of the most suitable formats for representing data over time is a chain index series, which can be used for calculating the mean development rate, i.e. the mean growth rate of a phenomenon. The graphic representation of chain index trends is shown on the graph below.

Graph 3. Trends in chain indexes (Vojvodina and the Republic of Serbia)



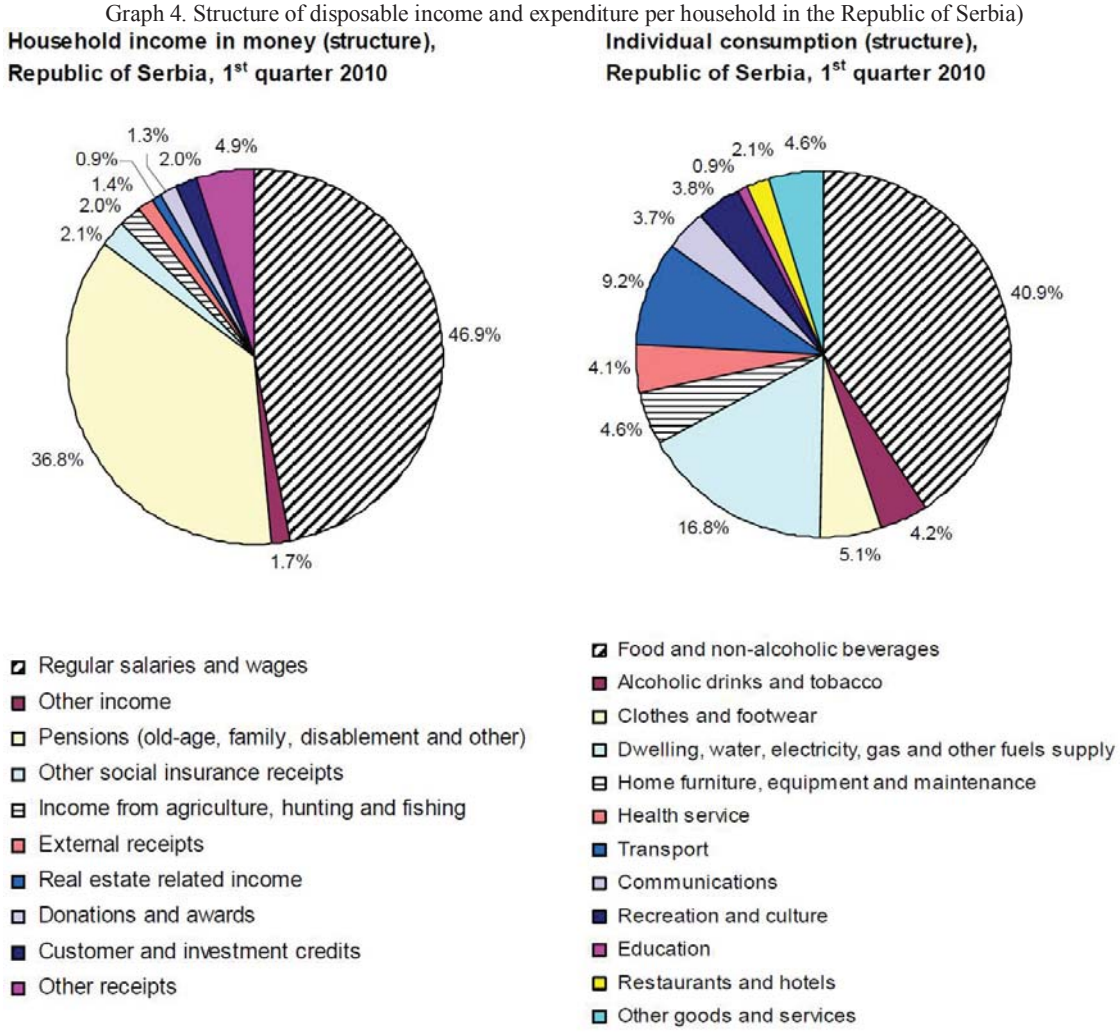

Source: Based on Table 1

Based on the above graphs, we can notice that trends in disposable income and expenditure on food show a growing tendency over the observed period both in Vojvodina and the Republic of Serbia. Moreover, we can see that, in the structure of the households' individual consumption structure, the expenditure on food and non-alcoholic beverages account for a high share - 36.64% in Vojvodina and 39.47% in Serbia (on the average for the observed period), and in addition, show a slight growing trend, which is a characteristic of undeveloped countries.

In most EU countries, the highest share in the individual consumption structure is taken up by expenditures on dwelling, water, power, gas and other fuels and transport, while expenditure on food and non-alcoholic drinks takes up the third place.

According to the latest Communication of the Serbian Republic Office of Statistics, in the fourth quarter of 2009, an average household in Serbia has the disposable monthly income amounting to 47582 RSD. At the same time, the average household expenditure amounts to 44886 RSD, where expenditure on food and non-alcoholic beverages amounts to 18249 RSD, i.e. 40.7%, which points to a fall in the population's living standards. The structure of the expenditure is shown on the graph below.

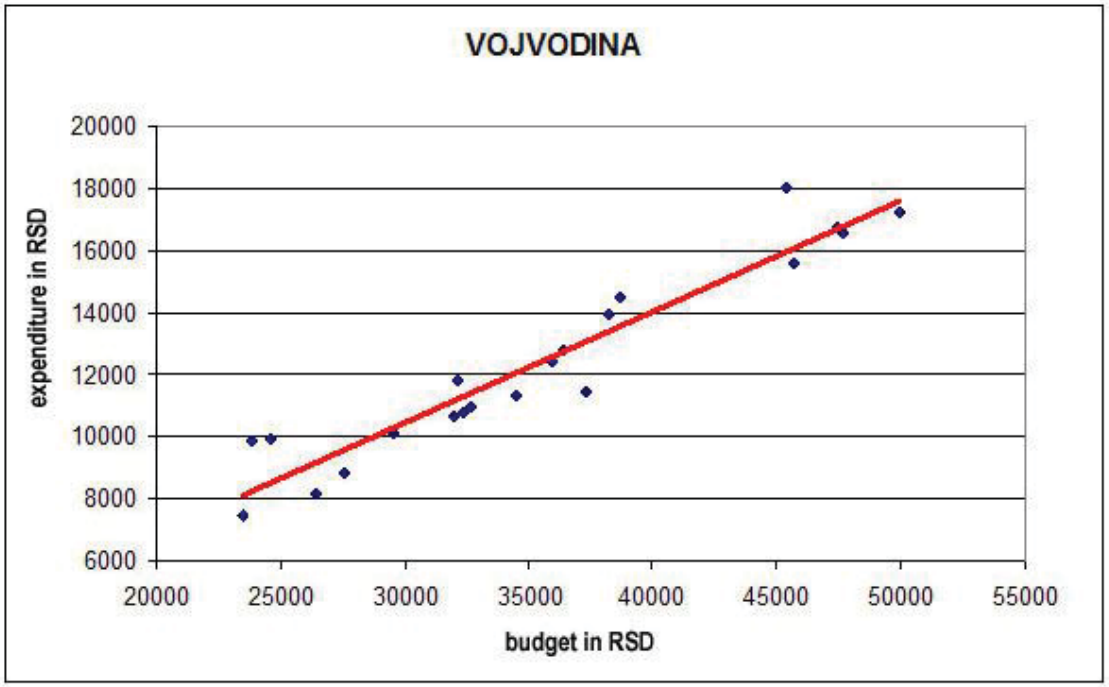
Graph 4. Structure of disposable income and expenditure per household in the Republic of Serbia)




Source: www.stat.gov.rs (Communication LP12, 16 June 2010)

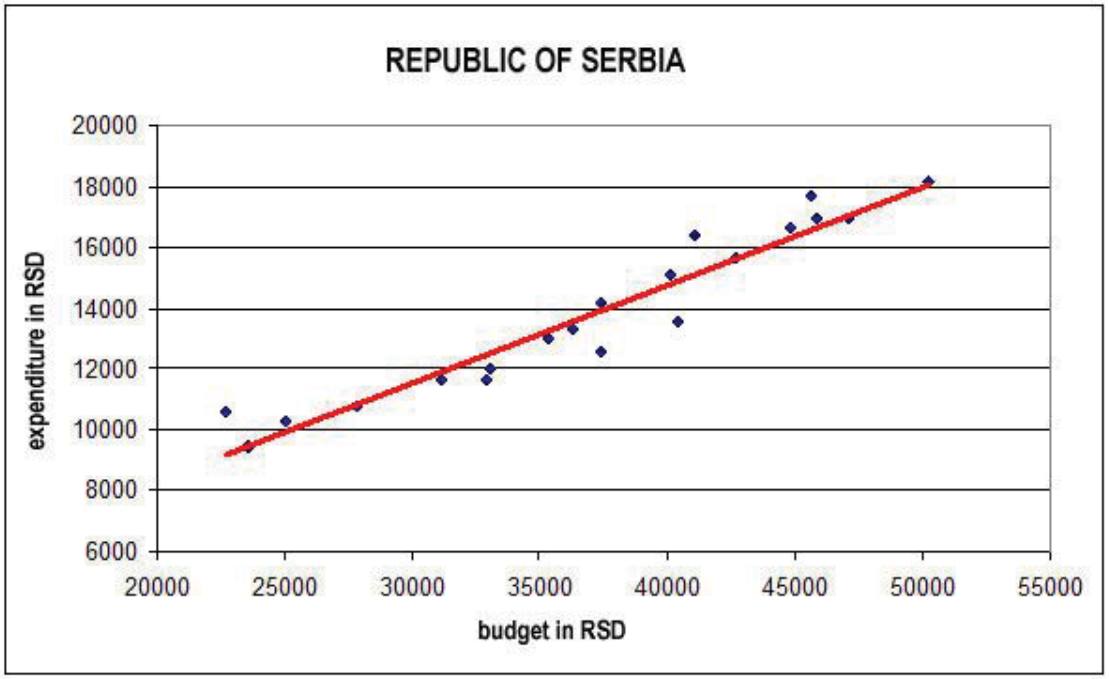
The graphs below show the average overlapping variance between data for disposable income and expenditure on food, used in econometric research.

Graph 5. Scatter diagram and regression line for Vojvodina




Source: Based on Table 1

Graph 6. Scatter diagram and regression line for the Republic of Serbia




Source: Based on Table 1

Table 2. Comparative overview – Research results

| No. | Model | Model equation for Vojvodina | FORECAST FOR VOJVODINA | Model equation for the Republic of Serbia | FORECAST FOR THE REPUBLIC OF SERBIA |
|---|---|---|---|---|---|
| 1. | **Linear trend for Y** | $Y_t$=7276.35+460.89t ($R^2$=0.8644) | $Y_{22}$=17415.8374 | $Y_t$=8584.752+438.62t ($R^2$=0.9457) | $Y_{22}$=18434.2952 |
| 2. | **Linear trend for X** | $X_t$=221350.18+1270.30t ($R^2$=0.9291) | $X_{22}$=4296.78 | $X_t$=21696.74+1355.10t ($R^2$=9720) | $X_{22}$=51068.96 |
| 3. | **Linear trend for Share** | $U_t$=34.33+0.2t ($R^2$=0.3802) | $U_{22}$=38.96% | $U_t$=37.02+0.22t ($R^2$=4968) | $U_{22}$=41.92% |
| 4. | **Exponential trend for Y** | $Y_t$=7949.78*1.038$^t$ ($R^2$=0.8651) | $Y_{22}$=18077.96 | $Y_t$=9299.88*1.033$^t$ ($R^2$=9433) | $Y_{22}$=19113.02 |
| 5. | **Exponential trend for X** | $X_t$=23102.59*1.037$^t$ ($R^2$=0.9316) | $X_{22}$=51304.92 | $Y_t$=32153.47*1.039$^t$ ($R^2$=9409) | $X_{22}$=54093.105 |
| 6. | **Mean development rate for Y** | 1.0378 | $Y_{22}$=17849.588 | 10332 | $Y_{22}$=18781.1126 |
| 7. | **Mean development rate for X** | 1.0322 | $X_{22}$=51492.13 | 10385 | $X_{22}$=52127.55 |
| 8. | **Linear econometric model (for Y)** | $Y_t$=504.7299+0.3644$X_t$ ($R^2$=0.9001) (Adj. $R^2$=0.8945) | $Y_{21}$=17676.40 (mdr. $X_{21}$) $Y_{22}$=18193.14 (exp. trend $X_{22}$) | $Y_t$=1390.192+0.3208$X_t$ ($R^2$=0.9262) (Adj. $R^2$=0.9221) | $Y_{21}$=18033.65 (mdr. $X_{21}$) $Y_{22}$=18314.34 (lin. Trend $X_{22}$) |
| 9. | **LOG-LOG model (for Y)** | $Y_t$=0.2978*$X_t^{1.0151}$ ($R^2$=0.8793) | $Y_{21}$=17481.53 | $Y_t$=2.7913*$X_t^{0.8087}$ ($R^2$=0.9198) | $Y_{21}$=1766.21 |
| 10. | **Multiple model (+time)** | $Y_t$=382.085+0.32$X_t$+58.68t ($R^2$=0.9012) (Adj. $R^2$=0.8896) | $Y_{21}$=17627.995 | $Y_t$=6517.91+0.105$X_t$+296.08t ($R^2$=0.9404 (Adj. $R^2$=0.9334) | $Y_{21}$=18015.51 |

Source: Based on the author's own research

Table 3. Testing the model's forecasting ability by t-test

| t-test | actual value | forecast value | forecasting interval | t* | t-table (0.05;18) |
|---|---|---|---|---|---|
| **VOJVODINA** | $Y_{21}$=17200 | $Y_{21,p}$=17676.40 | (15423.10; 19929.71) | 0.4439 | 2.101 |
| **THE REPUBLIC OF SERBIA** | $Y_{21}$=18177 | $Y_{21,p}$=18033.65 | (16306.24; 19761.05) | 0.1744 | 2.101 |

Table 4. Testing the model's forecasting ability by F-test

| F-test | Model for t=1-18 (shorter sample) | Model for t=1-21 (longer sample) | F* | F-table (0.05;3;18 |
|---|---|---|---|---|
| **VOJVODINA** | $Y_t$=-738.50+0.37$X_t$ | $Y_t$=-344.15+0.36$X_t$ | 0.1144 | 3.16 |
| **THE REPUBLIC OF SERBIA** | $Y_t$=1980.31+0.32$X_t$ | $Y_t$=1886.03+032$X_t$ | 0.0655 | 3.16 |

Based on the t-test, we can conclude that there is no statistically significant difference between the actual value of expenditure on food and the ones forecast by econometric model for the 3rd quarter of 2009, which means that we can consider the model's forecasting ability to be satisfactory with the probability of 95%, both in case of Vojvodina and the case of the Republic of Serbia

Based on the F-test, we can also conclude with the probability of 95% that the structural parameters are stable, both in case of Vojvodina and the case of the Republic of Serbia, i.e. that there is no statistically significant difference between the sums of squared residuals for the shorter (18 observations) and longer sample (21 observations), which means that the model's forecasting ability is satisfactory.

By this we have proven that **parameterized econometric models are also suitable for forecasting**; the results are in accordance with forecast values based on the trend and the mean development rate (see Table 2)

This research indicates that there is reason for concern, in view of the fall in the population's living standards, i.e. the growing trend in the share of expenditure on food in Total individual consumption.

## 2. CONCLUSION

The pace of change of events nowadays is too rapid for experience to be used as a guide for the future. New conditions are characterised by unpredictable and complex problems and possibilities as well as uncertain situations, so that planning and forecasting should secure better results in accomplishing an organisation's objectives.

Forecasting by econometric models is an area that provides for planning and selection of strategic objectives. A large number of forecasting methods and techniques are used nowadays.

Ideal forecasting is achieved by combining two types of methods:

1. the group of methods based on intuition and subjective assessment;
2. the group of methods relying on statistical and mathematical techniques, including econometric methods.

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# PROJECT MANAGEMENT – RISK MANAGEMENT – FINANCE

# Construction Project Managers' Cognitive Abilities, Personality and Knowledge

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**Abstract**

The tasks performed by a project manager are of special importance to the well being and economic prosperity of construction companies. This paper is aiming at defining the Greek project managers' dominant cognitive abilities, personality characteristics, skills and knowledge. This study is trying to examine and create a profile of the person, who is responsible for managing a construction project. Through this research an effort is made to identify available attributes connected with successful management performance and careers. This study is the second stage of an on-going research in this field which started in 2010 (Aretoulis et al. 2010). The survey was conducted based on a structured questionnaire. The effort led to 102 responses, originating from engineers and construction professionals, throughout the Greek Construction Industry, including public-work authorities. The questionnaire was based on international management literature, interviews and on the results of the first part of the research. The study moves on to present and discuss the results of the survey and provide a comparison with those attributes identified in the international literature.

**KEYWORDS**

Project Managers, Cognitive Abilities, Personality Characteristics, Knowledge, Construction Industry, Project Control.

## 1. INTRODUCTION

The importance of the project manager in the delivery of successful projects has generated a considerable amount of rhetoric and a smaller body of research based literature dealing with the knowledge, skills and personal attributes required of an effective project manager. With a few notable exceptions, findings have been based on opinions, primarily of project managers (Crawford 2000).

The nature and type of projects define the required attributes of the manager. Thus, it is important to focus on the special characteristics of construction projects. The latter ones are increasingly multidisciplinary and the role of a project manager has become critical for success (Andrew et al. 2003, Wolf and Jenkins 2006, Jha and Yger 2006). The effectiveness and productivity of organisations have been always depended heavily on the quality of their workforce, or their human capital, and there is general agreement that its importance relative to financial capital is steadily increasing (Wolf and Jenkins 2006, Andrew et al. 2003). Typically a large multidisciplinary project needs coordination among the personnel of different departments such as civil, electrical, mechanical, plant and machinery, HVAC, accounts, materials, design, construction method, quality, safety and HRD, totaling at least 12. The difficulty any project manager would have in coordinating resources for sites can easily be imagined. The complexity of the task will be demanding and the project manager may not be able to attend to other important project requirements (Jha and Yger 2006).

Furthermore, over the last decade, changes in organizational structure, through mergers and acquisitions as well as through decentralization, have encouraged the review of management responsibilities (Druker et al. 1996). Scrutinizing the duties of a project manager, it is undisputable that firstly they are faced with the need to have a greater understanding of all the relevant and associated economic aspects of their work. At the conceptual and feasibility stages of a construction project, a project manager needs to understand and to apply, the basic principles of investment and business finance in the context of the potential owner's current financial situation. A project manager needs to understand the different sources and forms of project funding.

The important relationship between time and money must also be understood (Pilcher 1994).

Moreover, a project manager must be able to implement a high degree of control, not only in respect of the day to day progress, cost and quality of the work, but also over the influences arising from the total external environment in which the project is being constructed. Consequently, at this initial stage, they also need to be able to gauge the risk of possible variability of external imponderable factors not controlled by the project manager (Pilcher 1994).

It is apparent that project managers' field of expertise is quite extensive. This fact creates difficulty in defining the manager's responsibilities and required attributes. Part of the problem of defining the project manager's role lies in the assumption that managers need only a single set of skills that help them manage any situation. In most organizations, two types of management exist-line management and project management. The responsibilities, tasks and required skills for each are very different, although the tendency is to treat them as synonymous, especially when the development aspects of the positions are considered (Carter 1988).

Trying to define Project Managers' agenda, one should bare in mind that project management is no longer just a sub-discipline of engineering, the management of projects – including programme management and portfolio management – is now the dominant model in many organisations for strategy implementation, business transformation, continuous improvement and new product development. Similarly, in areas such as infrastructure renewal, urban regeneration and community development, project management practices are becoming increasingly important, as more and more work is organized through projects and programmes (Winter et al. 2006).

At the same time in projects, crisis, uncertainty and suspense are continually recurring to test the quality of project managers. They always face the challenges of figuring out what to do with the implementation of their projects, despite uncertainty, great diversity and an enormous amount of potentially relevant information. Project managers try to get things done through a large and diverse set of people despite having little direct control over most of them. Knowing what project managers do, what kind of skills they demonstrate and what is their career path, would seem to constitute a very important step for the selection and development of an effective manager who is equipped to cope with any problem and accomplish unique outcomes with limited resources within critical time constraints (El-Sabaa 2001).

Similarly, Fryer (1997) cited in Haynes and Love (2004) suggests that the job of a manager has been defined as demanding, complex and varied. It includes the management of people, information and decision-making processes, and therefore is a critical human resource. Yet, the increasing demands and constraints being imposed on managers by the internal and external environment have resulted in longer working hours being experienced, which can have adverse psychological and physiological consequences (Haynes and Love 2004).

This study follows a rather traditional format. The paper firstly presents a literature review with the common held views of the project managers' attributes, then the research methodology is analyzed and its results' discussed. Finally, conclusions and future work are highlighted.

## 2. LITERATURE REVIEW

It is a fact that the issue of the project managers' profile has attracted and keeps attracting a lot of research interest, due to their key role in every project. That is the reason why, the available literature is quite extensive. The term project manager (PM) is used in the general sense of applying to the person, who has responsibility for managing the whole or some major part of a total project (Pilcher 1994). Typical responsibilities of a project manager are coordinating and integrating of subsystem tasks, assisting in determining technical and manpower requirements, schedules and budgets, measuring and analyzing project performance regarding technical progress, schedules and budgets (Jha and Iyer 2006, Royer 1974). In doing so, project managers devote attention to both the hard and soft systems, namely the formal system of rules and procedures and the potential informal / human system of motivation and leadership, in order to maximize the probability of achieving a successful project (Liu and Fang 2006).

Briner et al. (1996), cited in Dainty et al. (2004), suggest that managers must fulfill a number of roles including those of facilitator, coordinator, motivator and politician. This demanding and multifaceted role has necessitated the development of more sophisticated approaches to managing the performance of project managers. Notably, construction organizations have sought to develop evaluative criteria that can be used to

measure managers' performance, provide a basis for reward, determine training and development needs, provide a basis for succession planning and perhaps most significantly, facilitate goal setting amongst their project managers (Dainty et al. 2003).

El-Sabaa (2001) modified Katz's approach to study the skills and career path of the ideal project manager as perceived in Egypt. Based on a modified approach of Katz, the information about the skills of the best manager could be clustered into three main categories. These are: human skill, conceptual and organizational skill, and technical skill (El-Sabaa 2001).

Similarly, Jha and Yger (2006) focused their research on defining 24 required attributes for a successful project coordinator. Their research was based on literature and also interviews with top construction professionals based in India. A ranking has also been assigned to these attributes on the basis of a questionnaire survey. Statistical analysis of responses indicated a distinct difference between the attributes possessed by the project coordinators in projects that were considered successful, and those that were considered failures.

Liu et al. (2003) discuss the development and testing of a power-based model of project leadership from a behavior-performance-outcome approach in managing construction projects. The work of Fryer (Fraser 2000) epitomizes the approach taken generally to study the PM as an individual. He focused his work on what PMs actually do, as opposed to their personal characteristics. He found that his subjects spent a considerable amount of their time outdoors talking to people, mainly supervisors, about technical matters. Thus, they were mainly concerned with coordinating the people involved in performing the technical tasks, welding together effective teams. They attached rather less importance to planning and programming their projects and to their dealings with the design team and client. The PMs were asked to rank the managerial skills, allocated by Fryer (1979) that they thought they personally possessed, in order of importance for effective management. The combined order, from the most to the least important was (Fraser 2000):

- Social skill
- Decision making
- Handling problems
- Recognizing opportunities
- Managing change

This research project identified the personal factors which may facilitate the PMs' ability to perform these skills, and thus be effective. Another attempt at determining the components of a common core of criteria that can be used to assess the skill (and subsequently the PMs' effectiveness) was made by Brown (1983). His survey of 54 managers working in construction identified the following issues as being important factors:

- Achievement of quality
- Responsibility acceptance
- Decision-making skills
- Ability to handle stress
- Interpretation of design information
- Record of safety on site
- Utilization of resources available
- Communication
- Accuracy
- Judgment
- Organizational ability
- Technical knowledge

Another approach is that of examining behavior and skills in the context of specific tasks at hand and the stages in the construction process. Rowlinson et al. (1993) have examined variations of leadership style employed by the same construction managers in different circumstances plus the culture-specific results of managerial behavior modification (Lingard and Rowlinson, 1994). The surrounding circumstances have been documented and analyzed successfully but their impact is considered only in relation to one important personal characteristic: leadership style. The relevance of these studies is not just restricted to the obvious fact that leadership style is a crucial personal characteristic of an individual. The deeper relevance of their work to the present analysis is that they have demonstrated the dynamic nature of personal characteristics that are adaptable to job needs and circumstances and manageable attributes of the individual. What still remains to be done is a similar, comprehensive study of a wide range of characteristics (Fraser 2000).

Furthermore, El-Sabaa (2001) conducted an extensive research on the topic. The raw scores obtained

from 126 respondents of a questionnaire survey were summarized into three skill areas. Each area contained several critical factors. Based on Pinto's cited in El-Sabaa (2001) method, the percentile scores of the three main skills of all sectors include human skill with a percentile score of 85.3% represented the most essential project manager skill. Characteristics included in this category were mobilization, communication, coping with situations, delegation of authority, political sensitivity, high-self esteem and enthusiasm. Conceptual and organizational skill with a percentile score of 79.6% represented a second essential project manager skill. Characteristics included in this category were skills of planning, organizing, having strong goal orientation, ability to see the project as a whole, ability to visualize the relationship of the individual project to the industry and the community, and strong problem orientation. Technical skill with a percentile score of 50.46% represented, relatively, the least essential project manager skill. Characteristics included specialized knowledge in the use of tools and techniques, project knowledge, understanding methods, process and procedures, the technology required, and skill in the use of computer (El-Sabaa 2001).

Katz cited at El-Sabaa suggested that effective administration rests on three basic developable skills. These are human skill, conceptual skill and technical skill. Although these skills are interrelated, they can be developed independently (El-Sabaa 2001). Finally, Jha and Iyer (2006) summarized in the following table (1) the authors and skills identified in various research studies.

Table 1 Summary of project manager's skills identified in different studies (Jha and Iyer 2006)

| Authors | Description of Skills |
|---|---|
| Gaddis (1959) | Project manager needs solid basic experience in the relevant field and should be a leader able to carry out planning and follow-up activities |
| Katz and Kahn | Project manager should have technical skills, human skills and conceptual skills |
| Stuckenbruck (1976) | A proficient manager must be: multidisciplinary oriented, global problem oriented, an effective problem solver and decision maker, a good manager and administrator, possessing good analytical abilities, creative in dealing with information and problems, an effective communicator, able to motivate his team members to achieve fixed goals, flexible, and able to adapt to change, of the right temperament, and should be able to keep his calm |
| Adams and | Project manager must have planning, coordination and budgeting skills in addition to team management skills |
| Fryer (1979) cited in Odusami | Managing change, recognizing opportunities, handling problems, decision making, social skills are the skills needed in a project manager |
| Spitz (1982) cited in Pettersen (1991) | Project manager should have: interpersonal skills, synchronizing skills for different technology, technical expertise, and information processing skills |
| Goodwin (1993) | Project manager's effectiveness depends on conceptual, human, and negotiating skills as well as, to a lesser extent, on technical skills, besides verbal and written communication skills |
| Meredith et al. (1995) cited in El- | The skills needed for a project manager are categorized into six skills areas: communication, organizational, team building, leadership, coping, and technological skills |
| El-Sabaa (2001) | Three major skills groups: *human skills* such as communication, coping with situations, delegation of authority, high self-esteem and enthusiasm, *conceptual skills* such as planning and organizing skills, strong goal orientation, ability to see the project as a whole, and *technical skills* such as specialized knowledge of project, the technology required, and skill in the use of computer. |
| Project Management | The project manager should have the following characteristics: an open positive 'can do' attitude, common sense, open mindedness, adaptability, inventiveness, prudent risk taking, fairness and commitment |
| Kerzner (2002) | Team building, leadership, conflict resolution, technical expertise, planning, organization, entrepreneurship, administration, management support, resource allocation |

## 3. RESEARCH METHODOLOGY AND FINDINGS

The research methodology is quite straightforward. A structured questionnaire was used and a survey took place in order to evaluate the importance of various cognitive abilities, personality characteristics and body of knowledge. The current research represents the second stage of a survey which completed on 2010. The first stage (Aretoulis et al. 2010) included a number of open-ended questions and interviews with engineers, managers and professionals in the construction industry. During this initial attempt a number of attributes concerning the project managers were identified. The latter ones were compared with international literature and the final list of attributes and knowledge was defined, which was used in the second part of the research.

## 3.1 Questionnaire Structure

As mentioned before a list of cognitive abilities, personality characteristics and knowledge were defined. The questionnaire consists of the following three main parts:

- Profile of the survey's participant
- Personality characteristics and cognitive abilities of project managers
- Essential or required knowledge of project managers

The first part of the questionnaire is devoted to the participants, which have to deal with 13 personal questions, ranging from age to academic and professional background. The total characteristics regarding cognitive abilities and personality sum up to 47 elements. The Knowledge part includes 7 elements. It is chosen that the cognitive skills and personality characteristics are located together on the same part of the questionnaire and in the same list. This structure was preferred in order to achieve comparison between them as the survey participants' appoint points (1-5) at each attribute. The essential – required knowledge is individually presented in the questionnaire. This approach ensures that knowledge elements won't be compared with the items representing abilities and personality.

The majority of the answers are collected through check boxes, and use of a predefined ranking, ranging from 1 to 5. One point represents the lowest value and five points represent the highest value.

## 3.2 Survey - Research Findings

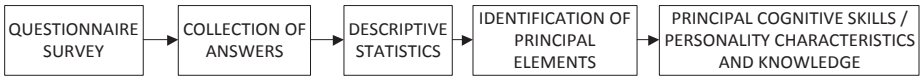
### 3.2.1 Sample Description

The survey collected 102 answers from engineers of various disciplines in Greek construction enterprises and public authorities. From the total of 102 participants 28 are engineers with a 2-year experience and 74 engineers with experience beyond 2 years. As far as their activity is concerned from the total of 102 participants, 52 are designers engineers, 12 are site engineers, 32 are project managers and 6 are contractors. Regarding the participants discipline, 80 are civil engineers, 1 is land surveyor, 5 are electrical engineers, 5 are chemical engineers, 3 are Architects, 6 are Mechanical engineers and 2 are listed as other engineers. Finally male participants are 70 and female ones are 32.

### 3.2.2 Descriptive Statistics

The goal of the questionnaire survey was firstly to identify the personality characteristics, the cognitive abilities and required knowledge. Then descriptive statistics are estimated using SPSS v.18 and the dominant elements are identified. The following figure briefly presents the whole process for the definition of principal cognitive skills, personality characteristics and knowledge.

Fig 1: The Process of Defining Principal Skills, Characteristics and Knowledge




A database was created using SPSS 18. The answers of the questionnaires were parameterized and inserted in the application. The results concerning the body of knowledge of project managers are presented in table 2 and the cognitive abilities and personality characteristics of project managers could be summarized in the following table 2:

Table 2. Descriptive statistics concerning project managers' body of knowledge

| | Range | Minimum | Maximum | Sum | Mean | Std. Deviation |
|---|---|---|---|---|---|---|
| KNOWLEDGE OF PROJECT STAGES | 2,00 | 3,00 | 5,00 | 475,00 | 4,6569 | ,58859 |
| KNOWLEDGE OF ECONOMICS | 2,00 | 3,00 | 5,00 | 432,00 | 4,2353 | ,73359 |
| KNOWLEDGE OF CONSTRUCTION EQUIPMENT | 3,00 | 2,00 | 5,00 | 431,00 | 4,2255 | ,86639 |
| KNOWLEDGE OF LEGISLATION | 4,00 | 1,00 | 5,00 | 425,00 | 4,1667 | ,93431 |
| KNOWLEDGE OF PROJECT MANAGEMENT SOFTWARE | 4,00 | 1,00 | 5,00 | 418,00 | 4,0980 | ,96994 |
| KNOWLEDGE OF WBS | 4,00 | 1,00 | 5,00 | 416,00 | 4,0784 | ,96150 |
| KNOWLEDGE OF TECHNICAL REGULATIONS | 4,00 | 1,00 | 5,00 | 404,00 | 3,9608 | 1,03332 |

Table 3. Descriptive statistics concerning project managers' cognitive abilities and personality characteristics

| COGNITIVE SKILLS AND PERSONALITY CHARACTERISTICS | Range | Min | Max | Sum | Mean | Std. Dev. |
|---|---|---|---|---|---|---|
| PROMPTNESS ON SOLUTION PROVISION | 2,00 | 3,00 | 5,00 | 464,00 | 4,5490 | ,62343 |
| COLLABORATIVE, TEAM SPIRIT | 2,00 | 3,00 | 5,00 | 453,00 | 4,4412 | ,62257 |
| CONFLICT MANAGEMENT | 2,00 | 3,00 | 5,00 | 452,00 | 4,4314 | ,63729 |
| CAPABILITY OF RISK EVALUATION | 3,00 | 2,00 | 5,00 | 447,00 | 4,3824 | ,74520 |
| EXPERIENCE | 2,00 | 3,00 | 5,00 | 447,00 | 4,3824 | ,73180 |
| SCHEDULING CAPABILITY | 4,00 | 1,00 | 5,00 | 447,00 | 4,3824 | ,77132 |
| DECISIVENESS | 3,00 | 2,00 | 5,00 | 446,00 | 4,3725 | ,67372 |
| LEADERSHIP CAPABILITIES | 4,00 | 1,00 | 5,00 | 446,00 | 4,3725 | ,75677 |
| ORGANIZATIONAL SKILL | 3,00 | 2,00 | 5,00 | 443,00 | 4,3431 | ,66742 |
| COMMUNICATION SKILLS | 2,00 | 3,00 | 5,00 | 442,00 | 4,3333 | ,69415 |
| EFFECTIVENESS | 4,00 | 1,00 | 5,00 | 441,00 | 4,3235 | ,75991 |
| PERCEPTION OF TIME | 4,00 | 1,00 | 5,00 | 439,00 | 4,3039 | ,78082 |
| PROMPTNESS ON DECISION MAKING | 2,00 | 3,00 | 5,00 | 436,00 | 4,2745 | ,66238 |
| PRACTICAL WAY OF THINKING | 3,00 | 2,00 | 5,00 | 435,00 | 4,2647 | ,75684 |
| CAPABILITY OF PREDICTING | 3,00 | 2,00 | 5,00 | 434,00 | 4,2549 | ,82877 |
| CAPABILITY OF CONSIDERING ALTERNATIVE SCENARIOS | 4,00 | 1,00 | 5,00 | 432,00 | 4,2353 | ,78572 |
| RESPONSIBLE | 3,00 | 2,00 | 5,00 | 432,00 | 4,2353 | ,66267 |
| METHODICAL | 2,00 | 3,00 | 5,00 | 428,00 | 4,1961 | ,67544 |
| PERCEPTION OF THE WHOLE PICTURE | 3,00 | 2,00 | 5,00 | 427,00 | 4,1863 | ,87575 |
| CONFLICT RESOLUTION | 3,00 | 2,00 | 5,00 | 427,00 | 4,1863 | ,78008 |
| PUNCTUALITY | 4,00 | 1,00 | 5,00 | 425,00 | 4,1667 | ,84529 |
| RESPECTED | 3,00 | 2,00 | 5,00 | 418,00 | 4,0980 | ,81459 |
| CAPABILITY OF OUTSOURCING | 4,00 | 1,00 | 5,00 | 417,00 | 4,0882 | ,93452 |
| HARD WORKING | 4,00 | 1,00 | 5,00 | 417,00 | 4,0882 | ,87995 |
| INTEGRITY | 3,00 | 2,00 | 5,00 | 416,00 | 4,0784 | ,84063 |
| DYNAMIC | 4,00 | 1,00 | 5,00 | 413,00 | 4,0490 | ,92680 |
| ΕΠΙΜΕΛΕΙΑ | 4,00 | 1,00 | 5,00 | 412,00 | 4,0392 | ,90018 |
| JUSTICE | 3,00 | 2,00 | 5,00 | 408,00 | 4,0000 | ,88999 |
| ETHICS | 4,00 | 1,00 | 5,00 | 408,00 | 4,0000 | ,88999 |
| CONFIDENCE | 3,00 | 2,00 | 5,00 | 407,00 | 3,9902 | ,76424 |
| CAPABILITY OF ASSIGNING RESPONSIBILITIES | 4,00 | 1,00 | 5,00 | 406,00 | 3,9804 | 1,00475 |
| FLEXIBLE | 4,00 | 1,00 | 5,00 | 405,00 | 3,9706 | ,87264 |
| PERCEPTION OF SCALE | 4,00 | 1,00 | 5,00 | 394,00 | 3,8627 | ,96513 |
| STRATEGIC CAPABILITY | 4,00 | 1,00 | 5,00 | 393,00 | 3,8529 | 1,12924 |
| DIPLOMACY | 4,00 | 1,00 | 5,00 | 389,00 | 3,8137 | 1,06919 |
| PERSISTENCE | 4,00 | 1,00 | 5,00 | 385,00 | 3,7745 | 1,03318 |
| PATIENT | 3,00 | 2,00 | 5,00 | 382,00 | 3,7451 | ,93009 |
| SELF CONTROL | 4,00 | 1,00 | 5,00 | 382,00 | 3,7451 | 1,03106 |
| INVENTIVE | 4,00 | 1,00 | 5,00 | 380,00 | 3,7255 | ,99660 |
| COMMITMENT | 4,00 | 1,00 | 5,00 | 364,00 | 3,5686 | ,98010 |
| CAPABLE OF PHYCHOLOGICAL EVALUATION | 4,00 | 1,00 | 5,00 | 363,00 | 3,5588 | 1,02032 |
| POLITENESS | 4,00 | 1,00 | 5,00 | 347,00 | 3,4020 | ,97757 |
| UNDERSTANDING | 4,00 | 1,00 | 5,00 | 343,00 | 3,3627 | ,94176 |
| SOCIAL CONSCIOUSNESS | 4,00 | 1,00 | 5,00 | 337,00 | 3,3039 | 1,01268 |
| CREATIVITY | 4,00 | 1,00 | 5,00 | 332,00 | 3,2549 | ,98188 |
| FRIENDLINESS | 4,00 | 1,00 | 5,00 | 324,00 | 3,1765 | 1,06624 |
| INSPIRATION | 4,00 | 1,00 | 5,00 | 313,00 | 3,0686 | 1,17110 |

Based on the sum points acquired by each attribute, the tables provide a ranking of each cognitive ability, personality characteristic and knowledge. The required knowledge has a limited number of elements. The study concluded that knowledge of the project stages is the most essential one. All the other elements of knowledge are very close regarding their total score. As far as the cognitive abilities are concerned promptness on solution provision is the most important and inspiration is the less essential one. The most significant attributes refer to the leadership capability of the project manager and more specifically his capability for managing personnel. Generally the attributes identified in this research appear similar to the attributes identified in the international literature. Fraser as mentioned earlier, provided a list from the most significant one to the less significant. Social Skills are also top of this study's list, and so does "decision making" = "promptness on solution provision" and finally "handling problems" = "conflict management". It seems that apart from the different cultural background, law and legislation the attributes' priority remains the same.

## 4. CONCLUSIONS AND FUTURE WORK

This research identified the profile of the Greek project managers, and more precisely their cognitive abilities, personality characteristics and knowledge that are required for a successful project manager career. The paper moved on to compare survey's findings with the international literature and concluded that apart from different cultural background and differences in legislation project managers' attributes remain the same and also the attributes' order of significance appears quite similar.

International literature mentions only "Technical Knowledge", at the end of the list. This paper identified in detail a body of knowledge required for a project manager. The contribution of this research as far as the knowledge is concerned could be to provide guidelines for academic program courses for efficient preparation of managers. The findings of this paper show that the selection and training of project managers could be conducted with greater reliability.

The future work could focus on evaluation of dominant cognitive abilities and personality characteristics. The latter ones could be measured using psychometric tests. These results could be used as a base for career orientation. Even in the case of promotions or assignments in the construction industry, the dominant attributes and their presence in an engineer could act as the criterion for his promotion and predict his future performance in managing the project.

It is a fact that cognitive abilities and personality characteristics could not be altered. Social skills and especially communication skills could be improved by adopting various communication techniques. On the other hand the area which leaves room for improvement is the required knowledge. This could be planned and organized in such a manner that creates efficient managers.

The research reported here has attempted to approach the profiling of the competent project manager from a potentially more objective viewpoint, by gathering data on project managers personality characteristics, using project management literature reviews and questionnaire surveys. The next step of the research apart from the use of psychometric tests, will try through correlation analysis, principal component analysis and regression analysis to extract more conclusions. The effort will be focused on grouping the attributes, investigate the notion that the conception of the PM's attributes is correlated with the profile of the participant in the survey. Finally, based on this last assumption, prediction models, that provide the desired profile of the project manager will be produced. The models will be based on the profile of the people responsible for the hiring of the project managers.

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# Regulating Time-Cost Project Targets via Monte Carlo Methods

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**Abstract**

In this study, we examine two currently used procedures for crashing the project completion time by additional budget and develop a new stochastic procedure for this purpose. Our initial aim is to define the various patterns of project activities and understand how each of the durations and costs of these patterns of activities is determined during expediting the activity execution. For this purpose a new generalized stochastic time-cost tradeoffs model is introduced. We also propose a new optimization procedure based on Monte Carlo simulations the purpose of which is to minimize a chance-constrained cost or budget under a chance-constrained completion time. By distributing the project budget among its activities, our particular objective is to crash any chance-constrained project completion time in the most economic manner. We apply this procedure on PERT-type projects and compare it with two methods existing in the literature.

**KEYWORDS**

Time-cost tradeoffs; statistical PERT; time chance constraint.

## 1. INTRODUCTION

The *Critical Path Method* (*CPM*) was developed in the late 1950s by the *DuPont Corporation* at about the same time that *General Dynamics* and the *US Navy* were developing the *Program Evaluation and Review Technique* (*PERT*), known today as *Statistical PERT*. Assuming that project activity durations do not vary, *CPM* calculates the completion time of the project paths (activity chains from the start to the end of the project) and then identifies the path with the longest completion time. Such a path is known as the *Critical Path (CP)*.

While the *CPM* assumes that activity durations do not vary, the *Statistical PERT* was originally developed for planning and controlling projects where there is uncertainty. Such uncertainty mainly concerns the time and the cost required by each activity.

In a paper dealing with tardiness risk assessment, Laslo (2003) defined two types of duration uncertainty. The first, which derives from worker absenteeism, equipment failure and weather conditions, is typical for some types of tasks such as production and construction. The second derives from estimates of work content, and is typical for other types of tasks such as research and development. Elmaghraby (2005) indicated that traditionally the focus has been on the first type, the *external duration uncertainty*, with little attention paid to the second, the *internal duration uncertainty*.

While the *Statistical PERT* provides a greater level of information to be analyzed and less unbiased estimates of the expectations than deterministic methods such as *CPM*, it requires multiple duration estimates that can be time consuming to produce (Abernathy (1971)). In addition, the analysis is based on the duration of activities along a single path and results can be skewed if there are multiple critical paths on the project. To deal with these problems, several analytical approximations of the distribution of project completion time

in traditional *PERT* networks have been developed (e.g., Fulkerson (1962); Hardie (2001); Azaron and Fatemi-Ghomi (2008)).

When it is infeasible or impossible to compute an exact result with a deterministic algorithm the *Monte Carlo* method tends to be used (Metropolis and Ulam (1949)). Simulative methods are useful for modeling phenomena with significant uncertainty in inputs. *Monte Carlo* simulations can quantify the effects of risk and uncertainty in project schedules and budgets, giving the project manager a statistical indicator of project performance such as target project completion date and budget.

Project managers are often asked to shorten the project completion time in order to meet contractual due dates. This can sometimes be obtained by: 1) pruning CP activities; 2) detailing activities and redistributing workloads in order to perform more activities in parallel; and mostly, 3) crashing the CP, i.e., shortening the durations of CP activities by adding resources. Siemens (1971) described a procedure for crashing PERT projects until the activity durations on at least one of the CPs exceed a predetermined limit. Golenko-Ginzburg (1993) considered *PERT*-type projects with random durations and introduced a procedure based on Monte Carlo simulations for crashing their expected completion times. Laslo and Gurevich (2007) developed a procedure that optimizes the crashing of a chance-constrained path when both external duration uncertainty and internal duration uncertainty are involved. The objective of all these procedures was to minimize the total cost of expediting the execution of activities in order to complete the project in the desired time.

Our initial aim in this study is to define the various types of *PERT* project activities and understand how each of the durations and costs of these types of activities is determined during expediting the activity performance. For this purpose a new generalized stochastic time-cost tradeoffs model is introduced. This model is an extension of the model developed by Laslo (2003) who considered four types of activities. The suggested model is applicable to fourteen types of activities presented by a common formulation. So that it enables the evaluation of the 'realistic' outcome of crashing *PERT*-type projects. We introduce a new optimization procedure based on *Monte Carlo* simulations the purpose of which is to minimize a chance-constrained cost or budget under a chance- constrained completion time. By distributing the project budget among its activities, our particular objective is to crash any chance-constrained project completion time in the most economic manner. This differs from other methods which aim to crash the expected project completion time (Siemens (1971); Golenko-Ginzburg (1993)). We implement on a *PERT*-type project and compare the proposed optimization procedure with those of Siemens (1971) and Golenko-Ginzburg (1993).

## 2. CLASSIFICATION OF PROJECT ACTIVITIES ON THE BASIS OF DURATION AND COST DISTRIBUTIONS

The *Statistical PERT* considers random activity durations. In most cases the distribution of activity durations are purely hypothetical. The expected activity duration derives from the allocated budget that determines the average performance speed. Laslo (2003) showed that when the *external uncertainty* has an extremely dominant share within the accumulative uncertainty of the activity duration, the standard deviation of the activity duration is kept during the crashing of the activity duration. However, when the extreme dominance belongs to the *internal uncertainty*'s share, the coefficient of variation of the activity duration is kept. Till now, the case when both types of uncertainty are significant has not been studied.

The combination of resources and the material lead-times assigned to an activity for obtaining satisfactory expected or chance-constrained activity duration requires an adequate budget. However, this budget is based on costing estimates and will mostly be unequal to the actual activity cost. The uncertainty of the activity cost may derive from two sources. The first is the *time-independent cost uncertainty*, and it derives from an inaccurate estimate of prices, materials, and wages. The standard deviation of this uncertainty increases proportionally with the increase of the allocated budget, i.e., its coefficient of variation is kept. The second is the *time-dependent cost uncertainty*, and it derives from the possible dependency between the activity duration and the actual activity cost.

We specify fourteen types of activities with the following attributes that can be included in real projects:

(1) deterministic duration and deterministic cost,

(2) deterministic duration and time-independent cost uncertainty,

(3) external duration uncertainty and deterministic cost,

(4) external duration uncertainty and time-dependent cost uncertainty,

(5) external duration uncertainty and time-independent cost uncertainty,

(6) external duration uncertainty, time-independent and time-dependent cost uncertainties,

(7) internal duration uncertainty and deterministic cost,

(8) internal duration uncertainty and time-dependent cost uncertainty,

(9) internal duration uncertainty and time-independent cost uncertainty,

(10) internal duration uncertainty, time-independent and time-dependent cost uncertainties,

(11) internal and external duration uncertainties and deterministic cost,

(12) internal and external duration uncertainties and time-dependent cost uncertainty,

(13) internal and external duration uncertainties and time-independent cost uncertainty,

(14) internal and external duration uncertainties, time-independent and time-dependent cost uncertainties.

## 3. GENERALIZED MODEL FOR ACTIVITY TIME-COST TRADEOFFS

As described in Section 2, via the fourteen types of activities, the effect of budget allocations on the distributions of activity durations and costs is essential for any time or cost risk analysis. The proposed generalized activity time-cost tradeoffs model which applicable for any type of activity is formulated as follows:

$$\begin{aligned} t_{i,j}^{B_{i,j}} &= g_{i,j}(B_{i,j}) + X_{i,j}^{B_{i,j}} + Y_{i,j}, \\ c_{i,j}^{B_{i,j}} &= B_{i,j} + Z_{i,j}^{B_{i,j}} + V_{i,j}^{B_{i,j}} \cdot \left( t_{i,j}^{B_{i,j}} - g_{i,j}(B_{i,j}) \right) \\ &= B_{i,j} + Z_{i,j}^{B_{i,j}} + V_{i,j}^{B_{i,j}} \cdot (X_{i,j}^{B_{i,j}} + Y_{i,j}) \end{aligned}, \tag{1}$$

where:

- $(i, j)$ is the index of the project activity,
- $t_{i,j}^{B_{i,j}}$ is the actual $(i, j)$ activity duration related to the budget allocated to the activity $(i, j)$, $B_{i,j}$ (a random variable with expectation $E(t_{i,j}^{B_{i,j}}) = g_{i,j}(B_{i,j})$, $E(t_{i,j}^{B_{i,j}^C}) \le E(t_{i,j}^{B_{i,j}}) \le E(t_{i,j}^{B_{i,j}^N})$),
- $B_{i,j}$ is the budget allocated to activity $(i, j)$, $B_{i,j}^N \le B_{i,j} \le B_{i,j}^C$,
- $g_{i,j}(B_{i,j})$ is the known decreasing continuous function that characterizes an expectation of the actual $(i, j)$ activity duration related to the budget allocated to the activity $(i, j)$, $B_{i,j}$, $g_{i,j}(B_{i,j}) = E(t_{i,j}^{B_{i,j}})$,
- $t_{i,j}^{B_{i,j}^C}$ is the actual $(i, j)$ activity duration related to the allocated budget for crash performance $B_{i,j}^C$ (a random variable with expectation $g_{i,j}(B_{i,j}^C)$),
- $B_{i,j}^C$ is the known budget allocated to the activity $(i, j)$ for crash performance,
- $t_{i,j}^{B_{i,j}^N}$ is the actual $(i, j)$ activity duration related to the allocated budget for normal performance $B_{i,j}^N$ (a random variable with expectation $g_{i,j}(B_{i,j}^N)$),
- $B_{i,j}^N$ is the known budget allocated to activity $(i, j)$ for normal performance,

- $X_{i,j}^{B_{i,j}}$ is the random variable with zero expectation, with the standard deviation being proportional to $g_{i,j}\left(B_{i,j}\right)$ ($E(X_{i,j}^{B_{i,j}})=0$, $\sigma(X_{i,j}^{B_{i,j}})=m_{i,j}\cdot g_{i,j}(B_{i,j})\geq 0$, $\sigma(X_{i,j}^{B_{i,j}})=0\Leftrightarrow X_{i,j}^{B_{i,j}}=0$). This variable characterizes the internal uncertainty of $(i,j)$'s activity duration,
- $m_{i,j}=\sigma(X_{i,j}^{B_{i,j}})/E(t_{i,j}^{B_{i,j}})$ is a known constant which is independent of allocated budget $B_{i,j}$ related to activity $(i,j)$,
- $Y_{i,j}$ is the random variable with zero expectation and known standard deviation ($E(Y_{i,j})=0$ and $\sigma(Y_{i,j})\geq 0$ is a known constant, $\sigma(Y_{i,j})=0\Leftrightarrow Y_{i,j}=0$). This variable characterizes the external uncertainty of $(i,j)$'s activity duration,
- $c_{i,j}^{B_{i,j}}$ is the actual $(i,j)$ activity cost related to the budget allocated to the activity $(i,j)$, $B_{i,j}$ (a random variable with expectation $E(c_{i,j}^{B_{i,j}})=B_{i,j}$),
- $Z_{i,j}^{B_{i,j}}$ is the random variable with zero expectation, with the standard deviation being proportional to $B_{i,j}$ ($E(Z_{i,j}^{B_{i,j}})=0$, $\sigma(Z_{i,j}^{B_{i,j}})=n_{i,j}\cdot B_{i,j}\geq 0$, $\sigma(Z_{i,j}^{B_{i,j}})=0\Leftrightarrow Z_{i,j}^{B_{i,j}}=0$). This variable characterizes the time-independent activity $(i,j)$'s cost,
- $n_{i,j}=\sigma(Z_{i,j}^{B_{i,j}})/B_{i,j}$ is a known constant independent of allocated budget $B_{i,j}$ related to activity $(i,j)$,
- $V_{i,j}^{B_{i,j}}=(B_{i,j}/B_{i,j}^{N})\cdot\left(g_{i,j}(B_{i,j}^{N})/g_{i,j}(B_{i,j})\right)\cdot V_{i,j}^{B_{i,j}^{N}}$ is the positive cost-per-time unit of executing activity $(i,j)$ related to the allocated budget $B_{i,j}$,
- $V_{i,j}^{B_{i,j}^{N}}$ is the known positive cost-per-time unit of executing activity $(i,j)$ related to the allocated budget for normal performance $B_{i,j}^{N}$. If the activity $(i,j)$'s cost is dependent on the random activity $(i,j)$'s duration then $V_{i,j}^{B_{i,j}^{N}}>0$, otherwise $V_{i,j}^{B_{i,j}^{N}}=0$.

We assume as well that $X_{i,j}^{B_{i,j}}\sim F_{X_{i,j}^{B_{i,j}}}$, $Y_{i,j}\sim F_{Y_{i,j}}$, $Z_{i,j}^{B_{i,j}}\sim F_{Z_{i,j}^{B_{i,j}}}$, where $F_{X_{i,j}^{B_{i,j}}}(x)$, $F_{Y_{i,j}}(x)$ and $F_{Z_{i,j}^{B_{i,j}}}(x)$ are known distribution functions and the random variables $X_{i,j}^{B_{i,j}}$, $Y_{i,j}$, $Z_{i,j}^{B_{i,j}}$ are independent. Then the analytic evaluation of the distribution of $(i,j)$'s activity duration $t_{i,j}^{B_{i,j}}$ and the actual activity cost $c_{i,j}^{B_{i,j}}$ is simple and straightforward.

## 4. SUGGESTED PROCEDURE FOR CRASHING A *PERT*-TYPE NETWORK

Let us consider a project denoted by a known directed acyclic graph (*DAG*) with a partial ordered set (*POSET*) of activities $G=\{(i,j)\}$, where $P_k=\{(i,j)\}$ is the *DAG* $k$ th path and $P\equiv\{P_k,\ P_k\subset G\}$ is the set of all the *DAG* paths.

Let $B=\{B_{i,j}\}_{(i,j)\in G}$ be the set of the budgets allocated to the project activities. We assume that the actual activity duration and the actual activity cost of each activity $(i,j)$, $t_{i,j}^{B_{i,j}}$ and $c_{i,j}^{B_{i,j}}$, respectively,

following the proposed model (1). The completion time of the $k$ th path, $T_{P_k}^B$ related to the set of budgets allocated to the project activities, $B$, is $T_{P_k}^B = \sum_{(i,j)\in P_k} t_{i,j}^{B_{i,j}}$, and the completion time of the critical path(s) $P_{k*}^B$ related to the set of budgets allocated to the project activities, $B$, is $T_{P_{k*}^B}^B = \underset{P_k \in P}{Max}(T_{P_k}^B) = T^B$, where $T^B$ is the project completion time related to the set of budgets allocated to the project activities, $B$. The actual project cost is $C^B = \sum_{(i,j)\in G} c_{i,j}^{B_{i,j}}$. Since we are interested in evaluating of a chance-constrained project completion time and a chance-constrained project cost, we let $T_{1-\alpha_1}^B$ define the $1-\alpha_1$ fractile of the actual project completion time and $C_{1-\alpha_2}^B$ the $1-\alpha_2$ fractile of the actual project cost. The analytical evaluation of the distribution of the project completion time $T^B$ is a very complicated and frequently impossible task. However, for any given set of budgets allocated to the project activities, $B$, following model (1) it is possible to evaluate simulated fractiles $T_{1-\alpha_1}^B$ and $C_{1-\alpha_2}^B$ for any given significant levels $0 \le \alpha_1 < 1$ and $0 \le \alpha_2 < 1$.

With the terms defined above, we present here a new suggested procedure for crashing a *PERT*-type network that completely utilizes a stochastic nature of activity durations. This procedure recommends to crash the project completion time in terms of its desired fractile, giving priority to that project activity which enables the greatest shortening of the pre-given fractile of the project completion time. The procedure can be considered as a 'stochastic procedure' since it utilizes the distributions of the stochastic activity durations. The suggested procedure has the following steps:

1. determine the desired probability confidence (chance constraint) of the project completion time $T^B$, $0 < 1-\alpha_1 \le 1$,
2. start the procedure by considering for each activity $(i,j) \in G$ the budget allocated for normal performance, $B_{i,j}^N$, and the distribution of the activity's duration related to the allocated budget for normal performance,
3. finish the procedure if there is no remaining budget for allocation among the activities, i.e., $B_{i,j} = B_{i,j}^C$, for each activity $(i,j) \in G$; otherwise, continue with Step 4,
4. allocate one unit of budget to one of the project's activities $(i,j) \in G$ that can be shortened (i.e., $B_{i,j} < B_{i,j}^C$) and update its duration distribution. Then, from the activity duration distributions related to the current budgets $B_{i,j}$, $(i,j) \in G$, generate samples of 10,000 observations and according to Equations (2)-(3*)* calculate the simulated $1-\alpha_1$ fractile of the actual project completion time, $T_{1-\alpha_1}^B$, and the simulated expected project completion time, $E\left(T^B\right)$,
5. repeat Step 4 for each of the project's activities that can be shortened and continue with Step 6,
6. select the activity to which the allocation of one unit of budget provides the minimal chance-constrained project completion time, $(i,j) = Arg\ \underset{(i,j)\in G}{Min}(T_{1-\alpha_1}^B)$. In the case of several minimal chance-constrained project completion times, choose the activity to which the allocation of one unit of cost provides also the minimal expected project completion time, $(i,j) = Arg\ \underset{(i,j)\in G}{Min}(T_{1-\alpha_1}^B)$ and $(i,j) = Arg\ \underset{(i,j)\in G}{Min}\left(E(T^B)\right)$. In the case of several minimal expected project completion times, choose one activity randomly,
7. allocate one unit of budget to the chosen activity and update its duration distribution. Then return to

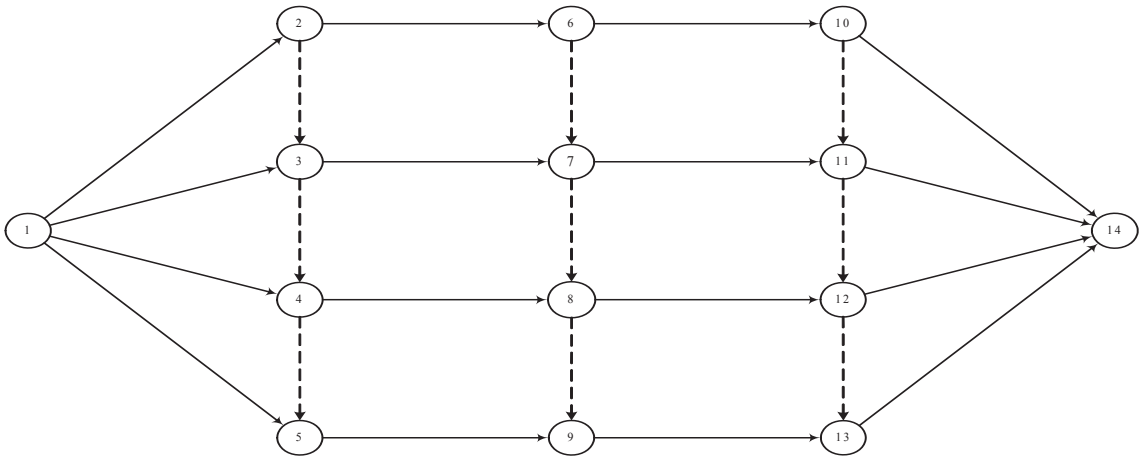
Step 4.

In Section 5 we examine and compare this procedure, which we denote as *Procedure C*, with the deterministic procedure introduced by Siemens (1971)-*Procedure A* and the semi-stochastic procedure introduced by Golenko-Ginzburg (1993)-*Procedure B*, by considering a typical *PERT*-type network.

## 5. EXAMINATION AND COMPARISON OF THE PROCEDURES FOR CRASHING THE PROJECT COMPLETION TIME

To examine and compare the procedures for crashing the project completion time an *activity on arc* (*AOA*) *PERT*-type project network was taken from Tavares et al. (1998) (Figure 1). The considered project $G$ includes twenty five activities, nine of which are dummies. Via lexicographic scanning thirty five paths are identified in this acyclic graph, $P \equiv \{P_k, \mathrm{k} = 1,...,35\}$.

Fig. 1: An AOA PERT-type project network




We assume that the time and the cost distributions for each of sixteen non-dummy activities $(i, j)$ of the project following model (1), where $g_{i,j}(B_{i,j})$ is considered as a decreasing linear function $g_{i,j}(B_{i,j}) = \phi_{i,j} \cdot B_{i,j} + \gamma_{i,j}, \phi_{ij} = -\dfrac{E(t_{i,j}^{B_{i,j}^N}) - E(t_{i,j}^{B_{i,j}^C})}{B_{i,j}^C - B_{i,j}^N}$, $\gamma_{ij} = \dfrac{B_{i,j}^C \cdot E(t_{i,j}^{B_{i,j}^N}) - B_{i,j}^N \cdot E(t_{i,j}^{B_{i,j}^C})}{B_{i,j}^C - B_{i,j}^N}$. The independent random variables $X_{i,j}^{B_{i,j}}$, $Y_{i,j}$ and $Z_{i,j}^{B_{i,j}}$ are normally distributed and the parameters of the model for each $(i, j)$ are determined in Table 1.

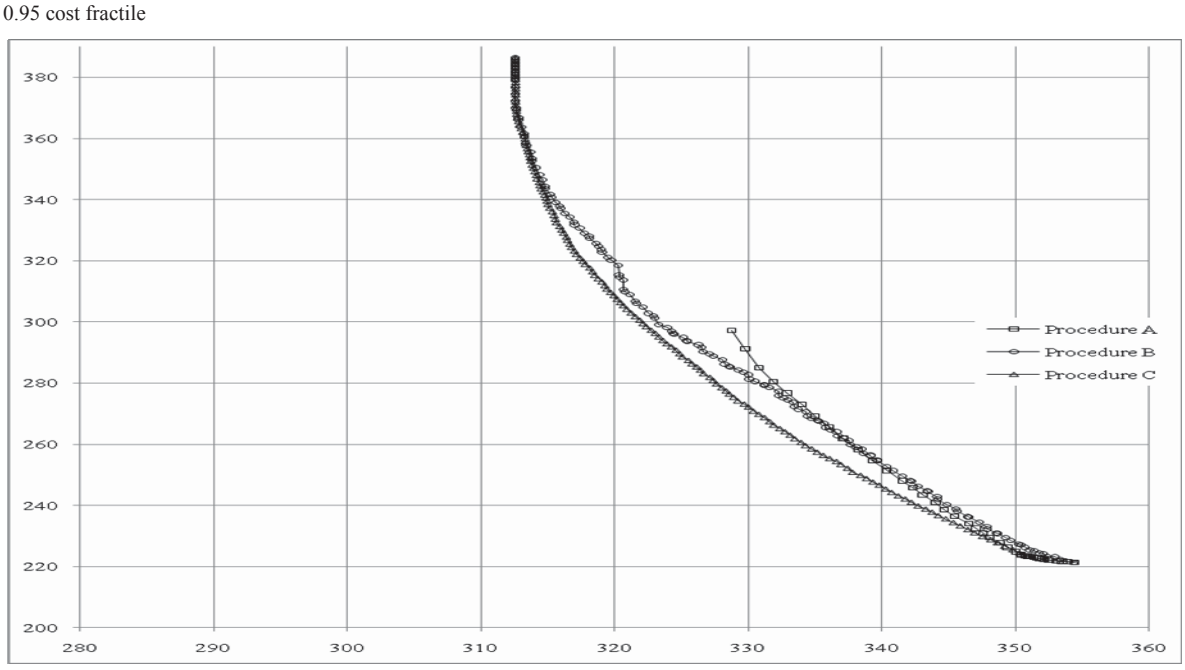
Figure 2 demonstrates the 'optimal' project completion time crashing according to the alternative procedures in the context of chance-constrained cost versus chance-constrained project completion time. The figure presents the simulated 0.95 fractiles of the actual project cost versus the simulated 0.95 fractiles of the

Table 1: The project activities

| **Project operation** | **Normal performance** | | | | | | **Crash performance** | |
|---|---|---|---|---|---|---|---|---|
| $(i,j)$ | $g_{i,j}\left(B_{i,j}^{N}\right)$ | $B_{i,j}^{N}$ | $\sigma\left(X_{i,j}^{B_{i,j}^{N}}\right)$ | $\sigma\left(Y_{i,j}\right)$ | $\sigma\left(Z_{i,j}^{B_{i,j}^{N}}\right)$ | $V_{i,j}^{B_{i,j}^{N}}$ | $g_{i,j}\left(B_{i,j}^{C}\right)$ | $B_{i,j}^{C}$ |
| 1 | 67.00 | 14.00 | 12.57 | 3.14 | 0 | 0.26 | 60.00 | 25.00 |
| 2 | 75.00 | 2.00 | 0 | 7.78 | 0 | 0 | 66.00 | 9.00 |
| 3 | 149.00 | 22.00 | 9.60 | 2.20 | 5.20 | 0 | 139.00 | 34.00 |
| 4 | 142.00 | 12.00 | 4.67 | 1.17 | 0.12 | 0.63 | 130.00 | 19.00 |
| 5 | 55.00 | 22.00 | 0 | 6.00 | 0 | 0.6 | 40.00 | 31.00 |
| 6 | 86.00 | 9.00 | 9.78 | 2.44 | 0.2 | 0.24 | 77.00 | 20.00 |
| 7 | 11.00 | 2.00 | 0 | 0 | 0.20 | 0 | 9.00 | 8.00 |
| 8 | 25.00 | 9.00 | 6.57 | 0 | 0 | 0 | 21.00 | 18.00 |
| 9 | 51.00 | 16.00 | 0 | 13.71 | 0 | 0.3 | 45.00 | 27.00 |
| 10 | 55.00 | 4.00 | 11.25 | 0 | 0.40 | 0.08 | 47.00 | 13.00 |
| 11 | 70.00 | 2.00 | 0 | 0 | 0 | 0 | 54.00 | 5.00 |
| 12 | 38.00 | 20.00 | 7.33 | 0 | 0 | 0.31 | 23.00 | 31.00 |
| 13 | 134.00 | 14.00 | 7.33 | 1.83 | 0.50 | 0.54 | 122.00 | 25.00 |
| 14 | 86.00 | 18.00 | 0 | 14.44 | 0.28 | 0.31 | 77.00 | 31.00 |
| 15 | 66.00 | 28.00 | 14.40 | 3.60 | 0.56 | 0.46 | 60.00 | 40.00 |
| 16 | 90.00 | 6.00 | 8.00 | 0 | 0.60 | 0 | 80.00 | 14.00 |

actual project completion time. Figure 2 clearly demonstrates the advantage of *Procedure C* in comparison with the other procedures. Note that *Procedure C* was constructed to maximize the 0.95 fractile of the actual project completion time for each given budget. Therefore, in this sense, the results of *Procedure C* confirm the correctness of this procedure and are straightforwardly explainable.

Fig. 2: The 'optimal' crashing: the simulated 0.95 fractiles of the actual project cost versus the simulated 0.95 fractiles of the actual project completion time



Considering these results, the *Monte Carlo* simulations confirm that the proposed *Procedure C* can be recommended for practical applications.

## 6. CONCLUSION

It is well known that forecasting the project completion time when the activity durations and costs are stochastic is a problematic task. The 'naïve' stochastic extension of the *CPM* suggested by Siemens (1971) does not take into account information about uncertainty and as a result provides an ineffective budget allocation among the project activities for meeting any pre-given chance-constrained project due date. Being aware that the analytical evaluation of the project completion time under uncertainty is very complicated and must be based on assumptions that impair the accuracy of results, we recommend the implementations of *Monte Carlo* simulations for evaluating the realistic distribution of the project completion time.

Golenko-Ginzburg (1993) suggested a procedure for crashing the project completion time by additional budget that evaluates the project completion time distribution via a frame of *Monte Carlo* simulations. We should note that this procedure is oriented specifically for crashing the expected project completion time and ignores the feasibility of decreasing the variance of the project completion time which also has a positive impact on the probability confidence to meet the project due date.

In this paper we suggested a generalized model for activity time-cost tradeoffs and proposed a new procedure for crashing a *PERT*-type network which completely utilizes a stochastic nature of activity durations and is based on *Monte Carlo* simulations. As demonstrated via the project under study by presented *Monte Carlo* results, the proposed procedure compares favorably with that suggested by Golenko-Ginzburg (1993).

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# Real Options and Emerging Markets: An Overview

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**Abstract**

Uncertainties regarding investment projects create options that are available to the management of the company. Uncertainty can add value to the company, if the management recognizes those options. Emerging markets, partly due to incomplete information, are sources of uncertainty and therefore they are enabling corporation to generate extra value. In order to, appropriately account for uncertainty and flexibility, real options valuation should be used.

**KEYWORDS**

Real options, Strategic flexibility, emerging markets

## 1. INTRODUCTION

Risk management is of increasing importance. Markets are volatile, and market conditions are uncertain. Risk represents measure of uncertainty and it is essential for corporation to integrate it in their decision-making process. Most important corporate decisions are investment decisions, financing decisions and dividend policy decision. Interactions among these decisions have implications on the value of corporation (Damondaran, 2002). Thus, investment decisions enhance value of the company and they will be focus of this article.

Traditional discounted cash flow (DCF) techniques incorporate risk in decision-making process by using discount rate that is adjusted for risk. However, there are numerous problems with this approach. Could only one discount rate be used for multi-period projects or several discount rates are needed? Discounted cash flow techniques do not capture managerial flexibility. During the completion of a project, new information can be introduced that can alter the course of a project, add value; but they are not captured with discounted cash flow techniques. Therefore, traditional valuation methods do not consider uncertainties with future projects outcomes (Kodukula and Papudesu, 2006). Furthermore, traditional discounted cash flow techniques are static and used alone do not represent adequate decision-making tool. Alternative project valuation framework is real options analysis (Cao and Leggio, 2008). Real options approach integrates risk in investment decision-making and allows proper valuation (Kodukula and Papudesu, 2006). This framework explicitly values managerial flexibility. Therefore, real options analysis is dynamic, and allows adjusting corporate strategy and its product portfolio. In real options analysis, volatility creates value, so managers should employ risk to some degree, which in return can add more value to the corporation (Cao and Leggio, 2008).

Reality is that risk should not be eliminated, it should be managed. By accepting risk, managers could create premium. In order to manage risk and create value, managers should be flexible in their decision-making. New flexible business models that are going to include emerging markets in their corporate strategy, will allow flexible and value-adding decision-making. Those kinds of strategies will involve realization of projects that will either involve outsourcing to emerging markets either will allow vertical integration of a company from emerging market country. Either way, flexible decision-making is allowed. Hence, there are more alternatives managers could implement and accept the most valuable projects. Real option valuation is applicable to various kinds of industries and projects, hence it should be considered as primary tool in investment decision-making, especially when emerging markets are involved in completion of a project. This

paper presents main properties of real options approach, its applicability to various industries and represents minimum guide for implementation of this approach to different types of investment projects in emerging markets.

# 2. REAL OPTIONS

The application of option concepts for valuing real assets represents an important and growth area in the theory and practice of finance (Schwartz and Trigeorgis, 2001). Concepts for valuing financial assets can be extended to assess capital investment opportunities in real assets (Hull, 2006). Frequently there are options embedded in such investment opportunities, like option to expand, to defer, to contract, to abandon the investment (Hull, 2006). One of the major differences between financial and real options is in the underlying risky asset. Underlying risky asset for real options is a physical asset whose value can be changed by managerial decisions (Copeland et al, 2005).

Real options approach incorporates managerial flexibility into analysis (Schwartz and Trigeorgis, 2001). Management's flexibility to adapt its future actions as a reaction to changed market conditions, enhance investment opportunity's value by increasing its upside potential while limiting downside losses (Trigeorgis, 1996). An option approach to capital budgeting quantifies the value of options from active management. Many of the real options arise naturally; others are being planned and built in at some extra cost (Trigeorgis, 1996).

## 2.1 Inputs in Real Options models

Real options analysis assumes implementation of valuation models for financial options on real assets. Furthermore, realization of a project can be assumed as an American type of an option (Shibata, 2008). In order to be able to implement option pricing models on a real world projects, it is important to define its inputs. Inputs that are required are value of a real asset and its volatility, information on investment costs, time to complete a project, and interest rate. Opponents of real options valuation framework state that estimation of model inputs is difficult (Chance and Peterson, 2002). Nevertheless, information needed to implement real options analysis is no more specific than information for net present value (Chance and Peterson, 2002).

First input, in option pricing model, is value of an asset. Underlying asset within real options approach is a project. Under asset value, market value is assumed. Hence, market value of a project is an input in valuation model. Valuation of financial options assumes markets are efficient and market price of the underlying security is respectable input in the model (Chance and Peterson, 2002). On the other hand, asset, when real option is assumed, is a project, which is not a tradable asset. Real asset value could be obtained as free cash flow that is expected to be generated from the project, discounted with risk-adjusted discount rate. Asset value must be adjusted if any event suddenly occurs, and have an influence on the cash flows from a project (Kodukula and Papudesu, 2006). Hence, value of a project is a net present value, not cost to enter a project and gain access to a real option (Chance and Peterson, 2002). Also, profit, demand, price could be assumed as underlying variables (Shibata, 2008).

Another input in option pricing model is volatility. Volatility represents uncertainty of the return realized on an asset, and is an only unobservable variable in financial option pricing. Valuation accuracy depends on estimation of volatility (Chance and Peterson, 2002). With real options, volatility represents uncertainty about cash flows received from the project. Uncertainty associated with the cash flows is important for asset value and, hence, real option value. Volatility also represents uncertainty related to unit price, quantity of products that are expected to be sold, margin etc., which determines cash flows of a project. Since, there can be few sources of volatility, it is important to know if they are correlated. If there is no correlation between different sources of volatility, option can be assumed rainbow option and valued as such. On the other hand, all volatilities can be combined and used together as aggregated value (Kodukula and Papudesu, 2006). Real options are long-term options; thus, volatility can change during the course of its life. Shifts in volatility are,

for example, due to unexpected global events. There are several approaches for estimating volatility. Volatility can be estimated from past records, obtained through simulation or from similar projects (Kodukula and Papudesu, 2006).

In the market, complete or incomplete information can be assumed. Underlying state variable is uncertain, and therefore can be modeled as random variable or stochastic process. If state variable is modeled as random variable, the effect of the real options value with respect to uncertainty of the underlying state variable is not examined. Traditional real options analysis assumes all relevant information is known in advance with certainty. However, case of incomplete information is more common. Besides uncertainty associated with underlying variable, other uncertainties could arise as well, like uncertainty about incomplete information and estimation uncertainty. They represent extension of the classical real options model. According to Shibata (2008), profit uncertainty has the same effect on the real option value under complete and incomplete information. If there is great uncertainty about profit, company will be less motivated to invest. If it is known that, the underlying state variable is more volatile, even if the realized value of the underlying state variable cannot be observed completely, the firm displays greater inertia in its investment behavior. An increase in information uncertainty decreases the real options value. In addition, an increase in estimation uncertainty increases the real options value, assuming stochastic process for the underlying variable (Shibata, 2008). Profit and information uncertainty decreases real options value, and estimation uncertainty increases real options value.

The exercise price, in a real options theory represents the amount that would be paid or would be received if a real option were exercised (Chance and Peterson, 2002). Exercising real options requires long period, since it involves for example, product development. Strike price of a real option involves investment costs, which are subject to change during the period of realization of a project. All changes in costs that are necessary to be made, in order to complete the project, must be accounted for, and according to those adjustments asset value should be corrected (Kodukula and Papudesu, 2006). Uncertainty about investment costs is especially important when large projects with long time to complete them are assumed. Examples of those kinds of projects involve petrochemical complexes, development of new aircraft line, construction projects, and development of a new drug. Uncertainties could arise due to fluctuation of construction prices or government regulations. For example, changing environmental regulations can enhance investment costs (Dixit and Pindyck, 1994).

Financial options have defined time to expiration. American style options can be exercised early, but not after its time to expiration (Chance and Peterson, 2002). With real options, time to maturity is not fixed and known with certainty. It is not known exactly for how long, opportunity to make decision will last. Options life should be long, but not too long, since competitors could use this option as well. In contrast to financial options, value of real options is not enhanced with longer time to maturity (Kodukula and Papudesu, 2006).

Risk free rate imputed in the model is usually reference spot rate from the market, with maturity that is the same as options life (Kodukula and Papudesu, 2006). Hence, risk free rate represents opportunity cost of the money (Chance and Peterson, 2002). Level of a risk free rate is uncertain, which means that their level can increase or decrease in the future. Uncertainty about interest rates increases the expected value of an investment. When level of interest rate is analyzed, public policies play an important role. If an objective of public policy is to stimulate investments, stability of interest rates is more important than its level. Lowering interest rates could make them more volatile, and discourage investment expenses (Dixit and Pindyck, 1994).

## 2.2 Examples of real options

Options are available in every investment decision environment, yet they are not easily recognized. Usually realization of one option reveals other options, which all have to be taken into account. In order to apply real options framework as accurately as possible, all options must be included in the analysis. Structure and flow of a project is important for proper valuation of an asset (Kodukula and Papudesu, 2006). Real options approach is not suitable for application when net present value is close to zero and when uncertainty and flexibility are low. Likewise, when uncertainty is high but risk is not associated with market risk, real options

framework will not help in finding real investment value. If real options framework were appropriate to apply in investment decision making, use of both binomial model and Black-Scholes model would be appropriate for valuation (Kodukula and Papudesu, 2006).

Real options theory has many practical applications. All uncertainties regarding inputs in real options model can create option that is available to the management. Waiting-to-invest option adds value to the investment since there is uncertainty about future interest rates. Interest rates could increase or decrease in the future, hence they can decrease or increase value of the project. By waiting interest rates to decrease, managers could take on projects that are more valuable and add value to the corporation. Project may be almost worthless today, yet, it might have some value if it would be completed in the future. However, this is true, only if to some extent project creates value in the present (Culp, 2001). On the other hand, there are situations in which corporation cannot wait, or cannot wait for a long time to complete investment. If there is some indication that competition may enter the market, or that patent or lease can expire, no real options will arise from waiting (Dixit and Pindyck, 1994).

Uncertainty about future value of interest rates is not the only one that can arise. There is also uncertainty about expected cash flow, expected capital expenditure, time to start a project. Consequently, option to defer project could be valuable. Deferment of an investment could be valuable if that option exists, given that by waiting, more favorable market conditions could emerge (Culp, 2001). Hence, option to defer an investment represents option to wait until new information is available, that is going to reduce investment uncertainty (Brach, 2003).

Option to abandon production of an asset or a research and development of a product, is an example of a real option (Culp, 2001). Option to abandon represents an option to get rid of an unprofitable project when market conditions change or expectation of market movements are not met. Thus, it is a put option (Brach, 2003). Abandonment decision is managerial decision to cease production and sell all of related equipment in the market, and in that way to recover salvage value of an asset. Decline in demand would create higher expenses if company would continue to produce an asset. In that case, liquidation value of an equipment is greater than the value company would create if would still produce an asset. Therefore, option to abandon can add value. This type of an option depends on the salvage value of an asset. This option is exercised when cash flows from production have lower value than salvage value. The abandonment option can be viewed as an option on maximum of two values, value of the assets in its current use and value of the assets in their alternative use (Culp, 2001). Managers are flexible in respect of their asset portfolio. They could develop or produce certain assets. Change in market conditions do not necessarily represents a bad thing. Salvage value could be obtained or new product model can be produced or completely different type of product.

Time to build represents combination of option to defer and option to abandon. Investment expenditures and time when new information could come to the market and result in abandonment of the project are variables. Each stage of a project can be viewed as an option combined with sequential option to abandon project, thus it can be viewed as a compound option. Corporation can continue completion of a project or it can abandon it. Either way, value is added. This type of real option does not oblige marketing assets. Thus, it can be valuable just to stop production of assets (Culp, 2001).

Beside change of capacity, alterations of inputs and outputs of the production process can be made as well. Switching option can be valuable if price of one input rises, and by flexibility of the production process, we switch to alternative lower-priced input. Output switching is alternative that is applied in industries where demand for assets is very volatile and scope of production is not large. This kind of option could be viewed as an exchange option or an option on the better of two assets (Culp, 2001). Switching option can be created when there is a task of defining an optimal production capacity. In response to demand shocks, corporation can outsource input production, fully integrate input production, or operate under mixed regime producing inputs internally until some threshold and outsourcing when that threshold is reached. If the purchase price for input is lower than average minimum cost of producing it internally, outsourcing is preferred to vertical integration. Opposite is also true. Corporation does not outsource production of inputs if high production capacity is assumed, rather it produces inputs partly in house and outsource units for which marginal cost of internal production exceeds the outsourcing price. Therefore, depending on production capacity of a

company, depends decision whether to outsource (assuming low production level), vertically integrate (assuming intermediate output level), or combine it (assuming high production level). Sharp drop in the supplier's prices can initiate outsourcing. Example represents the wave that has moved outsourcing to low cost emerging markets, like China and India. Decision regarding production of inputs represents trade-off between fixed costs of vertical integration and the option to outsource (Labrecht et al., 2010). According to Li and Wang (2010) when outsourcing to a supplier in offshore country there is additional uncertainty to be considered, that is uncertainty about exchange rates.

Shifts in demand can lead to decision to stop production for a specific period, or to decision to invest in expansion of capacities (Culp, 2001). Thus, option to expand (call option) or contract (American put option) could be assumed. Under this type of option, alteration of production capacity depends on market conditions (Brach, 2003). Importance of flexibility is always stressed. However, according to Fontes (2008), when strategy of flexible production capacity is applied to demand data set, sometimes, fixed capacity system is more valuable. Additional flexibility represents an advantage only when there is enough good and complete information about decision situation (Fontes, 2008).

If a project is executed in stages and if completion of one stage opens opportunities for growth in other areas, we can view that investment as an investment with an option for interactive growth (Culp, 2001). Thus, option to grow and option to stage could be assumed separately. Option to grow represents option to which will include future opportunities for development, and option to stage assumes investment is divided into stages that are conditional one on the other (Brach, 2003).

Real options can be simple or complex. Option to choose represents strategic option, under which management can choose between few mutually exclusive alternative decisions (Mun, 2002). For example, deregulation of electricity market has made this market, market with uncertain prices. Now there is more flexibility with making investment decisions regarding building power plants. Takashima et al. (2010) created a model that assumes that firm has the right but not the obligation to invest in power plant. After it is build, it will operate fixed number of years and create fixed operational costs depending on its size. Construction costs of the plant depend on its size as well. When to invest is not the only problem, but what capacity of the plant it will be represents another one. Furthermore, plant can operate under different technologies. Mutually exclusive projects assume completion of a power plant under one technology regime, with flexibility due to timing of investment and size of the plant (Takashima et al., 2010).

Compound option is a type of a real option where value of an option depends on the value of another option (Mun, 2002). Compound option is an option on an option, where with completion of each step of the investment, management buys an option to go to the next level with this investment, but they do not have the obligation to do so (Brach, 2003). For example, if company is planning to enter new market with its product, and it shows up to be successful that company can expand their production, or otherwise it can abandon it (Damondaran, 2002).

Number of factors influences the value of real options. Factors that have impact on the real option value are factors like value of the underlying asset, its volatility, the degree to which value of a real options is affected by behavior of competitors, the degree to which an exercise decision affects the value of other real options, time to maturity (Culp, 2001). The riskiness of the asset that is underlying for real option depends on the ability of the management to effectively respond to the option and take the full advantage of it. Realizing the benefit of the real option depends on the availability of the new information and the managerial ability to exercise an option (Brach, 2003). Examples of different industries and real option approach applicability are presented in table 1.

### 2.2.1 Real Options Optimization

Real options approach assumes optimization, usually maximization of returns to risk ratio. Every day, companies are faced with many difficult decisions, like, how to allocate financial resources, what to produce, do they need more facilities in order to accomplish target production. In order to make such decisions, reality needs to be simplified and modeled (Mun, 2002).

Table 1. Various types of Real options[1] and their explanation.

| Real options / industry | Industry and rationale | Industry and rationale | Industry and rationale | Industry and rationale |
|---|---|---|---|---|
| Option to defer | Oil, gas, and forestry/ investment in extraction tools | Leisure (seasonal demand)/ investment in expansion of capacity, defer to a specific season | Real estate / defer until that area is area with higher rents | Pharmaceutical, mining, automobile/ delay extraction or drug production |
| Option to abandon | Transportation/ abandon production of one vehicle and perhaps start production of another | Financial services/ sell assets in the market | | |
| Time to build option | Pharmaceutical/ R&D costs dependent on exact timing | Large scale construction and renewal/ costs are substantial, timing of lowest costs are essential (Reindrop and Fu, 2011) | Venture capital financing/ when to finance entrepreneur | |
| Option to alter operating scale (expand or contract) | Extraction of national resources/ amount will depend on demand | Real estate construction in different industries | Fashion, various production industries/ amount will depend on demand | Food/ amount will depend on demand |
| Switching option | Electricity | Air craft | Mining | Entering and exiting from the market in general |
| Option for interactive growth | Mineral resources | Every industry if M&A process is carried out | | |
| Option to grow | Biotech companies/ this industry is fast growing | Software/ this industry is fast growing | | |
| Compound option | R&D in prototypes, drug production/ these projects are done in sequences that are dependent | Investment in IT infrastructure/ these projects are done in sequences that are dependent | Investment in new distribution network/ these projects are done in sequences that are dependent | e-commerce/ these projects are done in sequences that are dependent |

If outsourcing is assumed, optimization can be done by maximization of utility function in two steps, with respect to companies risk preferences. First step will represent reserving capacities with two suppliers. Second step will assume realization of demand, and then making production decision according to the production costs (Li and Wang, 2010). According to Li and Wang (2010) exchange rate is assumed to be random variable, and demand is assumed either deterministic or stochastic.

[1] All examples are taken from referenced literature

## 3. OPPORTUNITIES IN EMERGING MARKETS

Emerging markets are not large, liquid, and transparent as developed markets. Different standards and policies are valid in emerging countries. Historically, emerging markets offered higher return for accepting higher risk, and there were low correlation with developed markets. In recent years, volatility in developed markets increased, and at the same time expected return decreased. Therefore, emerging markets became attractive for investment depending on the risk aversion (Bruner et al, 2003). These results are presented for securities that are included by institutional investors in their portfolios. Same argument could apply to the investment projects as well, since securities that are included in portfolios are securities issued by companies that are operating in emerging countries. Those companies are rapidly developing and offering excess return, and events are driving loss in developed market may not be the same factors that affect emerging markets. By incorporating those emerging companies in corporate strategies of developed corporations, more options could emerge. Those options could be viewed as real options, which can add value to the company.

According to Ioulianou et al. (2010), multinationality has a positive impact on firm performance. There is positive impact of operating and strategic flexibility on corporate value. Impact of multinational flexibility on company's performance is higher for corporations with a higher degree of real options capability in place. The more capability a corporation has in implementing its real options, the better is the company's performance (Ioulianou et al., 2010).

According to discussions of World Economic Forum, flexibility is of ad most importance. There is a need for new creative business models that are embodying emerging markets in its core strategy. Thus, emerging markets can represent solution for enhancing performance (Wyman, 2010). However, in emerging markets, there is a problem of acquiring valid information, so incomplete information framework could be assumed in valuing real options. Yet, it one should be caution because incomplete information and uncertainty can prevent corporation to enter emerging markets.

Since emerging markets are volatile, shifts in demand can arise. If products are supplied to an emerging market, that kind of project can be assumed as having real option. In addition, switching of production inputs to an emerging market country is possible. That also creates real option to the company. Acquisition of emerging market corporation, can create new value to the developed company. However, emerging markets are in process of developing government regulation and policies and if they change, they can incur investment costs. Exchange rates can be volatile, and they can lead in huge loss. Nevertheless, if emerging market conditions are studied in depth and if they are incorporated in business strategy of developed countries they can offer premium.

## 4. CONCLUSION

Real options are applicable to large-scale capital budgeting projects within different industries. In today's world, there is a need for decision flexibility. Flexibility allows managers to continually gather new information and change course of the companies' actions (Cao and Leggio, 2008).

In order to encompass risk in decision-making process successfully, real option approach should be applied. Under this framework, financial option valuation is applied to real assets. Real option approach assumes that payoff is adjusted to accommodate risk and is discounted at a risk-free rate (Kodukula and Papudesu, 2006). Variables that present source of an uncertainty like interest rate, expected cash flow, expected capital expenditure, time to start a project could be assumed as inputs in a stock option pricing model. Therefore, capital investment projects with their specificities could be viewed as a real option (Culp, 2001). Business flexibility is essential, because there is continuing request for creativity in developing new business models, which embodies emerging markets in its core strategy. Emerging markets can assist in surpassing benchmarks as measures of performance (Wyman, 2010). According to Rumelt et al. (1991) the main goal of the real option analysis is to align corporate strategy with financial markets when times are very uncertain

and complex. Emerging markets can enable flexibility and can create new value, but only if research is done properly.

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# Personalized Crisis Management Manuals

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**Abstract**

Human reaction during a Crisis is always a vital factor for the confrontation of a disaster and the minimizing of its catastrophic consequences. The lack of training and preparation for the System users against a possible threat increases the system incapability to face a disaster by minimizing the corresponding cost. Panic and confusion are some of the feelings that appear when a chaotic situation is materialized. Those feelings are enforced by the size of the crisis. As a result, the catastrophe effect is maximized leading in the majority of the situations to a total disaster of the System. Simulation and practice are important key factors for a successful crisis confrontation. The scope of this paper is to present a new approach where a crisis could be confronted. The construction of a variety of crisis manuals will be a helpful tool for the organizing of a successful crisis confrontation team. Specific manuals – Personalized Crisis Management Manuals- based on the skills, the knowledge of the staff as well as the kind of the disaster that the system has to face. Personalized Crisis Management Manuals could be created dynamically during a crisis focusing on the characteristic of the threat and the persons that must face the threat and of course the structure of the operational environment. Finally, a specific case is analyzed.

**KEYWORDS**

Informatics Management, Crisis Management, Manuals, Threats, Disasters, Digital Libraries.

## 1. INTRODUCTION

The rapid evolution of informatics in the last two decades has influenced many aspects of the global economy. This evolution was the key factor in the transformation of every well known business system. Most of the modern industrialized world is supported by complicated information systems. Many sensitive areas of the economy, health, power supply, transportation and defense have been totally mutated. In most cases, the mutation was a sign of progress, which on the other hand hides many *black operation holes,* where the danger of collapsing is more than certain. The basis of modern information systems has been altered. The structure of primitive economic areas has totally changed and new types of economic growth have appeared. Basic fields such as education, banking, national defense, and health care are transforming. Their main structure has been totally digitalized in order to adapt to the needs of the new century. Traditional operational areas where the human factor was necessary have been reconstructed minimizing or eliminating the participation of humans.

In order to better analyze the structure of a modern digitalized business model, but also to face the dangers that threaten it, we have to examine the problem under the light of the study of a complicated, constantly mutated and evolutionary information system**,** which interacts in a dynamic changed area**.**

By describing that complexity, the following two important characteristics are observed. The risk of the system collapsing is real. It becomes greater, if we consider the dangers that an information system faces by acting in an unstable, hostile environment, the data of which are constantly being transformed. The hostility that the modern systems face increases analogically or in many cases exponentially the irregularity of the Data Area. Those new methodologies are trying to face threats that until now, they were less significant, but they are proved to be the opposite exactly. In particular, after the huge amount of casualties and the destructions, it is more understandable that during crisis, in order to achieve the maximum of performance of a Disaster Recovery System, it is not preferable to use and count on the human factor. It seems that, Human

factor is unstable and in many cases unreliable. So, we suggest a way to minimize that problem. Each person authorized to manage a Crisis, could have a ***P**ersonalized **C**risis **M**anagement **M**anual* (PCMM) with all the instructions needed, corresponding both his individual characteristics and critical condition's variables.

# 2. PERSONALIZED CRISIS MANAGEMENT MANUAL

## 2.1. Crisis Management Plan

The literature describes plan Crisis Management Plan (CMP) (Devlin, 2006), as a documented plan detailing the actions the executives want to be taken when a crisis strikes a system. The text of an CMP, specify responsibilities and roles which managers are trained to "play" at all kind of emergencies, i.e. natural disasters, accidents with multiple victims or to threats of terrorism. Similarly, there are plans (FEMA, 2006), for "Restore" from such disasters. The organizations have adopted a methodology for managing such crises, including specific project, organization, guidance and control during disaster's period and following period.

However, most existing CMP, required a steady state, the smooth arrival of the human resources and delivery of basic services. When a crisis appeared, nobody can be sure that these sub-systems will remain undamaged (Chaiken, 2006) and will continue to function normally. The CMP is usually in written form and includes all of the risks and their assessment. In general, described the roles and responsibilities assigned to staff. The executives are trained to be prepared to react appropriately as described in the plan. Nevertheless, there are uncontrollable factors such as:

- Executives authorized to coordinate and manage the phenomenon, failed to be in place (due to loss, physical absence, or choking).
- Unable to update above the *critical point of information* for the development and the range of the crisis may lead to wrong actions with completely negative results.
- Incomplete training of people in all possible situations of crisis (e.g. a possible combination of threats), could confuse staff, because of the panic of the chaotic situation of crisis, and misinterpret the instructions that have been received.

These unexpected and uncontrollable factors are the main reasons making personnel unable to react satisfactory. So, instead the effects and the results of the crisis to be minimized, finally they ultimately maximized.

## 2.2. Dynamic Variables of PCMM

### 2.2.1. Features of Crisis

At the time of a crisis, we have a dynamic environment, constantly changing with combination of threats e.g. an earthquake followed by a tsunami. The new data from the emergency's state, as they are formed every second, should be observed, evaluated and used directly into a PCMM system.

### 2.2.2. Features of Users

All the users of the system, even with the same role, have different set of skills:

- its *relationship with a given knowledge space* (e.g. worked in other roles of a system)
- *psychological profile* and his reaction during crisis implementation (e.g. psychological collapse or vice versa, undertaking leadership initiatives)
- *educational profile* (different educational backgrounds such as secondary or university education) categorized on how user: *learned to learn* based on three Critical Learner-Difference Attributes: 1)emotional/intentional motivational aspects 2)self-directed strategic planning & committed learning effort , and 3)learning autonomy (Martinez, 2002), or how people *prefer to learn*, the four combinations of perceiving and processing determine one of four learning styles: 1.Concrete Experience, 2.Reflective Observation, 3.Abstract Conceptualization, 4.Active Experimentation (Kolb, 1984).
- *physiology*. Persons with inability of movement will not be selected for operations that require access to hard to reach places.

The PCMM system should take into account each of these special characteristics, which their combination is unique.

### 2.2.3. The resources of Roles

In a system, administrative users have roles with specific tasks and resources at normal circumstances. In crisis situation, usually continue to serve the same roles, but there will be cases where some of the users, should cover resources of a different role. It is necessary to give them instructions, at the best possible time, not the whole of knowledge space available for that role, but only those which made them capable to operate that specific moment.

### 2.2.4. Time

Time is the key factor during a critical situation. It's the variable which is always "limited." In a strike, a manager may need to take another role. He must accept the new duties through PCMM and dwell on them in the shortest time. In a change in the evolution of the crisis (e.g. the plane with the transplanted organ to show a significant delay, should the transport unit to be updated directly via PCMM to consider additional measures). A PCMM system should take into account the valuable time factor, someone needs to be updated.

## 2.3. Modelling

In general, in the case of real systems all the things that have been analyzed above are applicable. The structure of systems is complicated, as they are composed of more than two sub-systems. Suppose S is the set of the sub-systems of the analyzed system:

$$S=\{s_1,s_2,\ldots,si,\ldots,s_n\} \quad , \quad S^*=\{1,2,\ldots,n\}$$

As already been analyzed, an important step for the avoidance of a system collapse, is to locate its disadvantages, where usually appeared when external factors act on them and lead to a total destruction of the information system. Of course, the whole study does not have as its purpose the searching of the best or adequate solutions, which are applicable to the entire operation of the system, but our problem is concentrated inside a *programming timetable***.** For instance, the form of the business systems and of their information system obviously concerns us only for the duration of their specific mission. Of course, it is not known how the Information System will be changed inside a programming timetable of an unstable, dynamic environment, but it is possible to know what those elements are, that can act during the programming timetable on the system **S** by changing its structure and its functionality.

There is a basic consideration that makes every software engineer wonder if he has designed the perfect or even a good information system. A system that is more automated and less subjective to human activities. It seems that the answer to this thought is, in the majority of situations, negative. There is no system capable of avoiding any type of destruction, especially in cases where it operates in a hostile environment, and of course there is no system that could be operated totally without the surveillance and guidance of a significant number of human experts. Destructive forces are transformed constantly, and new types and combinations are being created. Nobody likes to spend money. In order to avoid any unpleasant and sudden destructive situations, the system administrators must always be alert. In order to minimise the danger, the system must be prepared to face any kind of danger, caused by human like terrorist acts, or natural disaster like Tsunami. The System administrators must be able to map their system and know its weaknesses exactly. The administrator determines the destructive forces that can be materialized in the sub-systems. The system analyser is responsible for the human resource and the operating staff. In addition, the Head Master must be able to report the abilities and skills of each one individual person that works and interacts with the system.

*Simulation job* is a well-known tool and in most cases is successful method to test and step-by-step improve the system. It is also a useful method for the system to self-improve and eliminate its vulnerabilities. The possible destruction scenarios are created by the different conditions that the system can be in.

## 2.4. Formulations

The appearance of a large number of threats leads Systems Administrators to think new ways of counter-measures. Every PCMM in order to work properly should fulfill the minimum Level of Functionality, i.e the

minimum number of necessary modules for the right operation of every system without any destruction. In the majority of crisis situations it is necessary in order to achieve the minimum Level of Functionality to activate well-trained human staff. Assuming that **SK** *is the set of skills* that are acquired for a person to operate a system in minimum Level of Functionality,

$$\mathbf{SK= \{sk_1, sk_2, \ldots, sk_i, \ldots, sk_n\}}, \quad \mathbf{SK^*=\{1,2,\ldots,n\}}$$

Also, assume that **P** *is the set of human resource* that are acquired for a system to operate in minimum Level of Functionality.

$$\mathbf{P = \{p_1, p_2, \ldots, p_i, \ldots, p_n\}}, \qquad \mathbf{P^*=\{1,2,\ldots,n\}}$$

And **X** *is a vector of the actual skills* that this person has:

$$\mathbf{X= (x_{ij})}, \ \forall i \in P^* \quad \forall j \in SK^*$$

so that,

$x_{ij}$ =1, if the person i has the needed skill $sk_j$.
$x_{ij}$ =0, otherwise.

A helpful tool in order to measure how a person is well-trained and idle for the specific position, is to get the minimum value of the *Skill Rate index (***SR***)***:**

$$\boldsymbol{SR = min\ |X_i| - |T|}$$

In any case, suppose n1 is the number of the skills that someone has and n2 the necessary skills. Obviously, the ***Suitability Proposition*** consists of the following three discrete cases***:***

1. *Identical Situation: SR = n1- n2 = 0.* This is the identical situation where the examined operator fulfils all the criteria. It is the right person for the right job.
2. *Worst Case: SR = n1 – n2 = n1 = n.* This is the worst-case scenario, where finally zero skills fulfilled**.** The disaster has materialized. Our trainee had absolutely failed to confront the crisis.
3. *Partial Skilled: 0< SR <n.* This is a medium situation, which in many cases depends on the training cost of the system operator. It is not the best scenario, but nor the worst.

Broadly speaking, dangers are constantly mutated with rapid speed. They manage to appear new shapes and functionalities more and more complicated and dangerous. That continuous transformation creates types of threats not easily categorized due to their complexity. As a result, system operators must be well-trained and well-informed for the new dynamic environment inside which they act.

A successful PCMM is a technique where a System is restored in his primitive state (i.e before the disaster) or to a new one similar to the original state by the use of an appropriate manual specified to the skills of each one individual. Assume **S** *is the System* under consideration. A *Basic Principle* for the PCMM creation is: "**The Restoration of a Disaster must be as similar as the Primitive Situation with the use of the maximum offered resources".**

Nobody likes to spend money. A PCMM must be able to use all the skills and abilities of the system user in order to be successful. An important criterion for the activation of the appropriate PCMM is the maximizing of user efficiency against a crisis, which is the calculation of the activation efficiency of the manual according to the skills and abilities of every human system operator. Activation is made with the aid of the **Efficiency-matrix:**

$$\mathbf{E= (e_{ij}),\ \ i \epsilon P^*,\ j \epsilon SK^*}$$

so that,

**$e_{ij}$ = cost, if the i-person satisfactory faces a crisis with the use of $sk_j$ skill,**
**$e_{ij}$ = infinite, otherwise.**

The main scope of a successful PCMM is the activation of system's recoveries with the maximum efficiency which is shown in the following proposition:

***Maximum Efficiency Proposition***

$$max \sum_i \sum_j \quad e_{ij}x_{ij}\ ,\ i \in D^*\ ,\ j \in SK^*$$

*so that,*

$$((\forall i) \in P^*): \sum_j \quad x_{ij} <= 1,\ j \in SK^*$$

*and* $$((\forall j) \in SK): \sum_i \quad x_{ij} <= 1,\ i \in P^*$$

The Recovery Model is obviously the well-known **Assignment Problem** with **n rows** and **m columns**, which are solved in a polynomial time for large n and m based on methods, which optimize the primitive method. Therefore, there is no any need to find methods to solve the Recovery Model.

## 2.5. Implementation of PCMM

Like any problem, the methodology to be followed to create PCMM divided into phases. We use the logical approach, the classic method of methodology's analysis for solving a task. The contents of each phase would be:

### 2.5.1. Phase 1: Diagnosis

A PCMM system has expert sub-systems to work on multiple objectives: i)while monitoring the phenomenon, will detect the type of threat and at each change would identify new countermeasures to be taken (Petrantonakis, Panayiotopoulos 2005), ii)determine the knowledge vector that user needs to learn, to deal with the crisis.

### 2.5.2. Phase 2: Composition

During this phase, PCMM system will identify all learning objects from the central knowledge base (Knowledge Repository) that required satisfying the digital feature vector obtained from previous phase (Northrup, 2007). These objects of knowledge have all the necessary information that will establish a PCMM. The designed PCMM could have more than one solution.

### 2.5.3. Phase 3: Presentation

During the presentation phase, the learning objects of PCMM, appropriately transformed to be presented to the user. The PCMM to be produced should be dependent on particular user's characteristics. There should be a process that isolates the information given to the user with those that he already possesses. Another intelligent model will follow, to define the specific nature of personality of the user. Finally, another model will plan, organize and format the content of information (Mavrommatis, 2008).

### 2.5.4. Phase 4: Evaluation

The evaluation of a PCMM may be using a survey questionnaire. The user will be asked about the effectiveness of PCMM after a crisis. The questionnaire will be completed, and user's critical comments used to improve the efficiency of the process. The findings may transform the first phase of PCMM. The intelligent model will learn from any mistakes, and the next version of PCMM will be closer to the original objectives.

## 2.6. Subsystems of PCMM

For a PCMM system to be completed, presupposes the existence of the following subsystems:

### 2.6.1. Knowledge-Based Education Works

For every threat, administrator should have built a knowledge base (knowledge repository), in electronic format, with all the resources needed for a role to deal with a critical situation. This material could as Learning Objects (IEEE 1484.12.1, 2002).

### 2.6.2. User History

Like patient's medical history, each user must have structured data about his all special characteristics: knowledge in a particular epistemic space, educational profile and generally all the information about the owned abilities, trough a model evaluation. This will be updated throughout the life of the individual. The information it contains will be available both to create a PCMM for the user, and for the optimal user selection for a specific position in the best possible response time.

## 3. CONCLUSION

Before the threat became a disaster, it will produced a dynamic Personalized Crisis Management Manual, which emphasize to both the crisis's characteristics and special features of each individual who is authorized with crisis management tasks. This would be unique for each user and will be reproduced every time, while the data of crisis be updated, with directions to address the new situation. The user will be informed immediately for changes to his role, without taking decisions and initiatives, delaying and having negative effect. The PCMM could include responsibilities not only for an individual but also for a group of users or any other resource, which has a specific profile, capabilities and skills, e.g. a mobile infantry unit, an ambulance unit.

Generally, when occur giant crisis, e.g. an earthquake, the public – government infrastructures and all available resources could be included in a system of PCMM to allow better coordination and greater correspondence for each sub-structure.

Today, the research of this study is continuing with the digital implementation within an Expert Information System.

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# CRISIS MANAGEMENT SCENARIOS IN IRREGULAR DYNAMIC DATA SPACES

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**Abstract**

The rapid evolution of informatics in the last years and the huge amount of their applications have enormously influenced many sensitive areas of the global economy. The applications of modern technology and especially of Electronics and Informatics have transformed the structure of the society. New technologies for the assistance of economic growth have been adopted by the majority of the modern industrialized world. The Modern Western economy is mainly supported by the vast usage of complicated computer applications. How can those systems guarantee their integrity and accuracy? Are they capable of preventing a possible disaster? The aim of this paper is to study the dangers that threat to destroy an Information System. Firstly, it is presenting the structure of the modern western business organizations. It is analyzing the operational environment, and the dangers that it faces during its operation. In addition, it is trying to analyze the potential risk that a digitalized business operation has, supported by electronic means and the lack of risk-minimizing methods. Finally, it is presenting a useful tool in order to point out the vulnerabilities that the system has. The main scope is to avoid collapsing by simulating catastrophe scenarios with which central points of destruction are highlighted. All the above are a part of an Applied Multi-Crisis Methodology. An algorithm of evaluating possible crisis scenarios, in order to minimize the possibility of destruction of the system, is also presented. Furthermore, the results of crisis scenarios offer a qualitative analysis of the vulnerabilities of the examined system.

**Keywords:** *Risk Management, Risk Analysis, Scenarios, Irregular Dynamic Data Spaces, OR in Military situations.*

## 1. Introduction

The rapid evolution of informatics in the last two decades has influenced many aspects of the global economy. This evolution was the key factor for the transformation of every well-known business system. The majority of modern industrialized world is supported by complicated information systems. Many sensitive areas of economy have been totally mutated. In most cases, the mutation was a sign of progress, which on the other hand, hides many Black Operation Areas, where the danger of collapsing is more than certain. The base of modern information systems has been altered. The structure of primitive economic areas has totally changed and new types of economic growth have appeared. Areas such as Education, Banking, National Defense, and Health Care have been fully transformed. Their main structure has been totally digitalized in order to be adapted to the needs of the new century. Traditional Operational Areas where the human factor was necessary have been totally reconstructured, minimizing or eliminating the participation of humans.

The digitalisation of Modern Information Systems offered many advantages not only to the system itself but also to the user. Unfortunately, that progressive digitalized revolution did not only benefit Information Systems but it was also a generator of a plethora of dangers that create a set of malfunctions in the Systems. A variety of unpredicted threats appeared and in many cases their materialization was destructive for the system. A vital question is how those systems can guarantee their accuracy and their reliability. Complicated Information Systems were created in order to cover the need for faster and low-cost services. As a result, a plethora of new types of vulnerabilities has appeared, threatening the normal operation of the existing digital Systems. Unpredictable forces have been materialized causing destructive results to the new complicated structure of the Systems. The operating environment became Irregular.

The continuous quest for the answer to the previous fundamental question has led us to the creation of a new methodology that predicts and confronts any type of threat for a given Information System, but also recovers the system from possible destructive situations.(Petrantonakis 2005a).

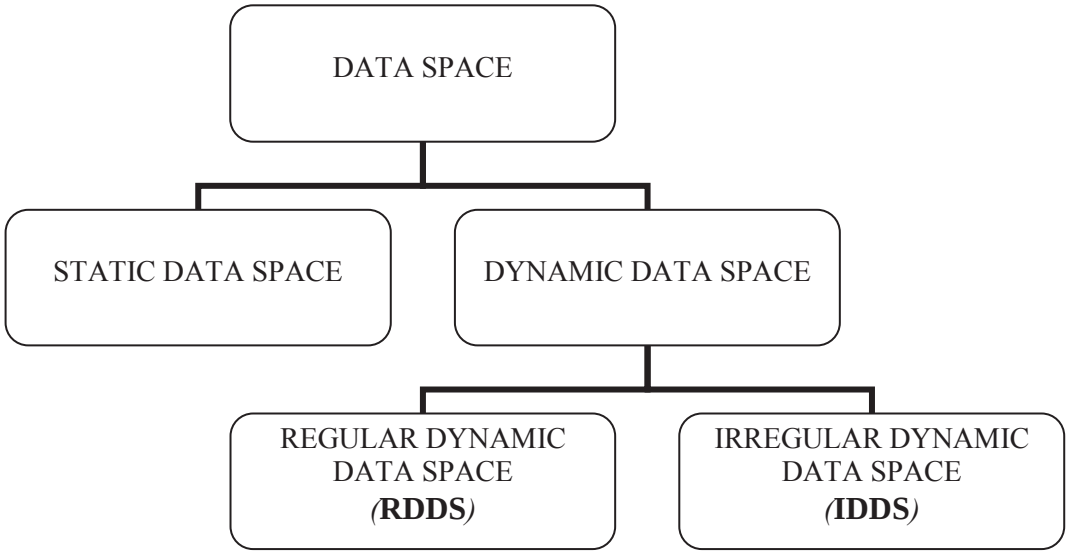
The existing recovery system offers solutions to classical cases of Software or Hardware failure. Moreover, if the data are exposed to web threats or the data area is characterized as Irregular Data Space, then the existence of efficient recovery methods is rare and the need of new ways of confrontation is obvious.

## 2. Data Spaces

In Crisis Management, it is necessary to make a decision at the present time $t_1$ in order to maximize "profit" and to avoid a possible unpleasant situation during a given planning horizon $\Delta t$ ($\Delta t = t_2-t_1$) of our System S. The Crisis is going to be faced with respect to a corresponding mathematical model based on a set of rules.

There are two general categories of Data Spaces in Crisis Management:

- **The Static Data Space**
- **The Dynamic Data Space**




***Schema 1***. *Data Space Categorization*

Static Data Space is an area where the data do not change within $\Delta t$. On the other hand, a dynamic data space is an area where the data are changing during the planning horizon $\Delta t$ (Panayiotopoulos, 1992).

Digitalized Information Systems are based on a set of complicated applications. The complexity is increased if the interaction between the System and the environment that operates is estimated. The new structure of Information Systems and Distributed Systems made them vulnerable to a variety of threats. Furthermore, in the worst-case scenario, there is a possibility those threats to be materialized and the organizational structure of the System to collapse. In addition, there is also the case for the System to be mutated and to be transformed to a new entity less advanced, less accurate and-in some cases-more lethal to its users!

Digital Systems operate in hostile environments, where a significant amount of threats and dangers is grown and can provoke crisis which leads the System to destruction. Moreover, this type of operating environment creates new Data Spaces - the main characteristic of those Data Spaces is their Dynamic Irregularity - where the structure of data continuously changes. The change is very often impossible to be predicted or calculated using well-known mathematical techniques. This is a phenomenon, which shows that the dynamic transformation of the structure of Data and the boundaries of the operating Areas are able to create irregularities. Those irregularities force the System to Abnormal Operation (Panayiotopoulos, 1992).

In order to better analyze the structure of a modern digitalized business model, but also to face the dangers that threat it, we have to examine the problem under the light of the study of a *complicated, constantly mutated and evolutionary Information System,* which interacts in a *constantly dynamic changed area.* The risk of the system to collapse is real. It becomes greater, if we consider the dangers that an Information System faces operating in an *unstable, hostile environment* where the data are constantly being transformed. The hostility that the modern Systems face increases *analogically* or in many cases *exponentially* to the Irregularity of the Data Space. Irregular Dynamic Data Spaces are sectors of an increasing number of risks and threats. A System that operates inside them is vulnerable to Crisis. **A Crisis is a situation where a threat is materialized by destroying the connecting nodes of the Digitalized System.** Moreover, new unspecified forms of the System with corrupted structures appear. The Irregularities that threat an Information System are categorized as **Internal** and **External** ones.

With the definition Internal, we mean the dangers that threat the Information System and they come from *internal operations and abnormal –also internal- mutated situations.* On the other hand, by External we mean the dangers which threat the Information System and are part of *external actions of the factors of the system which,* also lead us to abnormal mutated situations. The dangers that threat the System and they come from Internal operations and abnormal (internal also) mutated situations are identified as Internal ones. On the other hand, the dangers that threat the Information System and those they are provoked by external actions are named External ones. Those external factors are also capable of guiding the System to abnormal mutated situations (Petrantonakis, 2005a).

In both cases, the result is the same for a System like a distributed database of a huge economic organization which operates in a hostile environment. Many times, accidentally or on purpose, the partial collapsing of the System and its transmission to a different structural and functional level is achieved, where it is trapped in uncontrollable, chainable and chaotic situations. It is obvious that both risk categories lead us to the same result, the final collapsing of the Information System and the destruction of its *connectivity structures* between their subsystems.

In the previous paragraph, we introduced the structure of modern economic systems based on complicated computer applications. By describing that complexity, we observe the following two important parts. Firstly, the existence of a main computer System, which controls the peripheral digital devices is necessary and secondly, the use of appropriate software for the digitalization of the functions of its Information System is also important. The use of computer systems in a business model increases the possibility of a crisis appearance.

## 3.Modeling

The structure of modern digitalized information systems is complicated, as they are composed of a variety of sub-systems. Suppose S as the set of the sub-systems of the analyzed system:

$$S=\{s_1,s_2,\ldots,s_i,\ldots,s_n\} \quad , \quad S^*=\{1,2,\ldots,n\}$$

An important step for the avoidance of a system collapse is to locate its disadvantages. They usually appear when external factors act on them within Δt and lead either to a total destruction of the information system, or to its mutation.

**Mutation is a situation where a change that happens in specific time within Δt is capable of transforming the total structure and operational reliability of the system S.**

A mutative situation appears when the structure of the operational environment of the System alters (Mavrommatis 2005). Generally, a mutative situation is materialized when the resultant of the catastrophic forces that act on the system is greater than the reaction of the system S.

Of course, the whole study does not aim at the searching for the best or an adequate solution, which is applicable to the entire operation of the system. It is mainly focused on a *programming timetable Δt.* For example, the structure of the business systems and of their information systems obviously concerns us only for the duration of their specific mission. In addition, it is not known how the Information System will change within Δt of *an unstable*, *dynamic environment*, but it is possible for us to know those elements, that can act during Δt on the system S by changing its structure as well as the main functionality of the System.

Suppose C is the set of catastrophe *forces* (factors) that can appear (activated) within Δt:

$$C=\{c_1,c_2,\ldots,c_l,\ldots,c_m\} \quad , \quad C^*=\{1,2,\ldots,m\}$$

A method that helps to locate possible points of self-destruction is by the use of the **Danger Map**. That map registers the *danger status* of the information system and is expressed by the following matrix (Petrantonakis, 2006):

$$K=(k_{ij}), \quad \forall i \in S^*, \ j \in C^*$$

$k_{ij}=1$, *if the activation of* $c_j$ *changes the data of* $s_i$.
$=0$, *otherwise*.

Usually, certain conditions are activated inside our timetable. Suppose a subset of them $C^0$ is activated. The following propositions are given:

- *Total Destruction Proposition(**Black Operational Hole**)*

  If it is true*:*

$$(\forall i \in S^*)\ (\sum_j k_{ij} >= 1\ ,\ \forall j \in C^* : c_j \in C^0) \qquad (1)$$

then the System S has totally collapsed.

- *Operational Tranquility Proposition(**White Operational Hole**)*
  If it is true:

$$(\forall i \in S^*)\ (\sum_j k_{ij} = 0\ ,\ \forall j \in C^* : c_j \in C^0) \qquad (2)$$

then the System S is in operational tranquility.

- *Mutation Proposition(**Gray Operational Hole**)*
  If it is true:

$$(\exists m \in S^*)(\sum_j k_{mj}=0\ \forall j \in C^* : c_j \in C^0) \qquad (3)$$

then the System S is mutated.

A catastrophe index is a very useful **qualitative tool** that assists the system administrator to make a decision. The above three Propositions that were given previously define the catastrophe index as the ratio:

$$\mathbf{CI=CN/n} \qquad (4)$$

where**,** CN is the cardinality number of the set which contains all the subsystems that have been affected by the catastrophic forces, which have been activated during Δt. It is a qualitative tool that shows how vulnerable the system is. In addition, it urges the system administrator to activate countermeasures in order to avoid unpropitious situations. Generally, all the previous analysis is applicable, in case of real systems. As it has already been analyzed, an important step for the avoidance of a system collapsing, is the localization of its disadvantages, where usually external factors act on them and lead to a total destruction of the Information System, or to its defoliated mutation. Of course, the whole study does not aim at the searching of the best or good enough solutions, which are applicable for the entire operation of the System, but our problem is concentrated on a **programming timetable.**

Moreover, it is necessary to have the ability to locate not only vulnerabilities but also it is essential to point out how successful our Recovery Plan is. It is important to be able to ensure the success of a *Recovery Plan*. A helpful tool that will aid the measuring how successful the Recovery Plan is, is the **Final State Recovery index (FSR)**. Suppose that:

$$\mathbf{FSR=\ min}\ \left|P(S)-R(S)\right|$$

In the Primitive Situation (PS), before the destruction of the system, the Information System operates normally. All the subsystems of the system are well functional. Assume that **N is the number of the subsystems before the destruction**. If the Destruction takes place during the operation of the system, then there are three discrete cases:

**1. Total Recovery Situation**

$$\mathbf{FSR = min}\ \left|N-N\right| \mathbf{= 0}$$

This is the **identical situation,** which has achieved the total **Restoration of the System.** In other words, N subsystems are still operating after the Destruction. We have achieved to overcome the danger without any cost.

**2. Disaster Situation**

$$\mathbf{FSR = min}\ \left|N-0\right|$$

This is the worst-case scenario **where zero subsystems are working.** The Disaster has been materialized. Our Recovery has failed and the cost of Restoration is enormous, in many cases.

**3. Partial Recovery Situation**

$$\textbf{FSR = min } \left| \mathcal{N} - \mathcal{N}1 \right| < \left| N \right|$$

This is a medium situation, which-in many cases-depends **on the cost of the partial recovery**. It is not the best scenario but not even the worst. It is a situation where our system has been mutated into a new one, a new System with fewer capabilities and less operations. Moreover, if the system has been mutated, then it malfunctions with N1<N active subsystems **that are able to operate the whole system**. On the other hand, the destroyed subsystems **are not necessary for the normal operation of the System.**

Finally, it is easily understood that by the Final State Recovery index we only mentioned the mutation, the destruction or the operational normality of the examined Information System. Of course, there is no way to define the new mutated System with the use of the FSR.

It is obvious that the common points of the subsystems can be finally affected by even only one destructive condition inside our programming timetable. This destructive force will guide us to total destructions. The common elements of our examined system must be split to more subsystems. Our main target must be the minimizing of their common elements. Of course, the mutation of the information system, to avoid a collapse, is simply a temporary solution just only for gaining Recovery time. The way that the last one can be defined, is at the moment something that is under consideration by the authors of the present work, in their research field.

There is a basic consideration that makes every software engineer wonder if he has designed the perfect Information System. The answer to this consideration is, in the majority of the situations, negative. There is no System on earth that is capable of avoiding any type of destruction especially in cases where it operates in a Hostile environment. The catastrophic forces are constantly transformed and new types and combinations are being created. Nobody likes to spend money. In order to avoid any unpleasant and in most cases-sudden destructive situations, the system administrators must always be alert. In order to minimize the danger, the system must be prepared to face any kind of danger. System administrators must be able to map their system and know exactly its weaknesses. The administrator determines the catastrophic forces that are possible to be materialized in the subsystems of the examined System.

Simulation is a well-known and in the plethora of the cases successful method to improve the system. It is also a useful technique for the System to self-improve and eliminate its vulnerabilities (Wood 1996).

The possible catastrophe scenarios are created by the states in which the system can be. There are three primitive phases in which a subsystem can be. They are described above with the aid of the three propositions. A further analysis of the system states shows that there are more than those three phases. The Mutation phase hides a plethora of states that each subsystem faces.

## 4.Crisis Scenarios Algorithm

The dangers were presented in the previous paragraph reveal the hostility of the operational environment. It is such an unstable data space that a digitalized information system acts. Moreover, that instability is a critical factor for the normal operation of the system. Nowadays, it is necessary to create a mathematical model in order to avoid any critical situations that take place in an IDDS. The purpose of the model is not to calculate the irregularity but to predict its materialization and to eliminate its destructive consequences.

A well organized analysis is necessary so as the methodology is able to predict a crisis during a given period $\Delta t$. In the majority of the situations this is not a unique factor for success. A very important successful aspect is also, the ability that the methodology must have to be adaptive to any type of irregular dynamic data spaces and under any circumstances.

Moreover, all the above can be achieved under a continuous and complete simulation of the methodology. A set of scenarios must be executed and their results must be analyzed so as the system confronts any possible threat with success. Its vulnerabilities would be minimized if the crisis scenarios are accurate. Suppose that S is the System that operates in an IDDS and K its Danger Map as they have been analyzed in the previous paragraph. In addition, Scenario Vectors (SV) are created. Each Scenario

Vector contains the set of destructive forces that are materialized on the System within Δt. It is obvious that every possible scenario has its own Scenario Vector. So, the Scenario Vector is defined as follows:

$SV = (sv_j), \quad \forall j \in C^*$

$sv_j = 1$, *if the* $c_j$ *catastrophe is activated,*

$= 0$, *otherwise*.

The steps for a successful simulation of the crisis scenarios are presented as follows:

**Crisis Scenarios Algorithm** *(CSA)*

*Step1: Create Scenario Vector SV.*
*Step2: Find the corresponding Catastrophe Vector CV.*
*Step3: Calculate the Catastrophe Index of the System S from CV.*
*Step4: Check Catastrophe Index according to Mutation or Total Destruction or Operation Tranquility Proposition.*
*Step5: Repeat the above set of steps until a suitable number of different possible scenarios is created.*
*Step6: Represent graphically the extracted set of indexes.*

The catastrophe Vector CV is defined as:

$CV = (cv_i), \ \forall i \in S^*$

$cv_i = 1$ *if* $\exists j \in C^* : sv_j * k_{ij} = 1$,

$= 0$, *otherwise.*

The third important step is to calculate the *Catastrophe Index* of the System S as it has been calculated in (4). If CI =1, then the set of catastrophic forces that are presented in vector CV is a destructive combination, a lethal combination that will lead the System S to its total collapse. If CI=0, then the set of catastrophic forces that are presented in vector CV are not dangerous for the System S.

Finally if CI<1, then the set of catastrophic forces *might lead our system to a Gray Operational Hole*, where a part of the structure of the System will be mutated and transformed to a new undefined form.

By changing the original scenario vector SV, a new scenario will be executed and new results will be analyzed. The administrator of the system will be able to concentrate on the results and after an appropriate examination; he will be able to extract very useful qualitative conclusions.

## 5.An easy to follow example

The above general algorithm can be applied to any type of systems without constraints on the number of the catastrophic forces and the cardinality of the set of the subsystems. The complexity of calculating all possible scenarios in a system of m subsystems and n catastrophes is defined as the sum of all possible combinations:

$$\text{Scenarios\_Set} = \sum_{j=1}^{n} \frac{(j+1)(j+2)\ldots(n-1)n}{(n-j)!}$$

The following example demonstrates the Crisis Scenario Algorithm. Suppose a hypothetical system S with five subsystems and a set of eight possible catastrophes, the impact of which is presented in the Danger Map K below:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **1** | **0** | **0** | **1** | **0** | **0** | **0** | **1** |
| **1** | **0** | **1** | **0** | **0** | **0** | **0** | **0** |
| **0** | **1** | **0** | **0** | **0** | **1** | **0** | **0** |
| **1** | **0** | **0** | **1** | **1** | **0** | **0** | **1** |
| **0** | **0** | **1** | **0** | **1** | **0** | **1** | **0** |

Furthermore, random sets of Scenarios Vectors are created. 82 possible scenarios vectors for the execution of the CSA have been used, i.e a sample of which is presented below:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **0** | **0** | **0** | **0** | **0** | **0** | **1** | **1** |
| **0** | **0** | **1** | **0** | **1** | **1** | **0** | **1** |
| **0** | **1** | **1** | **0** | **1** | **1** | **1** | **0** |
| **0** | **1** | **0** | **1** | **0** | **0** | **0** | **1** |
| **1** | **0** | **0** | **0** | **1** | **0** | **0** | **0** |
| **0** | **0** | **1** | **1** | **1** | **0** | **0** | **1** |
| **0** | **0** | **1** | **0** | **0** | **1** | **0** | **0** |
| **0** | **0** | **0** | **0** | **1** | **0** | **1** | **1** |
| **0** | **0** | **1** | **1** | **0** | **1** | **1** | **0** |
| **1** | **1** | **1** | **0** | **0** | **1** | **0** | **1** |
| **0** | **0** | **1** | **0** | **0** | **1** | **0** | **1** |
| **1** | **1** | **0** | **1** | **0** | **0** | **0** | **0** |
| **1** | **1** | **1** | **0** | **1** | **0** | **0** | **1** |

In the first line of the above set of scenarios vectors two catastrophes are applied. The catastrophe vector for this scenario is:

CV= (1 0 0 1 1)

Moreover the Catastrophe Index CI for this vector is CI= 3/5=0.6. According to the proposition (2) analyzed in previous paragraphs, the system is in a Gray Area of Operation. It is mutated to a new type of system with inadequate functionality. The full 82 scenarios results are graphically presented in the plot-diagram below where CI are points which are joined through linear interpolation on the plot.

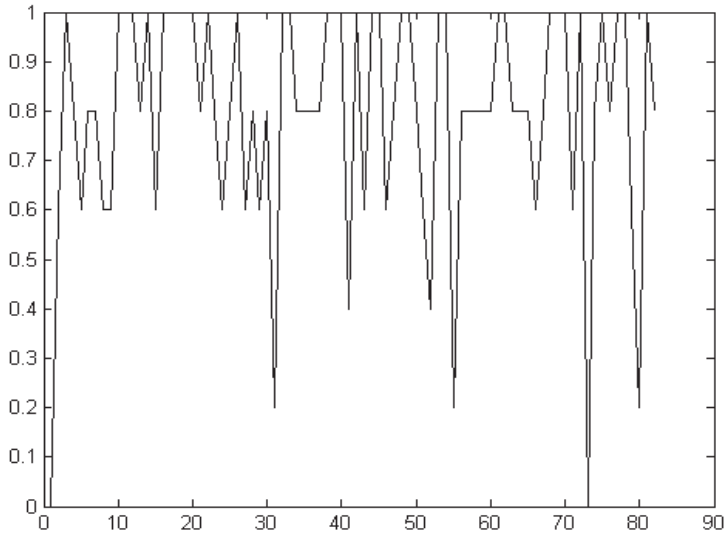



***Image 1****.Catastrophe Index*

More statistical conclusions are extracted with the aid of the following histogram of CI.

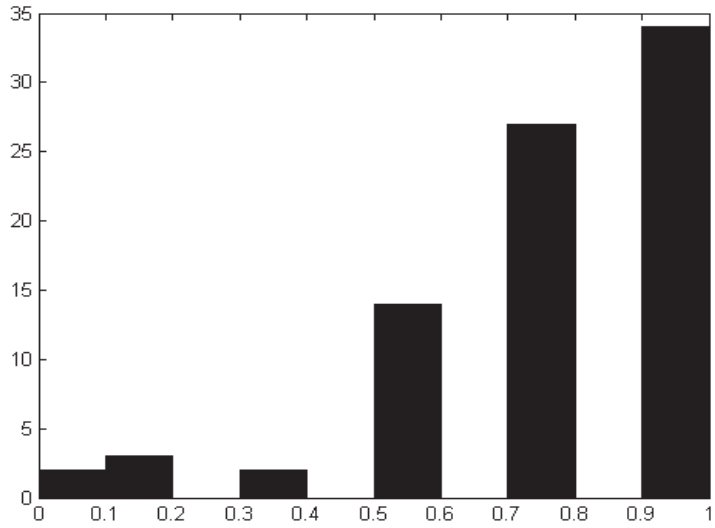



***Image 2**.Catastrophe Index Statistical Results*

The majority of the scenarios are extremely dangerous (34/82) for the System S. Almost 40% of the scenarios result in the total Destruction of the system. In addition, (28/82) approximately 34% of the scenarios lead the system to a Gray Area of Operation where 80% of the functionality of the system collapse. 20% of the scenarios (17/82) mutated the system to a possible medium functionality level. Finally, only 3/82, just 3.5% of the scenarios leave the system untouched or cause minor damages to its subsystem. The fact that almost 75% of the scenarios lead to the system collapsing or to a highest degree of mutation proves that the system is extremely vulnerable to the examined catastrophes.

Through this qualitative approach, the system Administrator is aware of the dangers that threat its system. He is able to use the appropriate countermeasures or even the most useful recoveries if the catastrophe has been materialized. The qualitative approach (as presented previously) can easily be transformed to a quantitative approach with the aid of weights (profits) for every subsystem.

## 6.Conclusions

The model that is presented is part of an *applied multi-crisis methodology***.** The methodology developed in this paper has one main area of focus which is the prediction and confrontation of the threats that appear in digitalized information systems, operating in IDDS.

We can see from the algorithm, a larger problem might lead to a significant number of scenarios and the value of $\Delta t$ would determine the amount of analysis. Values of $\Delta t$ might not lead to an increased computational effort before a crisis is materialized. However, the management of post-crisis situations, along with mutation characteristics might lead to a need for a significant increase in the computational effort. A set of small adjustments can easily transform the presented qualitative simulation model to a quantitative one with the aid of well known mathematical programming techniques.

The authors of the paper are concentrated on the study of the above conclusions, in order to improve the extracting results of the presented methodology. In addition, the authors are experimenting with new ways of simulating crisis in irregular dynamic data spaces so as to extract new, very important conclusions. Protecting a system is a daily fight against time and the unexpected. Any mistake costs a lot of money and (in the worst case) the total destruction of the organization.The Information System that it incorporates has disadvantages and imperfections that need protection. Digitalized Business Systems act in a hostile and dynamically changed environment. Its Information System has to observe those changes, so as to avoid cases of destruction. Their system Administrator must be ready to avoid any destruction by simulating all the possible worst case scenarios. The Catastrophe Scenario Algorithm is a useful tool for the prediction of irreversible situations.

Future work-study will be an effort to calculate of the objective risk of the Business System collapsing. We will also try to expand the ways of locating the central points of the self-destruction of the system, so as to protect it. Finally, we will present ways of recovering the system, in cases of collapsing or mutation and (most important of all) the application of the proposed methodology.

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# Energy Efficiency and Techno-economic Analyses of the Combined Heating System Using Gas-fired Boilers for Peak load Compensation

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**Abstract**

With natural gas penetrating into the heating markets of China, combined heating systems with combined heat and power (CHP) as basic heat source and gas-fired boilers for peak load compensation are starting to appear. However, due to their very different properties, coal and gas need to be evaluated from energetic and techno-economic aspects before engineering applications. Accordingly, the theory of energy quality factor (EQF) and dynamic techno-economics are presented to evaluate a combined heating system located in Yinchuan city of China. Firstly, we discuss the total energy and exergy consumption. Subsequently, we conduct a techno-economic analysis considering initial capital investment, operation and maintenance (O&M) expenses, etc. The results show that saving ratios for total energy and exergy increase with the decreasing basic heat load ratio $\beta$. Furthermore, the optimal basic heat load ratio for this combined heating system should be 0.80 from the perspective of economic performance. Finally, we also analyze the structure of annual costs of the combined heating system with different values for $\beta$. Fuel cost is the dominant factor that influences the combined heating system; it changes from 59.9% to 37.8% when $\beta$ varies from 0.50 to 1.00. However, for the second most important factor, initial capital investment, the trend is just reversed.

**KEYWORDS**

District Heating, Combined Heat and Power, Efficiency, Economic, Gas-fired Boiler, Peak load

## 1. INTRODUCTION

The district heating area in China has grown rapidly in recent years and the energy consumption in heating systems is always in a large-scale size, which causes aggravated environmental pollution. Meanwhile, the ultimate way to improve the atmospheric environment is to change the energy structure (Helgason et al., 1988; Marbe et al., 2006; Tromborg et al., 2007). Therefore, with the background of energy structure regulation in China (Zhou, 2008) and on the premise of coal-fired combined heat and power (CHP) plant as dominant heat sources, we propose application of gas-fired boilers in underperforming heating substations as peak load heat sources for effective adaptation to the regulation demands of heat load fluctuations and improvement of the energy efficiency.

CHP production is an increasingly important technology for clean provision of energy because of its higher efficiency (Rong and Lahdelma, 2005). On the other hand, gas is becoming more and more common in the market of heating systems. Barelli et al. (2006) modulated the heat load with auxiliary boilers fueled by natural gas, which resulted in savings in primary energy. Brkić and Tanasković (2008) carried out a systematic study of natural gas usage for domestic heating in urban areas including direct use with individual gas-fired boilers and indirect combustion through district heating systems.

In China, combined heating systems with coal-fired CHP as basic heat sources and gas-fired boilers as peak load heat sources (Liu, 2008;Wang et al., 2010) are starting to appear and may alleviate the huge economic differences between coal and gas as well as reduce primary energy consumption. Jiang found that residential heating energy consumption can be reduced by 20%~30% with the introduction of combined heating systems (Jiang, 2005). However, the cost of gas heating is normally about 2~3 times larger than that of coal. Besides, to what extent a combined heating system can save energy and how to evaluate this energy saving capability more reasonably and objectively still needs to be analyzed. Therefore, it is suggested that the energy efficiency and techno-economic analyses of the whole combined heating system be carried out so as to identify the optimal basic heat load ratio that will lead to an acceptable economic performance.

## 2. COMBINED HEATING SYSTEMS WITH GAS-FIRED BOILERS FOR PEAK LOAD COMPENSATION

The connecting mode of gas-fired boilers and heating network of a typical combined heating system are indicated in Figures 1 and 2.

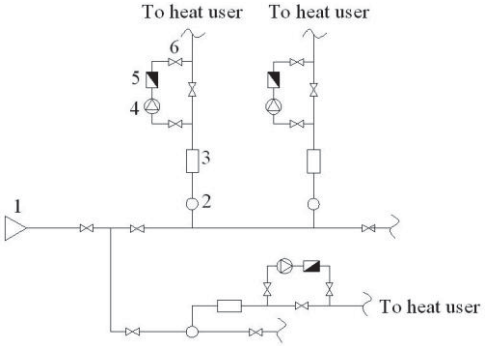


Figure 1. Connecting mode of gas-fired boilers and heating network. 1—CHP; 2—coal-fired boiler (if any); 3—heating substation; 4—peak heating circulating pump; 5—gas-fired boiler; 6—valve.

The CHP plant supplies basic heat load for the combined heating system. However, if the outdoor temperature declines to a certain extent, namely, it is lower than critical peak heating temperature, and then gas-fired boilers will be set into service.

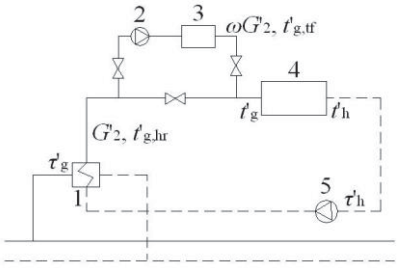


Figure 2. Connecting mode of peak load gas-fired boilers and secondary heating network. 1—heat exchanger; 2—peak heating circulating pump; 3—gas-fired boiler; 4—heat user; 5—secondary circulating pump.

In Figure 2, $\tau'_g$ and $\tau'_h$ are the design supply and return water temperatures of the primary heating network; $t'_g$ and $t'_h$ are the design supply and return water temperatures of heat user; $t'_{g,hr}$ is the design supply water temperature of heat exchanger; $t'_{w,tf}$ is the design supply water temperature of peak load heat source; $G'_2$ is the flow rate of the secondary heating network and $\omega$ is the peak load heating flow ratio. In a certain heating substation, return water from secondary heating network is first heated by the heat exchanger, and then a portion of the flow is sent to the gas-fired boiler to absorb heat again; the other part of the return water flows through a bypass pipe and is finally combined with the reheated water flow. Then, the mixed water is sent to heat users if its temperature also satisfies operation requirements.

Of great importance to combined heating systems is the heat load duration curve, which can be derived form the non-dimensional comprehensive equations. The heat load duration curve of a typical combined heating system with a CHP as basic heat source is shown in Figure 3.

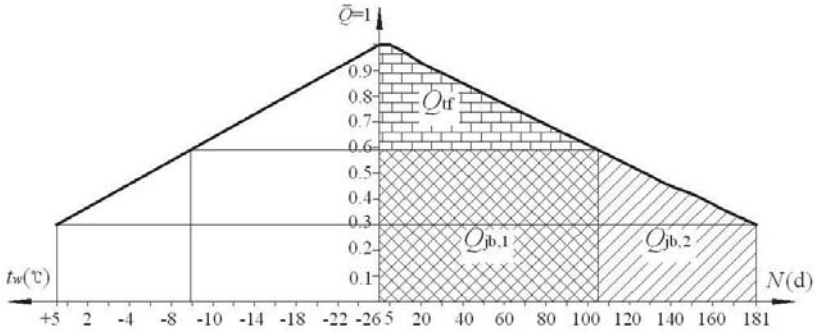


Figure 3. Heating load duration curve of combined heating system.

As can be seen in Figure 3, $t_w$ is the outdoor temperature; $N$ is the heating duration days and $\bar{Q}$ is the ratio of actual and design heat load. $Q_{jb,1}$ and $Q_{jb,2}$ are the full load and partial load heat supply of the CHP plant; $Q_{tf}$ is the heat supply of gas-fired boilers. Assume the design heat load of a combined heating system is $Q'$, then, the basic heat load ratio $\beta$ can be written as:

$$\beta=Q_{jb}/Q' \quad (1)$$

where $Q_{jb}$ means the actual heat load of basic heat source under a certain outdoor temperature. With the premise of coal as basic heat source, we assume that the value of $\beta$ varies from 0.5 to 1.0 here.

To conclude, combined heating systems are preferred to traditional central heating systems, because they can 1) regulate heat supplies in time and avoid excessive heating; 2) optimize hydraulic conditions; 3) extend heating load by peak load gas-fired boilers; 4) enhance the reliability of heating systems; 5) prolong the running time periods of heat sources under high energy efficiencies; and 6) alleviate environmental impacts.

# 3. METHODOLOGY

## 3.1 Energy Efficiency Model

The conventional statistical method only converts the thermal value of different kind of energies into the standard coal equivalent. But the second law of thermodynamics says that energy conversion accompanies entropy generation and exergy loss. Therefore, the evaluation of exergy utilization can be particularly useful in reflecting energy utilization and environmental impact (Cengel and Boles, 2002; Dincer et al., 2004). In general, energy utilization evaluation should take into account energy quantity and quality. Therefore, it is suggested that electric power should be the measurement standard for energy. On this basis, the ratio of the power transfer capability to the total energy is referred to as the energy quality factor (EQF) of a type of energy and is expressed as

$$\lambda=W/Q \quad (2)$$

where $W$ is the fraction of total energy that can be converted into power and $Q$ is the total energy.

The chemical energy of conventional energy resources, such as fossil fuels, is generally converted into thermal energy by direct combustion, and power is then derived in a thermodynamic cycle. Therefore, EQF can be expressed as (Yang, 1986):

$$\lambda = 1 - \frac{T_0}{T_{burn} - T_0} \ln \frac{T_{burn}}{T_0} \quad (3)$$

where $T_0$ is the environmental benchmark temperature and $T_{burn}$ is the theoretical combustion temperature of the fossil fuel.

Traditional energy utilization analysis normally uses total energy consumption to evaluate the performances of different heating scenarios; however, total energy consumption does not reflect the essence

of energy utilization, and thus, the evaluation is uncertain (Chang, 2006). Therefore, we propose exergy analysis of a heating system based on EQF to evaluate energy utilization more completely in terms of energy quantity and quality. The model adopted for calculating the total energy consumed by combined heating system is as follows:

$$Q_t = \sum_{i=1}^{n}\sum_{j=1}^{k}\left( m_j Q_{d,j}^{y} + \frac{3.6\times10^{-3} e_i}{\eta_{coal}} \right) \tag{4}$$

where $Q_t$ is the total energy of the combined heating system; $m_j$ stands for the consumption amount of $j$th kind of energy; $Q^y_{d,j}$ is the lower heat value of $j$th kind of energy; $e_i$ is the electricity consumption of $i$th electric equipment; $\eta_{coal}$ is the power generation efficiency of coal.

The exergy consumption of a combined heating system is expressed as

$$E = \sum_{i=1}^{n}\sum_{j=1}^{k}\left( m_j Q_{d,j}^{y} \lambda_j + 3.6\times10^{-3} e_i \right) \tag{5}$$

where $E$ is the exergy of the combined heating system; $\lambda_j$ is the EQF of $j$th kind of energy.

## 3.2 Techno-economic analysis Model

Dynamic techno-economics is adopted for evaluating a combined heating system from the perspective of economic performance. The annual cost of the whole heating system basically consists of two parts, including the initial investment of facilities and annual operating cost, as follows:

$$A_c = \left[ \frac{i(1+i)^n}{(1+i)^n - 1} \right] C_{\text{ini}} + C_{\text{ope}} \tag{6}$$

where $A_c$ is the annual cost; $C_{\text{ini}}$ is the initial investment, $C_{\text{ope}}$ is the annual operating cost, $i$ is the internal rate of return (IRR), and $n$ is the construction period.

### 3.2.1 Initial Capital Investment

The initial capital investment mainly includes three parts:

$$C_{\text{ini}} = C_{\text{ini+source}} + C_{\text{ini+net}} + C_z \tag{7}$$

where $C_{\text{ini+source}}$ is the initial cost of heat-sources; $C_{\text{ini+net}}$ is the initial cost of heating network; $C_z$ is the extra fees for water and gas supply increments. The initial cost of heat-sources contains the coal-fired CHP plant and gas-fired boilers; the former one can be calculated according to the amount and capacities of the CHP plants. Furthermore, the initial cost of piping arrangements of the whole heating network was calculated based on an estimate guideline (PRC Ministry of Construction, 2007), which presents the initial cost per kilometer of the piping arrangement with a nominal diameter from 100 mm to 1200 mm. Therefore

$$C_{\text{ini+net}} = \sum_{i=1}^{N} f(d_j) l_j \tag{8}$$

where $N$ is the number of pipelines; $f(d_j)$ is the capital cost per meter of pipeline with nominal diameter $d_j$; $l_j$ is the length of $j$th pipeline. The extra fees for water and gas supply increments were estimated according to some typical practical projects.

### 3.2.2 Annual Operation and Maintenance Cost

The annual operating cost of the boiler plants (coal- and gas-fired) can be expressed as

$$C_{\text{ope}} = C_{\text{fuel}} + C_{\text{electricity}} + C_{\text{water}} + C_{\text{servicing}} \tag{9}$$

where $C_{\text{fuel}}$, $C_{\text{electricity}}$ and $C_{\text{water}}$ is the fuel cost, electricity cost and water cost, respectively; $C_{\text{servicing}}$ is the maintenance and depreciation cost. In addition, salaries and welfare costs of employees were also considered in $C_{\text{servicing}}$.

Fuel cost includes coal and gas expenses, which are calculated in accordance to the basic heat load ratio $\beta$. We consider different ways of regulation for the heating network before electricity cost calculation. The electricity cost of fans is similar to that of the pumps. Moreover, some other equipment's electricity costs are also taken into account. The supply water supplementary rate of a closed heating system should be less than 1% (PRC Ministry of Construction, 2002), which is used to calculate the water cost. The maintenance and depreciation cost calculation is based on the following equation:

$$C_{\text{servicing}} = \alpha_1 C_{\text{ini+source}} + \alpha_2 C_{\text{ini+net}} \quad (10)$$

where $\alpha_1$, $\alpha_2$ are the depreciation rates of heat-sources and heating network, including the maintenance cost rate.

# 4. ENERGY EFFICIENCY AND TECHNO-ECONOMIC ANALYSES

## 4.1 Combined heating system in Yinchuan city, China

A newly-built combined heating system located in Yinchuan city of China is introduced to carry out the corresponding analyses. This hybrid heating system has 19 heating substations with a CHP plant as basic heat source and gas-fired boilers installed in some underperformed heating substations. Yinchuan has mild winters and belongs to continental zone in terms of climate. Some relevant design parameters are listed in Table 1.

Table 1. Design parameters of Yinchuan combined heating system (2007)

| Item | Value |
|---|---|
| Heating area | 3342300m$^2$ |
| Heating load | 182.89MW |
| square thermal index | 54.72w/m$^2$ |
| Specific fractional resistance of main pipelines | 30~70Pa/m |
| Heat production | 1562880GJ/a |
| Design indoor temperature | 18°C |
| Design outdoor temperature | －15 °C |
| Design supply and return water temperature | 130/80 °C |
| Heating period | 149d |

Besides, Yinchuan city is rich in coal resources and it lies on the main pipeline of West-East natural gas transmission project of China. Therefore, the prices of hydrocarbon resources in Yinchuan are relatively low. The main economic indices of this region are listed in Table 2.

Table 2. Main techno-economic indices of this region (2007)

| Item | Value |
|---|---|
| Coal price | 150Yuan /t |
| Low heating value of coal | 20.934MJ/kg |
| Heating value of standard coal | 29.306MJ/kg |
| Efficiency of coal-fired boiler | 76% |
| Efficiency of coal-fired boiler with partial load | 65% |
| Water price | 5Yuan/t |
| Gas price | 1.4Yuan /Nm$^3$ |
| Low heating value of gas | 35.587MJ/m$^3$ |
| Efficiency of gas-fired boiler | 88% |
| Construction period | 12a |
| Interest rate (Long-term loan) | 7.56% |
| Electricity price | 0.5Yuan /kWh |

## 4.2 Energy Efficiency Analysis Results

The energy consumption of Yinchuan combined heating system varies with different $\beta$, and can be calculated based on Eqs. (5) and (6). The results are shown in Table 3.

Table 3. Energy consumption of combined heating system ($10^3$GJ)

| $\beta$ | 0.50 | 0.55 | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 | 1.00 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Total energy | 2073 | 2120 | 2170 | 2216 | 2264 | 2310 | 2355 | 2395 | 2434 | 2463 | 2487 |
| Exergy | 1035 | 1040 | 1048 | 1056 | 1067 | 1079 | 1092 | 1105 | 1119 | 1130 | 1139 |

Table 3 clearly shows that total energy and exergy of Yinchuan combined heating system uniformly decline with the decreasing $\beta$. Although gas is superior to coal in energy quality, it doesn't mean that the more heat load gas supplies, the better a combined heating system is. Therefore, techno-economics should be taken in to account to evaluate and design a combined heating system, so as to make it economically reasonable. Besides, the energy saving ratios compared with traditional district heating systems of different $\beta$ are also illustrated in Figure 4.

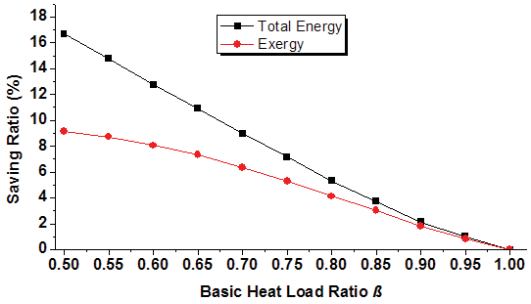


Figure 4. Saving ratios of total energy, exergy of the combined heating system with different $\beta$

It can be indicated from Figure 4 that all of the three energy saving ratios increase with the decreasing $\beta$. Total energy saving ratio is maximal among them, while exergy saving ratio is the least.

## 4.3 Techno-economic Analysis Results

According to the dynamic techno-economics, we calculate the annual cost with different $\beta$ to evaluate the combined heating system in terms of economic performance and to determine the optimal value of $\beta$. Table 4 and Figure 5 show the detailed calculation results.

Table 4. Techno-economic analysis results of Yinchuan combined heating system ($10^4$Yuan/a)

| $\beta$ | $C_{ope}$ | $C_{ini}$ | $A_c$ |
|---|---|---|---|
| 0.5 | 3757.73 | 7915.33 | 4784.35 |
| 0.55 | 3535.57 | 8219.41 | 4601.62 |
| 0.6 | 3358.82 | 8459.14 | 4455.97 |
| 0.65 | 3213.48 | 8770.85 | 4351.06 |
| 0.7 | 3107.95 | 8940.38 | 4267.52 |
| 0.75 | 3036.29 | 9169.96 | 4225.64 |
| 0.8 | 2999.52 | 9354.76 | 4212.83 |
| 0.85 | 2994.65 | 9789.52 | 4264.35 |
| 0.9 | 3022.09 | 10148.93 | 4338.4 |
| 0.95 | 3051.95 | 10646.52 | 4432.81 |
| 1 | 3103.84 | 10954.02 | 4524.57 |

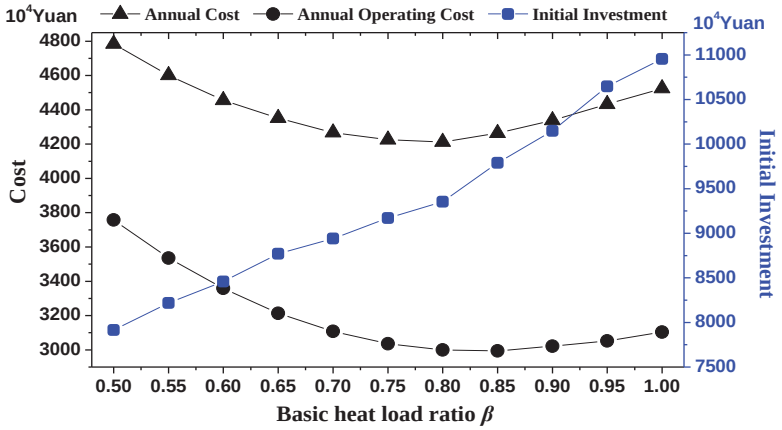


Figure 5. Economic performance of the combined heating system with different $\beta$.

Figure 6 also illustrates the annual cost when $\beta$ is changed from 0.50 to 1.00. As can be seen, the annual cost reaches its minimum level at $\beta$=0.80. In addition, the operation cost reaches the minimum when $\beta$ is 0.85; this is because as long as $\beta$ is decreasing, the relevant cost of primary heating network reduces. Namely, the increased cost caused by gas peak heating can be counterbalanced by the above mentioned investment reduction. However, when $\beta$ continues to decline, the operation cost of gas-fired boilers will ascend dramatically even to the extent that the combined heating scenario cannot be afforded. Furthermore, the total energy and exergy consumption saving ratios when $\beta$ is 0.8 are 5.31% and 4.12%, respectively.

Besides, we also analyze the structure of annual costs of the combined heating system with different $\beta$. Figure 6 shows the cost proportion of fuel, electricity, supplementary water, depreciation and maintenance as well as initial investment when $\beta$ is changed from 0.50 to 1.00.

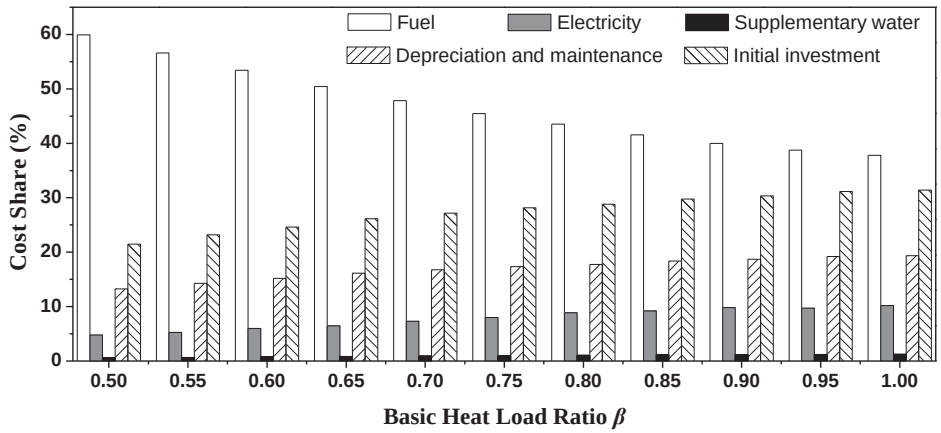


Figure 6. Cost proportion of Yinchuan combined heating system with different $\beta$

It can be seen form Figure 6 that fuel cost is the dominant factor that influences a combined heating system; it changes from 59.9% to 37.8% when $\beta$ varies from 0.50 to 1.00. Fuel cost of the combined heating system declines steadily with the increasing $\beta$. But as the second most important factor, initial investment's development trend is just reversed. In addition, supplementary water and depreciation and maintenance cost rise up moderately along with increasing $\beta$. Moreover, electricity cost fluctuates slightly with $\beta$.

## 5. CONCLUSION

Combined heating systems are regarded as the development trend of central heating in the near future. But there still exist some problems before their applications due to the energy quality and economic differences of coal and gas. Therefore, this paper evaluate the energy efficiency and economic performance of a combined heating system in Yinchuan city according to the theory of EQF and dynamic techno-economics so as to develop an evaluation framework for combined heating system. The results suggest that total energy and exergy of Yinchuan combined heating system uniformly decline with the decreasing $\beta$, and their energy saving ratios increase with the decreasing $\beta$. Furthermore, total energy saving ratio is maximal among them, while exergy saving ratio is the least. In addition, the system's economic performance is relatively good when $\beta$ is 0.80. Finally, it is found that fuel cost of the combined heating system declines steadily with the increasing $\beta$, but the development trend of initial investment is just reversed. Supplementary water and depreciation and maintenance cost rise up moderately along with increasing $\beta$, moreover, electricity cost fluctuates slightly with $\beta$.

## ACKNOWLEDGEMENT

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# OPTIMIZATION

# Optimization of student personal course schedules with Evolutionary Algorithms

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**Abstract**

In this paper we present the optimization part of a course registration system through the web (EvoWebReg). The system consists of three parts. The first one is a web application which allows the students to submit their course preferences to the system's database, through Internet. The second part is an administrative tool which controls the whole system and allows its smooth operation. The third part is an evolutionary algorithm which is responsible for the optimization of the student course schedules. The schedules are optimized according to the student's submitted preferences, and the constraints imposed by the department. The results of the experimental tests of the evolutionary algorithm prove that our initial objectives to provide an open generic and effective tool, which can satisfactory implement the course registration procedure, were achieved.

**KEYWORDS**

Optimization, Evolutionary Algorithms, Web course registration, educational timetabling.

## 1. INTRODUCTION

According to Carter and Laporte (1998) the course scheduling problem can be decomposed into the following sub-problems: course timetabling (assign courses or course sections to time periods satisfying some requirements and/or constraints), teacher assignment (assign teachers to courses maximizing a preference function), classroom assignment (events must be assigned to specific rooms to satisfy some criteria), and student scheduling (arises when courses are taught in multiple sections). The course timetabling systems are also distinguished as *Master Timetables* or *Demand Driven*, with their primary difference lying in the sequence in which the various sub-problems are being solved.

In this paper we consider the student scheduling problem with the Master Timetable approach, adapted to the requirements and characteristics of the Dept. of Informatics of ATEITh. Although we discuss a specific case study, most methods are general enough and should be applicable to modeling and solving other instances of the problem. In our case study, a semester is the academic time unit. Course timetables and student sectioning must be done in the beginning of every semester.

A course can be either *simple* or *mixed*. A course is considered *simple* when it is taught only by lectures in big enough lecture rooms, and *mixed* when it is taught by lectures (theoretical part) and laboratory exercises (lab part). When a course is mixed, in order for a student to pass the course he/she must enrol in both parts of the course and succeed a passing mark in both parts. A simple course or the theoretical part of a mixed course is taught by lectures in rooms that are large enough to accommodate all enrolled students. It's very rare to have two sections of a simple course. On the other hand, all laboratory rooms have limited capacity which is usually much smaller than the number of students registered in the theoretical part of the course. So the students of most mixed courses must be separated into lab groups (course sections for the lab part). The available number of lab groups for each mixed course is decided before the timetable creation, according to the number of students who are anticipated to enrol. Each lab group is assigned the following data: lab room, day of the week and time, teachers (1 or 2). The course timetable which is constructed and announced at the beginning of each semester contains the above data.

Every student prepares his own personal schedule selecting not only courses, but lab groups for the mixed courses as well, according to the already announced timetable. Every student's schedule must satisfy a number of rules and constraints set by the department and by the available facilities (room number and sizes), and also by the student's own preferences, concerning days, time and staff members.

The program of study of the department includes a limited number of lab rooms with specific places. The places are equivalent. The lab groups are considered as collections of places. Each place must be assigned to an appropriate student.

## 2. PROBLEM DESCRIPTION – CASE STUDY

As already mentioned, in this paper we consider the student scheduling problem with the Master Timetable approach, adapted to the requirements and characteristics of the Dept. of Informatics of ATEITh.

In our case study, a semester is the academic time unit. Course timetables and student sectioning must be done in the beginning of every semester. Studies last for 8 semesters.

A course can be either *simple* or *mixed*. A course is considered *simple* when it is taught only by lectures in big enough lecture rooms, and *mixed* when it is taught by lectures (theoretical part) and laboratory exercises (lab part). When a course is mixed, in order for a student to pass the course he/she must enrol in both parts of the course and succeed a passing mark in both parts. A simple course or the theoretical part of a mixed course is taught by lectures in rooms that are large enough to accommodate all enrolled students. It's very rare to have two sections of a simple course. On the other hand, all laboratory rooms have limited capacity which is usually much smaller than the number of students registered in the theoretical part of the course. So the students of most mixed courses must be separated into lab groups (course sections for the lab part). The available number of lab groups for each mixed course is decided before the timetable creation, according to the number of students who are anticipated to enrol, judging from the previous semester. Each lab group is assigned the following data: lab room, day of the week and time, teachers (1 or 2). Although not necessary, usually lab groups for the same course are assigned on different days and times.

The course timetable which is constructed and announced at the beginning of each semester contains the above data.

Every student prepares his own personal schedule selecting not only courses, but lab groups for the mixed courses as well, according to the already announced timetable. Every student's schedule must satisfy a number of rules and constraints set by the department and by the available facilities (room number and sizes), and also by the student's own preferences, concerning days, time and staff members.

The program of study of the department includes a limited number of lab rooms with specific places. The places are equivalent. The lab groups are considered as collections of places. Each place must be assigned to an appropriate student.

## 3. COURSE SECTIONING: HISTORICAL PERSPECTIVE

The first attempts to solve the student scheduling problem were trying to produce conflict-free schedules and balance course section sizes. The algorithms proposed by Abell (1965) and Winters (1971), often produced solutions with invalid schedules. Faulkner (1965) introduced the idea of preference instead of only choosing a course.

Macon and Walker (1966) used a "Monte Carlo" algorithm to examine the case of assigning students to a fixed course schedule. Some of the courses were divided into sections and section choices for assignment must be made. Their algorithm was trying to reduce conflicts and balance the load between sections. A more stochastic algorithm was used by Busam (1967). They compared a non-preference and a preference (ordering of sections within the courses) algorithm and they concluded that both algorithms meet the requirements of machine sectioning equally well.

Later the approaches of Colijn (1973) and Miyaji et al. (1981) produced better results. Colijin's algorithm produced conflict-free schedules, while preserving the balance between sections and satisfying some requirements. The approach of Miyaji et al. was based on goal programming, and was tested successfully on realistic examples. Sabin and Winter (1986) proposed the use of two programs using "greedy" algorithms.

Laporte and Desroches (1986) treated the student scheduling problem as an optimization problem satisfying a number of criteria. Some of the criteria were treated as constraints, while others were considered as objectives. They introduced the use of hard and soft constraints and used a "branch and bound" algorithm divided into the three phases (constructing student schedules, balancing section enrollments, and respecting room capacities), but it did not give students the possibility of section preference.

Aubin and Ferlan (1989) used the "Mater Timetable" approach. First a timetable is generated and then students are assigned to course sections. They used the integer programming model improving the objective by working successively on the timetabling and the grouping sub-problems until a solution that cannot be improved, is reached. This approach was also used by Hertzh (1991) with tabu search techniques to handle sub-problems. Her innovation was to accept as valid, schedules in which courses shared common teachers or students. Tabu search was also used by Alvarez-Valdes et al. (2000) in a two step student registration system. Initially students select courses but not sections, according to their priority (depending on their record and seniority).

Carter (2001) describes a system based on a "demand-driven" approach where students first choose their courses, and the system tries to find the best timetable to maximize the number of satisfied requests. The problem is decomposed into sub-problems which are solved using a greedy heuristic to assign times to sections, and a Lagrancian relaxation algorithm to assign classrooms.

Graph theory was used by Burke et al. (2004) given that for each course there was only one lecture per week. The problem was modeled as an edge-coloring problem in a bipartite graph with students represented as left nodes and teachers by right nodes. Course sections (collections of time periods and resources) are represented by edges between left and right nodes. The approach was taking into consideration the balance of sections, the student preferences and the policy of the institute.

Amintoosi and Haddadnia (2005) proposed a new fuzzy sectioning algorithm. They use fuzzy clustering, a fuzzy evaluator and a novel feature selection method. Students are classified by a Fuzzy c-Means classifier after best feature selection for sectioning. A fuzzy function evaluates the produced clusters based on section balancing and similarity of student schedules within each section.

The most integrated solution was proposed by Murray et al. (2007). They introduced a university wide real time sectioning system for Purdue University where optimization happens at real time. The timetable is created and most students are sectioned based on demand data in student schedule requests, except from the beginning students, since their requests are unknown. Timetable construction considers actual course demand from students who have submitted schedule requests plus the projected demand for students who are anticipated to enroll.

## 4. SCHEDULING SUB-SYSTEM AND EVOLUTIONARY ALGORITHM

The operation of the system starts with the public announcement of the master timetable on the website. The timetable includes the usual necessary information in order for the students to be able to construct their personal schedules.

After the system accepts the student preferences, an evolutionary algorithm undertakes the responsibility of course sectioning (scheduling students to different lab groups).

The evolutionary algorithm produces the final personal schedules for each student. The results apart from the feasibility of the solution will also comply with the departmental policy. The requirements of the system, that concern the scheduling algorithm, are the following:

1. A student is assigned only to one lab group of the same course.
2. A student is assigned only one place of a lab group.
3. A student is assigned only one lab group at the same day and time (timeslot).
4. A student declares a course only when he has passed the prerequisite courses.
5. Industrial placement is declared only when all prerequisite courses are passed.
6. Final year thesis can be assigned only after a specific semester.
7. The number of TU that is assigned to a student is limited, depending on their seniority.
8. In special cases the TU of assigned courses are ignored.
9. No overlap of theoretical part and lab part of a course is allowed for a student, without the consent of the student.

10. Lab preferences are mandatory for all students.
11. The system supports lab rooms with different capacities.
12. The length of lecture times must be an integer number.
13. The system must fill as much as possible the available lab places.
14. Students have priority over the courses taught on their *typical semester* (typical semester is the number of semesters that have elapsed since they first registered in the Department).
15. The system provides the students with the capability to give a preference order of the lab groups for each and every course that they want to enrol (basic preferences).
16. The system gives the capability of declaring redundant courses as well. These will be used in case that, basic preferences cannot be assigned.
17. The system gives bonus to students according to their performance over the previous semester (do not withdraw from lab groups, and have better marks) and over all passed semesters (higher mark average)

The requirements 7, 8, and 9 are checked by the system when the students declare their preferences.

We've chosen to use evolutionary algorithms (EAs) as a search and optimization technique that can effectively search the space of feasible solutions and find the best of them. Moreover the quality function of EAs can easily accommodate new optimization criteria that can emerge in the future.

The evolutionary algorithm (EA) operates and communicates with other parts of the system as follows:

- Step 1. The root island starts operating. It communicates with the database to acquire all necessary data. The data are saved as objects in the main memory of the node that hosts the root island.
- Step 2. The root island caters for the initialization of the other islands. It is responsible for a) the creation of the number of islands defined in the parameters of the EA, and b) the initialization of the inhabitants (individuals) of the new islands with the data acquired from the database.

Requirements 4, 5, 6, and 10 are considered hard constraints and they must be satisfied in order for the solution to be valid. The requirements 16-20 are considered soft and are taken into consideration by the quality function of the EA. Other requirements are implemented in the others parts of EvoWebReg system.

## 4.1 Chromosome representation and Evaluation

Since the lack of sufficient lab rooms and accordingly lab places is the major problem of the department, this problem is central in designing the appropriate representation. A lab place can be assigned to a student or remain empty. The data that distinguish each lab place are: course, lab group of the course, student to be assigned the lab place, and student preference for this lab group. The number of lab groups is limited and defined in the Master timetable. The final chromosome is depicted in Fig. 4. It consists of a series of the lab part of mixed courses.

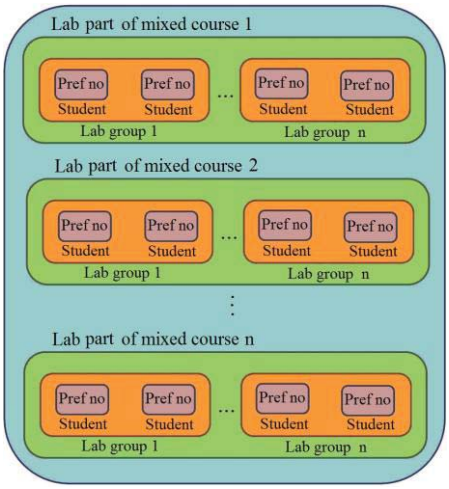


Fig. 4 Chromosome representation

Special care has been given to initialization which is biased towards the first lab group preference of the students.

The quality of an individual equals to the sum of the qualities of every gene:

$$indivFitness = \sum_{i=0}^{N} G_i \quad (1)$$

where $N$ is the number of genes, and $G_i$ the quality of gene i.

We distinguish between two cases for each gene: a) the lab place that corresponds to a gene is assigned to a student and b) the lab place remains empty. In the first case the quality of the gene is:

$$G = penTyp + p * prefNo + m*(11-avg) + d*\left(\frac{regTU}{passTU}\right) + penWD + penRed \quad (2)$$

where $p$, $m$, and $d$ are some weight coefficients, set by the administrator, depending on the policy of the department. For example, if the policy is to reward the students with good performance in the previous semester then the weight of d must increase.

*penTyp,* is a penalty concerned with the typical semester of the student. According to the policy of the department the students have absolute priority over the courses of their typical semester. If places are left then priority is conveyed upon the students of semester "*typical+1*", then if there are still available places priority is conveyed to students with typical semester after the 7th, and then to the rest of the students independent of semester.

*prefNo* is the preference number of the student for this lab group, *avg* is the average mark of all the courses that the student has passed successfully, *regTU* is the number of teaching units (TU) registered on previous semester, *passTU* is the number of TU passed successfully on previous semester, *penWD* is the penalty for a student that withdraws from a lab group in the middle of the semester, and *penRed* is the penalty for the use of a redundant declaration.

In the second case (the lab place remains empty) the quality of the gene is: $G = emptyPlace\ Penalty$

## 4.2 Genetic operators, mechanisms and parameters

We need to design appropriate reproduction operators (recombination & mutation) in order to keep solutions valid. We've implemented four different recombination operators: *synchronous (1), asynchronous (2), independent (3)* and *place-based (4)*. All recombination operators affect a block according to a probability (recombination per block rate). In the first three operators, a block corresponds to a lab group. In the forth, it corresponds to a lab place.

The operators are applied in two phases. In the first phase, these operators recombine characteristics of the two parents, blocking exchanges that violate the validity of the chromosome. In the second phase the empty places are filled with appropriate students, i.e. students that have declared these lab groups in their preferences. The operators differ in the first phase and mainly in the way that they deal with exchanges which will result in invalid chromosomes. When the operators "synchronous" and "place-based" find an invalid exchange, they keep the previous values. The "independent" leaves empty places and "asynchronous" is a combination of the other operators.

We've implemented two mutation operators which are based on the same general idea: replacement of a student in some lab group place with another student that has also shown his preference on this lab group. Of course the mutation operators must check and verify that this replacement does not violate the validity of the chromosome. The first operator makes the student replacement without checking anything else. The second mutation operator does not allow replacements that deteriorate the fitness of the chromosome.

Elitism is implemented as follows: the best parent can replace the worst offspring and not only a randomly selected one. Moreover the new mechanism was extended to support elitism for more than one individuals from the parent population.

The termination criteria of the EA are: maximum number of generations, or best (or goal) solution found, or maximum execution time, or best individual does not improve for a number of generations.

The system also offers the possibility of immigration of individuals with either constant migration probability or an alternative that increases the selection intensity (Cantu-Paz 2001).

# 5. DATA DESCRIPTION, EXPERIMENTS AND RESULTS

The scheduling system was tested on an experimental database created according to the courses offered by the Dept. of Informatics. It includes 51 courses, 14 simple courses and 37 mixed courses. The lab part of the 37 courses is separated into 113 lab groups. Each lab group has at most 25 places. The total number of available lab places in all mixed courses is 2825. The master timetable of the department offers courses in five days, and every day has 12 time periods (from 08:00 to 20:00).

The total number of students is 800 who are distributed in 10 semesters. The number of declared lab group places is 2845 (20 places more than the available). Considering the student preferences, the fact that the demand for places in some lab groups was extremely high and in others extremely low, and that the students did not declared their interest for 281 lab places, the maximum number of lab places that could assigned to students was 2544. Also, in some mixed courses the available places were more than the required (31 lab places would anyway stay empty), and the EA could not assign more than 2796 lab places to students in any case. From the previous semester data, we found that the students had withdraw from 347 lab places in the previous semester.

A factor of success of the students in the previous semester (TU passed against registered TU) was also taken into consideration.

The preferences of the students were randomly created using student declarations from previous semesters in order to have a simulation of the real conditions.

## 5.1 Parameter values of the evolutionary algorithm

We run eight experiments. Some of the operators and parameters of the EA were the same for all experiments and some were different. The values were set after some preliminary short experiments. Table 4 shows the parameters and their values that were the same over all experiments and table 5 the operators and parameters that were different.

Table 4. Parameters of the EA that are the same for all experiments

| Parameter | Value |
|---|---|
| Recombination rate | 0.80 |
| Recombination per block rate | 0.02 |
| Mutation rate | 0.20 |
| Mutation per block rate | 0.10 |
| Elitism | 1 individual |

Table 5. Parameters/operators of the EA that are different

| Parameter/Operators | Exp1 | Exp2 | Exp3 | Exp4 | Exp5 | Exp6 | Exp7 | Exp8 |
|---|---|---|---|---|---|---|---|---|
| Recombination type | 1 | 2 | 3 | 4 | 1 | 1 | 3 | 1 |
| Mutation type | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 1 |
| Population size | 100 | 100 | 100 | 100 | 50 | 200 | 100 | 100 |
| Selection method | tour | tour | tour | tour | tour | tour | tour | RW |

The parameter “Recombination per block rate” refers to the recombination probability of every block (it can be either a lab group or a lab place) of the chromosome. The parameter “Mutation per gene rate” refers to the mutation probability of each lab place.

Numbers of recombination and mutation types correspond to their definition in section 6.3.

Selection methods used are either tournament selection of size 2 (tour) and roulette wheel. In the preliminary short experiments, ranking selection was also tested. The results were similar, so in the final experiments, we’ve decided to use only one of them, i.e. tournament selection.

Each experiment run for two hours, or for 5000 generations, or to the point that the best individual did not improve for 1000 generations.

## 5.2 Results

Before presenting the results, we have to remind that the EA searches only the space of feasible solutions, keeping the chromosomes valid during the evolution.

Results are presented in table 6. The stopping criterion that was mainly applied by the algorithm was the time limit (2 hours). The limit generation limit (5000 generations) was never applied. The EA was only occasionally stuck in local optima for 30 generations.

Table 6. Results of the eight experiments

| | Exp 1 | Exp 2 | Exp 3 | Exp 4 | Exp 5 | Exp 6 | Exp 7 | Exp 8 |
|---|---|---|---|---|---|---|---|---|
| Execution time | 7227 | 7214 | 7211 | 7242 | 7202 | 7206 | 7203 | 7206 |
| Generation number | 780 | 759 | 720 | 322 | 1419 | 380 | 813 | 715 |
| Average fitness | **1,073** | 1,098 | 1,097 | 1,236 | **1,043** | 1,121 | 1,233 | 1,210 |
| Required places | 2796 | 2796 | 2796 | 2796 | 2796 | 2796 | 2796 | 2796 |
| Assigned places | **2718** | 2716 | 2717 | 2702 | **2721** | 2714 | 2702 | 2712 |
| 1st semester | **600** | 599 | 591 | 598 | **600** | **600** | 558 | 595 |
| Typical semester | **1049** | **1047** | 1046 | 1011 | **1047** | 1033 | 1011 | 1031 |
| Typical+1 semester | 913 | 914 | 920 | 907 | 918 | 906 | 937 | 910 |
| Other semester | 156 | 156 | 160 | 186 | 156 | 175 | 196 | 176 |
| 1st preference | **2238** | 2228 | 2181 | 1371 | **2332** | 1984 | 941 | 2016 |
| 2nd preference | 350 | 352 | 414 | 592 | 291 | 463 | 472 | 460 |
| 3rd preference | 72 | 70 | 75 | 318 | 52 | 156 | 407 | 128 |
| 4th preference | 38 | 41 | 21 | **222** | 25 | 69 | **390** | 60 |
| 5th preference | 11 | 11 | 13 | **115** | 10 | 27 | **269** | 32 |
| 6th preference | 2 | 5 | 3 | 55 | 1 | 5 | **142** | 6 |
| 7th preference | 1 | 2 | 2 | 13 | 3 | 4 | 57 | 4 |
| 8th preference | 6 | 7 | 8 | 16 | 7 | 6 | 24 | 6 |
| Basic prefs | 2469 | 2480 | 2480 | 2513 | 2470 | 2485 | 2518 | 2495 |
| Redundant prefs | 249 | 236 | 237 | **189** | 251 | 229 | **184** | 217 |
| Withdraw: false | 2589 | 2581 | 2584 | 2550 | 2585 | 2567 | 2537 | 2566 |
| Withdraw: true | **129** | 135 | **133** | 152 | 136 | 147 | 165 | 146 |
| Average of registred TU. | 29,90 | **29,89** | 29,91 | 29,90 | 29,92 | 29,90 | 29,91 | 29,89 |
| Average of passed TU | 14,94 | **14,96** | 14,93 | 14,95 | 14,95 | 14,91 | 14,95 | 14,94 |

Comparing the results of the first 4 experiments we observe that using the 4th recombination operator the EA becomes considerably slower than using the others. This is due to the fact that it is applied at the level of lab places, so it has to do more checks in order to verify that no student has been assigned two places of the same lab.

The best results were obtained using the 1st recombination operator. This operator is the only one that succeeded in assigning all available places to students of the first semester. Besides, it managed to satisfy the student preferences to a higher degree than the others.

Comparing the results of experiments 3 and 7, we observe that the 1st mutation type (random mutation) has better results that the 2nd type.

The results on the population size agree with what is reported in the literature: small populations have better performance when the time is limited.

## 6. CONCLUSION

In this paper we presented the scheduling algorithm of a complete web registration system which was implemented by an evolutionary algorithm.

The results show that the proposed system succeeded in fulfilling our expectations. The evolutionary algorithm managed to:

- allocate most of the places (percentage of filled places is higher than 97%),
- assign to the students of the first semester all corresponding places,

- maintain the “typical semester” rule, the students of each semester are assigned the lab places from courses of their semester that they required.
- The preferences of the students to enroll to specific lab groups were satisfied in percent higher than 85%
- Basic requests were satisfied in percent higher than 90%
- There was a small bonus to students with better performance.

At the same time the EA ensures the feasibility of the solutions.

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# Determining the Minimum Number of Warehouses and their Space-Size for Storing Compatible Items

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**Abstract**

We present an exact procedure for determining the smallest number of necessary warehouses and their space-size for storing compatible items. The required floor space housing of every item is known. The method developed here refers to store compatible items in the same warehouse in order to diminish the maximum necessary space-size of every warehouse and consequently to the determination of the minimum number of needed warehouses. The problem is formulated in the context of graph theory. Compatible items stored in the same warehouse are the elements of a color class of a specific coloring of a weighted conflict graph *G = (V, E, W),* where the vertices of *V* represent the items to be stored and all couples of non-compatible items define the edge set *E*. The elements of *W* are the numbers assigned to the vertices of *V* that express the required storing space of every corresponding item. That is the problem is reduced to find a coloring of *G* that correspond to an optimal solution.

**KEYWORDS**

Graph coloring, inventory, algorithm

## 1. INTRODUCTION

The problem of avoiding the storing of non-compatible items in the same warehouse occurs in diverse practical situations related to the inventory.
This eventuality arises when the warehousing of a particular set of items can cause their deterioration, or be a fire hazard, as well as for reasons of organizational scheduling as it is the case, for example, in the construction industry where the time interval for the removal or for the storing of each item is closely estimated.
The warehouse storing policies are directly related to the well known bin-packing problem, see [3], [8]. Nevertheless, limited studies appear in the literature that take into account the simultaneous housing of only compatible items, the methods in [1], [7], [5], [6] are approximation procedures, while an exact approach is given in [9] based on a set covering formulation. In this paper we develop an exact method that finds the smallest size as well as the minimum number of needed warehouses.
The problem is formulated in the context of graph theory. A conflict weighted graph $G=(V,E,W)$ is stated, the vertices of which represent the items to be stored, while all couples of non-compatible items are the elements of edge set $E$. To every vertex $v_i \in V=\{v_1,v_2,\ldots,v_n\}$ corresponds a number $w_i \in W=\{w_1,w_2,\ldots,w_n\}$ that expresses the storing space required for the item associated to $v_i$.
An *independent set* of $G$ is a subset of $V$ so that no two elements of $V$ are adjacent. Conclusively, a subset of compatible items allowable to be stored simultaneously in the same warehouse corresponds to a subset $S \subset V$ of the conflict graph $G$ for which $\Gamma(S) \cap S = \varnothing$ where $\Gamma(S)=\bigcup_{v\in S}\Gamma(v)$, $\Gamma(v)$ denote the set of adjacent vertices of vertex $v$. Thereby our problem turns out to partition the elements $V$ of $G$ into the smallest number $L$ of independent sets $S_i \subset V$, that is, to determine the sets of family

$$\wp = \{S_1, S_2, \ldots, S_L\} \text{ such that:}$$
$$L = \min\{K\} \text{ and For } i, j \in \{1,2,\ldots,K\} \text{ it holds that}$$

$$S_i \cap S_j = \varnothing, i \neq j \quad \wedge \quad \Gamma(S_i) \cap S_i = \varnothing \quad \wedge \quad \bigcup_{i=1}^{K} S_i = V \qquad (1.1)$$

An assignment of $k$ colors to the vertices of a graph $G$ so that no two adjacent nodes share the same color is a *$k$-coloring* of $G$. The smallest number of colors needed to color $G$ is its *chromatic number* $x(G)$. The vertices assigned the same color evidently form an independent set and constitute a *color class*, thus the compatible items stored in the same warehouse at the same time correspond to a color class of the conflict graph $G$.

The proposed problem is dealt within the framework of coloring graph $G$ taking into account the relationships that implicate the parameters involved.

A coloring can be represented by the set family $\wp = \{S_1, S_2, \ldots, S_L\}$ where every set $S_i, i = 1,2,\ldots,L$ denotes a color class. Following, a natural number is assigned to every vertex, so the elements of a color class are expressed by the corresponding numbers.

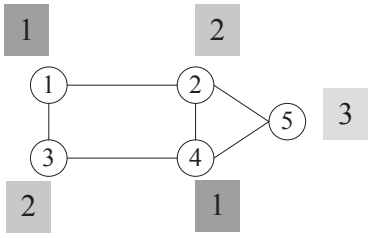
The next section deals with the development of the proposed algorithm AGC (Alternative Graph Coloring) that determines the sets of the family $\wp$ with the smallest number $L$ of color classes $S_i, i = 1,2,\ldots,L$ that concurrently satisfy relation 1.1.

## 2. ALGORITHM AGC

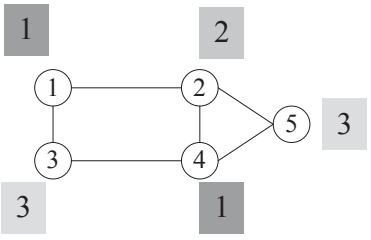
Algorithm AGC is an implicit enumeration backtracking algorithm that generates the entire alternative non-isomorphic $k$ - colorings of a graph. The inclusion of an optimality test produces the non-isomorphic $x(G)$ colorings of $G$. Two colorings of $G$ are said to be isomorphic if they contain exactly the same color classes.

Example:

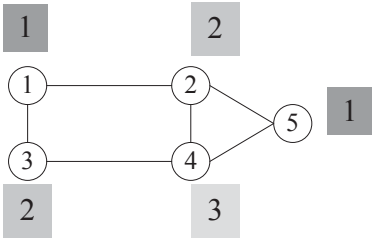
The $x(G) = 3-$ colorings $\wp_1$, $\wp_2$ and $\wp_3$ of graph $G$ shown in Figure 1 are non- isomorphic, while coloring $\wp_4$ is isomorphic to $\wp_1$.



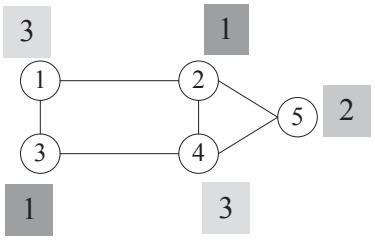

$\wp_1 = \{\{1,4\},\{2,3\},\{5\}\}$




$\wp_2 = \{\{1,4\},\{2\},\{3,5\}\}$




$\wp_3 = \{\{1,5\},\{2,3\},\{4\}\}$




$\wp_4 = \{\{2,3\},\{5\},\{1,4\}\}$

Figure 1

## 2.1 Algorithm: General Description

Before outlining the basic operations of AGC some necessary notations are introduced.
$S_L = \{s_{L_1}, s_{L_2}, \ldots, s_{L_{n_L}}\}$ with $s_{L_j} < s_{L_{j+1}}$, $j \in \{1,2,\ldots,n_L - 1\}$ is a generated independent set representing a color class in a lexicographic order, $L = \{1,2,\ldots,m\}$ where $L$ is the number of distinct colors used and $s_{L_1}$ is the smallest vertex of class $S_L$. Clearly $L$ is the number of color classes and $S_i \cap S_j = \varnothing, i \neq j, i, j \in \{1,2,\ldots,L\}$.

$F$: contains the colored vertices i.e., $F = \bigcup_{i=1}^{m} S_i$ and $\bar{F} = V - F$

$\bar{\Gamma}(y) = \{y_1, y_2, \ldots, y_{n_y}\}$, $y_i < y_{i+1}$, $i \in \{1,2,\ldots,n_y - 1\}$ represents the nonadjacent vertices of vertex $y$ that are greater than $y$, i.e. $y_i \notin \Gamma(y), y_i > y, i \in \{1,2,\ldots,n_y\}$

$n_L$ : The number of colored vertices that belong to the same color class $L$.
$MNL$ : The smallest number of used colors in a coloring found so far by the algorithmic process.
$k$ : Number of colorings $\wp_k$ generated so far.

### 2.1.1 Branching

The starting stage in order to generate a color class $S_L$ consists of placing in $S_L$ the smallest uncolored vertex. Let $v$ be the latest vertex inserted in $S_L$. The process augments $S_L$ repeatedly by a vertex $y$ greater than $v$, where $y$ is the smallest uncolored non-adjacent vertex to $v$ as well as to any element of $S_i$ and which has not been used to extend $S_L$ in a previous branching phase, namely, $S_i = \{i_1, i_2, \ldots, i_k\} \subseteq S_L$ is augmented by vertex $i_{k+1}$, where

$$i_{k+1} = \min\{y \in (\bar{\Gamma}(i_k) \cap S_i \cap \bar{F}),\ y \text{ not used in a previous phase to augment } S_L\}$$

At that particular instant the vertices used in an earlier stage to extend $S_L$ are uncolored due to a backtracking operation and they are contained in a subsequent generated color class. The process of extending $S_L$ ends when one of the two following cases appears.

*i) All vertices of $G$ are colored*

If the current coloring contains a smaller number of the $MNL$ color classes previously retained, then AGC sets $k \leftarrow 1$, otherwise it sets $k \leftarrow k+1$ and the current coloring is retained representing the family set $\wp_k$.

*ii) If such a vertex does not exist*

The algorithm goes on to form the next color class. In the case where the number $L$ of the currently generated color classes exceeds the corresponding number $MNL$ retained so far, the procedure passes to the backtracking operations.

### 2.1.2 Backtracking

The backtracking operation is performed when one of the two following cases arises.

- The procedure attains the instance in order to form a new color class , $S_{L+1}$ with $L+1 > MNL$
- A new $L$ - coloring $\wp_L$ with $L = MNL$ is detected.

In the occurrence of either of the two cases above, the un-coloring of all vertices in the last color class $S_L$ is actuated and the procedure backtracks from the last vertex $s_{L_{n_L}}$ inserted in $L \leftarrow L-1$, if $L < 1$ the algorithm ends.

## 2.2 Formal Statement of Algorithm AGC

$FV$ : The number of colored vertices

$h(s_{L_i})$ : The position in $\bar{\Gamma}(s_{L_{i-1}})$ of vertex $s_{L_i} \in S_L$ for $i > 1$, and $h(s_{L_1}) = 0$. The vertices in the preceding positions $h(s_{L_j}) < h(s_{L_i})$ in $\bar{\Gamma}(s_{L_i})$ have been used in a previous stage so as to extend $S_L$.

$\overline{\Gamma_{h(y)}}(y)$ : Contain the vertices of $\bar{\Gamma}(y)$ placed in positions $h(y)+1,\ h(y)+2,\ \ldots,\ h(y)+n_y$

$$\overline{\Gamma_{h(y)}}(y) = \{y_{h(y)+1}, y_{h(y)+2}, \ldots, y_{h(y)+n_y}\} \subset \bar{\Gamma}(y)$$

$Bool \leftarrow 1$
While $Bool = 1$
Let $v$ be the smallest uncolored vertex $v \notin F$
$L \leftarrow L+1$ {*next color class*}
If $L \le MNL$ Then { *coloring number optimality test* }
Set
$n_L \leftarrow 1,\ s_{L_1} \leftarrow v,\ S_L = \{s_{L_1}\},\ F \leftarrow F + \{v\},\ FV \leftarrow FV + 1,\ h_v \leftarrow 0$
If $FV = N$ then
CALL *COL* , CALL *BACK*
End if
Else
DO
$L \leftarrow L-1$ : CALL *BACK*
End If
DO
CALL *CHECK* {extend $S_L$ if possible}
If $Bool = 1$ then EXIT DO
Set $FV \leftarrow FV+1,\ n_L \leftarrow n_L+1,\ S_L \leftarrow S_L + \{y\},\ F \leftarrow F + \{y\}$
If $FV = N$ then
CALL *COL* , CALL *BACK*
End If
End If
LOOP
END While

*BACK* { Backtracking }
DO $i = 1, n_L$
Set $F \leftarrow F - \{s_{L_i}\},\ h_{s_{L_i}} \leftarrow 0$
LOOP
Set $FV \leftarrow FV - n_L$
DO
$L \leftarrow L-1$, If $L < 1$ CALL *WRT* {yield result and STOP}
Set $y \leftarrow s_{L_{n_L}},\ F \leftarrow F - \{y\},\ h(y) \leftarrow 0,\ FV \leftarrow FV - 1$
$n_L \leftarrow n_L - 1,\ y \leftarrow s_{L_{n_L}}$
LOOP While $n_L = 0$
*Return*

*CHECK* {independency test }

Set $r \leftarrow s_{n_L-1}$

If $S_L \cap \{y\} \neq \varnothing, \forall y \in \{\overline{\Gamma_{h(r)}}(r) - \overline{F}\}$

Set $Bool \leftarrow 1$

Else

Let $q$ be the smallest index for which $S_L \cap \{y_q\} = \varnothing, y_q \in \overline{\Gamma_{h(r)}}(r)$

$y \leftarrow y_{h_q}, h(y) \leftarrow q$

*Return*

*COL* {Store a new coloring $\wp_k$ }

If $L < MNL$ then

Set $k \leftarrow 0$, $MNL \leftarrow L$

End if

Set $k \leftarrow k+1$ and $\wp_k \leftarrow \{S_1, S_2, \ldots, S_L\}$

*Return*

## 3. NUMBER AND SIZE OF WAREHOUSES

Without loss of the generality it is assumed that all used warehouses are the same size. The housing of compatible items in the minimum number of warehouses generally leads to numerous disparate housing systems, since every distinct housing method corresponds to a discrete $k$-coloring $\wp$ of the conflict graph $G$, where $k$ is the smallest possible number of warehouses needed. Apparently $k$ is the chromatic number $x(G)$ of $G$ when no restriction is given to warehouse size. To a specific coloring class $S_L = \{s_{L_1}, s_{L_2}, \ldots, s_{L_{n_L}}\}$ we associate the *class-capacity* $WS(L) = \bigcup_{i=1}^{n_L} w_{L_i}$, $w_{L_i} \in W$ is the required storing space of a specific item, as mentioned in the introduction. We state the *coloring-capacity* $WC(\wp_q)$ to be the greatest *class-capacity* of a color class $\{S_1^q, S_2^q, \ldots, S_L^q\} = \wp_q$. That is, $WC(\wp_q) = \max_{i \in [1,L]} \{WS(S_i^q)\}$.

Since it is assumed that all warehouses are the same size, the selected warehouse with the minimum size is derived from coloring $\wp_{opt}$ with the smallest *coloring-capacity* among the non-isomorphic colorings, therefore,

$$WC(\wp_{opt}) = \min_{i \in [1,k]} \{WC(\wp_i)\}$$

Next a small example of the above conceptions on the graph in Figure 1 is shown in Table 1, where the sizes of the storing surface of the associated items are the elements of $W$.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| $W =$ {12, | 15, | 6, | 10, | 20} |

Table 1

| $\wp$ | $L$ | $S_L$ | $W_{L_i}, i \in [1, n_L]$ | $WS(L)$ | $WC(\wp)$ | $WC(\wp_{opt})$ |
|---|---|---|---|---|---|---|
| $\wp_1$ | 1 | $S_1^1 = \{1,4\}$ | {12 , 10} | 22 | **22** | **22** |
| | 2 | $S_2^1 = \{2,3\}$ | {15 , 6} | 21 | | |
| | 3 | $S_3^1 = \{5\}$ | {20} | 20 | | |
| $\wp_2$ | 1 | $S_1^2 = \{1,4\}$ | {12 , 10} | 22 | **26** | |
| | 2 | $S_2^2 = \{2\}$ | {15} | 15 | | |
| | 3 | $S_3^2 = \{3,5\}$ | {6 , 20} | 26 | | |
| $\wp_3$ | 1 | $S_1^3 = \{1,5\}$ | {12 , 20} | 32 | **32** | |
| | 2 | $S_1^3 = \{2,3\}$ | {15 , 6} | 21 | | |
| | 3 | $S_1^3 = \{4\}$ | {10} | 10 | | |

The optimal coloring is $\wp_{opt} = \wp_1 = \{ \{1,4\}, \{2,3\}, \{5\} \}$ that corresponds to a minimum warehouse size of $WC(\wp_{opt}) = 22$.

It is observed that the warehouse sizes (*class-capacity*) that correspond to an optimal coloring lead to a uniform distribution; this is an added advantage that concerns the decrement of the unused warehouse space $U(\wp_i)$ of a coloring $\wp_i$. In the short example, the unused space for each generated distinct coloring can easily be deduced in Table 1 as follows

$U(\wp_1) = (22-22)+(22-21)+(22-20) = \mathbf{3}$, $U(\wp_2) = (26-22)+(26-15)+(26-26) = \mathbf{15}$ and

$U(\wp_3) = (32-32)+(32-21)+32-10) = \mathbf{33}$

In all the above it was assumed that there were no restrictions concerning the size of the warehouses. In real life situations, however, the case may arise that, for the given conflict graph $G$, the size of the available warehouses is smaller than those obtained by the corresponding $WC(\wp_{opt})$. In this eventuality, clearly the smallest number of warehouses needed is greater than the chromatic number $x(G)$.

A size optimality test concerning the magnitude of *class-capacity* $WS(L)$ is incorporated in AGC during the extension phase of a color class $S_L$ preventing the inclusion of an item in a coloring class $S_L$. The procedure, thus, advances to the backtracking operations whenever its associated capacity $WS(L)$ reaches the situation of being greater than a given surface space restriction, say $SPR$. A more illustrative example in given in graph $G$ of Figure 2, where the numbers in italic near the vertices represent the space needed for storing the corresponding items.

The chromatic number of $G$ is $x(G) = 3$ and there are 14 distinct 3-colorings of $G$, therefore there exist 14 different ways for storing all items without any restrictions on the sizes of the warehouses, under this assumption the optimal solution concerning the warehouses size is the coloring

$$\wp_{opt} = \{S_1 = \{1,6,7,9\}, S_2 = \{2,5\}, S_3 = \{3,4,8\}\} \Rightarrow$$
$$WC(\wp_{opt}) = \max\{WS(S_1) = 49, WS(S_2) = 46, WS(S_3) = 45\} = \mathbf{49}$$

The worst solution is the coloring $\wp_w = \{S_1 = \{1,7\}, S_2 = \{2,5\}, S_3 = \{3,4,6,8,9\}\}$ with

$$WC(\wp_w) = \max\{S_1 = 17, S_2 = 46, S_3 = 77\} = \mathbf{77}$$

Let $SP = 48$ be the maximum warehouse space capacity. In this case there exist 114 distinct 4 – colorings of $G$ for which $WC(\wp_i) \le 48, i = 1,2,\ldots,114$ and

$$\wp_{opt} = \{S_1 = \{1,6,7\}, S_2 = \{2,4\}, S_3 = \{3,8\}, S_4 = \{5,9\}\}$$

$$WC(\wp_{opt}) = \max\{WS(S_1) = 36, WS(S_2) = 31, WS(S_3) = 35, WS(S_4) = 38\} = \mathbf{38}$$

The worst solution is the coloring

$$\wp_w = \{S_1 = \{1\}, S_2 = \{2,5\}, S_3 = \{3,8,9\}, S_4 = \{4,6,7\}\}$$

$$WC(\wp_w) = \max\{WS(S_1) = 9, WS(S_2) = 46, WS(S_3) = 48, WS(S_4) = 37\} = \mathbf{48}$$

It is important to observe the uniform storing distribution in the optimal solution of the items in the warehouses. Also to note that the algorithm works exclusively with the colorings that may lead to an optimal solution due to the size optimality test on *class-capacity* $WS(L)$ and of course does not generate all 114 distinct 4 – colorings.

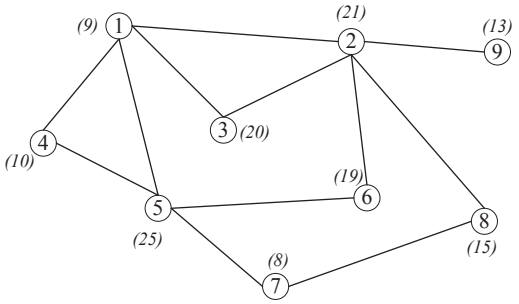



Figure 2

## 4. CONCLUSIONS

Inventory management is needed in diverse types of enterprises. The problem of concurrently determining the minimum number and size of the warehouses necessary for storing distinct items is evidently a meaningful task.

In the previous sections, a new method was presented in order to confront the problem where a subset of couples of non-compatible items must not be stored simultaneously in the same warehouse.

The problem is modeled in the context of graph theory, more specifically, the procedure developed is a depth-first branch and bound technique controlled by feasibility and optimality tests seeking an optimal solution among the distinct coloring of a conflict weighted graph G.

Although the problem is classified as *NP-hard*, see [2], [4] the density of a conflict graph related to compatible items is in general low enough $(d \le 0.1)$ permitting the successful application of the proposed algorithm to problems of which the size are met in real world situations.

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# Multistart approach for exact vehicle routing

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**Abstract**

The asymmetric distance–constrained vehicle routing problem consists of finding vehicle tours to connect all customers with a depot such that the total distance is minimum and the travelled distance by each vehicle is less than or equal to the given maximum value. In order to solve it exactly we develop new tolerance based branching rules within branch and bound method. Since our method is fast but memory consuming, it could stop before optimality is proven. Therefore we introduce the idea of randomness in choosing the node of search tree that will be next chosen. If an optimal solution is not found, we restart our method. In that way we get multi-start with branch-and-bound method. Computational experiments show that with our multi-start branch and bound approach we are able to solve exactly large test instances with up to 1000 customers.

keywords: Branch and Bound; Tolerance; Vehicle routing; CPLEX.

## 1 Introduction

The problem of distance-costrained vehicle routing problem (DVRP) is defined as follows: find optimal tours to connect the depot to $n$ customers with $m$ vehicles, such that every customer is visited exactly once; every vehicle starts and ends its tour at the depot; the total travelled distance by each vehicle in the solution is less than or equal to the maximum possible travelled distance ($D_{max}$). The problem is asymmetric (ADVRP) if the distance from node $i$ to node $j$ is different than the distance from node $j$ to node $i$. Otherwise the symmetric DVRP is defined. Surprisingly ADVRP is not studied as other types of vehicle routing problems (VRPs). To the best of our knowledge, the last such article was published in 1987 by Laporte, Nobert and Taillefer [6].

In this paper we suggest new exact algorithms for solving ADVRP. Two different tie-breaking rules are used in two tolerance-based Branch and Bound methods. One tie-breaking rule is deterministic and already presented in our previous conference version [1]. It is interesting that the limitation of that approach is not CPU time, but the memory requirements. That is why we suggest here multi-start approach with random tie-breaking rule in branching, it is called (`MSBB-ADVRP`). We use assignment problem as a lower bounding problem in each node of our best-first search tree. With our method we are able to solve large problems with up to 1000 customers.

The structure of this paper is as follows. In section 2 we give details of our deterministic branch and bound method for ADVRP. In section 3 we present our Multi-Start method that use random tie-breaking rule. Computational results is provided in section 4. Section 5 contains conclusions, and possible future research directions.

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## 2 Branch and Bound for ADVRP

Let $N = \{1, 2..., n-1\}$ denotes the set of customers and $V = N \cup \{0\}$ denotes the set of nodes where 0 is index of the depot. The set of arcs is denoted by $A = \{(i,j) \in V \times V : i \neq j\}$. The traveled distance from customer $i$ to customer $j$ is denoted by $d_{ij}$. The number of vehicles is denoted by $m$. The decision binary variables $x_{ij}$ is defined as follows:

$$x_{ij} = \begin{cases} 1 & \text{if the arc } (i,j) \text{ belongs to an optimal tour} \\ 0 & \text{otherwise} \end{cases}$$

The continuous variables $z_{ij}$ present the total length traveled from the depot to customer $j$, where $i$ is the predecessor of $j$. The formulation of ADVRP, that will be later used with CPLEX solver (`CPLEX-ADVRP`, is given below (Demir and Kara [4]):

$$f = \min \sum_{(i,j)\in A} d_{ij}x_{ij} \tag{1}$$

subject to

$$\sum_{j\in V} x_{ij} = \sum_{j\in V} x_{ji} = 1 \qquad \forall \ \ i \in N \tag{2}$$

$$\sum_{j\in V} x_{j0} = \sum_{j\in V} x_{0j} = m \tag{3}$$

$$\sum_{(i,j)\in A} z_{ij} - \sum_{(i,j)\in A} z_{ji} = \sum_{j\in V} d_{ij}x_{ij} \qquad \forall\ i \in N \tag{4}$$

$$d_{ij}x_{ij} \leq z_{ij} \leq D_{max}x_{ij} \qquad \forall\ (i,j) \in A, i \neq 0 \tag{5}$$

$$z_{0i} = d_{0i}x_{0i} \qquad \forall\ i \in N \tag{6}$$

$$x_{ij} \in \{0,1\} \qquad \forall\ (i,j) \in A. \tag{7}$$

Obviously, there are 2-index variables and thus polynomial number of constraints. This model is known as flow based model since constraint (4) is typical flow constraint. It says that the distance from node $i$ to any other node $j$ on the tour should be equal to the difference between distance from depot to $i$ and distance from depot to $j$. The constraints (2) ensure that the in and out degrees of each vertex equal to 1. Constraints (3) express that in and out degrees of the depot equal to the number of vehicles $m$. The constraints (5) present that the total distance from depot to customer $j$ is less than or equal to the maximum distance allowed and it should be greater than or equal to the distance from $i$ to $j$. Last constraint (6) gives the initial value for $z_{0i}$ which equals to the distance from depot to customer $i$. This flow based formulation (1)-(7) gives better result than node based formulation as noted in [4].

**Lower Bounding Procedure.** The assignment problem (AP) which is defined with (1),(2) and (7) provides a lower bound on the optimal value of ADVRP. However, in order to solve the AP relaxation of ADVRP, we made $m-1$ copies of the depot 0 and added to vertex set $V$ [7, 6], i.e. $V' = V \cup \{n+1, \ldots, n+m-2\}$. Thus we formulate ADVRP as multiple traveling salesman problem (m-TSP) before relaxation. Any feasible AP solution of reformulated ADVRP is not necessary ADVRP feasible. Therefore three types of cycles may occur in relaxed solution such obtained:(i)*a served cycle* (contains exactly one depot);(ii)*unserved cycle* (contains no depot);(iii) *a path* (a cycle contains more than one depot, the number of paths in the cycle

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equals to the number of depots). A tour is called *infeasible* if its total distance is larger than $D_{max}$ or if it contains no depot.

**Tolerance based Branching Rule.** If the tour is infeasible, it must be destroyed, i.e., one arc should be excluded (deleted). There are several ways to do this. In this paper we use the concept of *tolerance*. Tolerance is one of sensitivity analysis techniques [5, 8]. We use it as a branching rule in our `B&B` method. The difference between the value of the objective function before and after the exclusion of an arc in the current solution is called *upper tolerance* of the arc. The arc to be removed corresponds to the smallest objective function such obtained. Therefore the arc which has the smallest upper tolerance is chosen to be excluded within `B&B` technique. This approach has already been tested in [11] for solving traveling salesman problem (TSP).

When the infeasible tour is destroyed we get new subproblems. The number of new subproblems equals to the number of arcs in the destroyed tour. Each new subproblem has to be checked, if its value is less than ubber bound ($UB$). Then we add it to the list of active subproblems ($L$) for further branching. Otherwise we ignore it. There are two possible ways to choose the next node from $L$ when there are more than one choice: (i) at random and (ii) among subproblems that have the smallest objective function value, we choose the last one to branch next. We call that *Deterministic tie-breaking rule*.

Another possibility to destroy infeasible tour is to exclude the arc with the largest cost. `B&B` method that uses such a branching rule we call `Cost-ADVRP`. Based on extensive computational analysis, the results obtained with `Cost-ADVRP` were slightly worse than the quality of those obtained by our `Tol-ADVRP`. That was the reason why we will not report on `Cost-ADVRP` results in computational analysis section .

**Stopping Rule.** The basic deterministic algorithm we call `Tol-ADVRP`. It stops if one of these 3 stopping rules are met:

1. An optimal solution is found;
2. There is no feasible solution at all to the ADVRP;
3. The number of subproblems in the search tree is larger than 100 000 and hence there are two possibilities:
   (a) there is an incumbent solution;
   (b) no feasible solution has been found.

**Data Strcture.** In this program (`Tol-ADVRP`) we store the information of each node in the search tree by using two matrices. Each row in each matrix represents a node in the search tree. First row represents the root node, second row represents second generated node, and so on. The columns of first and second matrix represent pointers to the excluded arcs for each node in the search tree. More details could be found in [2]

## 3 Multi–Start Branch and Bound

As mention earlier, motivation for developing multi-start `B&B` method for solving ADVRP is based on the fact that `Tol-ADVRP` is very fast but requires a lot of memory. It usually stops after a few seconds, for solving even small size instances reaching the maximum number of nodes (100,000) generated in the search tree. The main idea of Multi–start `B&B` method for ADVRP (`MSBB-ADVRP`) is to select randomly the next subproblem among those with the same smallest objective function value. This random choice may cause the generation of smaller search

tree. Therefore, if we reach the maximum number of subproblems allowed we restart exact B&B method hoping that in the next attempt we will get an optimal solution. For that we rerun the experiment many times in case an optimal solution is not found. If it is found we stop. The main points of (MSBB-ADVRP) in this article is as follows:

1. Re-run each instance given number of times ( in this paper-twice); stop when an optimal solution is found. This will increase the chance to find an optimal solution or at least improve the value of best feasible solution found so far.

2. Choose randomly one subproblem from the list of active subproblems ($L$):

   Generate random number $\alpha \in [0, 1]$; find number of nodes in the list ($ns$) which has value equal to the smallest objective function value in this list; find $k \in [1, ns]$ as $k = 1 + ns * \alpha$; branch the node corresponding to $k$ in the list $L$.

3. Change the value of $\alpha$ each time the node from the list has to be chosen.

4. change the seed each time you run the program.

The stopping rule and data structure in (MSBB-ADVRP) are the same as in (Tol-ADVRP). We will consider these notations in the Algorithm which is given in (MSBB Algorithm 1):

**(i)** the number of tours in the solution $S$ is $M(S)$ where $M(S) \geq m$;

**(ii)** the set of tours in the solution S is $\Gamma(S) = \{T_1, T_2, ..., T_M\}$;

**(iii)** the total travelled distance in a tour $k$ is $d(T_k)$ where:

$$d(T_k) = \sum_{(i,j)\in T_k} d_{ij}$$

**(iv)** the number of arcs in a tour $k$ is $t(T_k)$ where:

$$t(T_k) = \sum_{(i,j)\in T_k} x_{ij}$$

**(v)** the value of an optimal solution to $AP$ at any node can be written by using the previous notation as $f(S) = \sum_k d(T_k)$ where $1 \leq k \leq M(S)$.

**(i)** $f^*$ is an optimal solution to ADVRP;

**(ii)** L is the list of active subproblems (or unfathomed nodes in a search tree), it is updated during the running of the code;

**(iii)** UB denotes an upper bound value of an optimal solution $f^*$ of ADVRP (initially set to $m * D_{max}$). Generally UB equals the value of the incumbent;

**(iv)** LB denotes a lower bound of $f^*$,i.e., it is the smallest value of the objective function to AP during the search (initially set to the value at root node to AP)

$$LB = \min_{S\in L}\{f(S)\} \tag{8}$$

**(v)** $APcnt$ is the number of times we have applied $AP$ during the search. The maximum number of nodes used in the B&B tree is chosen according to the computer memory available. In this article we allow to generate 100 000 subproblems as the limit to the number of subproblems. So $APcnt$ it should be less than or equal to 100 000.

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**Algorithm:** MSBB;
1 $UB \leftarrow m * D_{max}$,$APcnt \leftarrow 1$ ,$Maxnodes \leftarrow 100000$;
2 find $\Gamma(S) = \{T_k | k = 1, ..., M\}$ at the root node (with index 1);
3 $L = \{1\}$;
4 calculate $d(T_k)$ for every tour of $T_k \in \Gamma(S)$ ;
5 **if** $S$ *feasible* **then**
  $S$ optimal, Stop;
**else**
  **repeat**
6  choose the subproblem $S \in L$ with the smallest value of the objective function $LB = f(S)$ in case there is more than one subproblem has the smallest value choose randomly subproblem in the list;
7  find the ratio $\frac{t(T_k)}{d(T_k)}$ for every infeasible tour $k = 1, .., M'$;
8  choose the tour $k^*$ with the smallest ratio to branch;
9  calculate upper tolerances for all arcs in this tour $k^*$;
10  choose the subproblem with the smallest upper tolerance to develop further;
11  check feasibility of all new generated nodes;
12  update $UB$ (if necessary);
13  update L, and APcnt;
  **until** $L = \emptyset$ *or* $APcnt = Maxnodes$ ;
**end**
14 **if** $UB \neq m * D$ **then**
  **if** $L = \emptyset$ **then**
    $S^*$ is an optimal;
  **end**
  **if** $APcnt = Maxnodes$ **then**
    $S^*$ is the new incumbent ;
  **end**
**else**
  no feasible solution ;
**end**

**Algorithm 1**: (MSBB-ADVRP) algorithm with tolerance based Branching

We try to keep the best feasible solution, it order to use it as upper bound to the next run, we found that updating the upper bound each time does not affect next run because the program find first feasible solution quickly with good quality. Therefore we keep the upper bound as it is when we rerun the experiment ($UB = m * D_{max}$).

# 4 Computational Results

**Hardware.** All experiments were implemented under windows XP and on intel(R) Core(TM)2 CPU 6600@2.40GHz, with 3.24 GB of RAM. The code is written in C++ language. It has been derived from publically available source code Turkensteen [10].
**Test Instances.** Random test instances are generated as in [1]. It generates a full asymmetric distance matrix. This program needs the following input information: the size of distance matrix $n$; the parameter $\alpha$ controlling the degree of symmetry in the distance matrix ($\alpha \in [0, 1]$):0 means

completely random and asymmetric; 1 means completely symmetric; 0.5 means 50% symmetric, etc.; the seed value. When $n \leq 200$, four different seeds were chosen to generate four different distance matrices for each combination of $(n, m)$. Note that only one distance matrix is generated for $200 < n \leq 1000$

The size of distance matrix $(n)$ for each example belongs to the first set of instances which contains random problems with $n = 40, 60, ..., 200$. For the second set of large test problems $n \in \{240, 280, ..., 1000\}$. For the first set, two different number of vehicles $m(1)$ and $m(2)$ were defined as follows:

$$m(1) = \frac{n}{20} \qquad m(2) = \frac{n}{10} \tag{9}$$

However for the second set, we choose the number of vehicles to be $m(1)$ defined as above. Every example has been run at least twice. The first run has the value of maximum distance allowed $D_{max} = \infty$. This case produces as output the length of the longest tour $(LT)$ in the optimal solution. In the next run the value of maximum distance allowed is chosen to be $D_{max} = 0.9 * LT$. Then we reduce $D_{max}$ as $D_{max} \leftarrow 0.9 * LT'$ where $LT'$ is the longest tour found in the optimal solution. We continue as long as an optimal solution is found, but not more than four times. It appears that the number of all experiments is 256.

**Methods Compared.** Three programs ((`MSBB-ADVRP`), `Tol-ADVRP`, and `CPLEX-ADVRP`) are used to find an optimal solution to ADVRP. In the first two programs when the problem solution requires the generation of subproblems more than a specific number (a pre user defined limit), the value of optimal solution to ADVRP is not found. In this case the program only produces an upper bound, defined as the best feasible solution found so far. `MSBB-ADVRP` in general, is repeated until an optimal solution is found. If the optimal solution is found then the time of all runs is taken. In all our experiments reported below, we run `MSBB-ADVRP` only two times. We note that increasing the number of restarts might improve chances to find an optimal solution, but with the cost of larger CPU time. In addition multistart approach suffers from so-called *centeral-limit catastrophe*[3]: when problems grow large, random local minima are almost surely of 'average' quality and increasing the number of restarts does not help . In `CPLEX-ADVRP` the process will continue until an optimal solution is found or the time limit is reached. We choose the time limit as 3600 seconds when the number of customers is up to 200, while we increase the time limit to 3 hours (10800 seconds) when the number of customers is up to 1000.

**Comparison.** Table 1 contains summary results for small test problems from $n = 40$ up to $n = 200$ customers, the time limit is 1 hour. Table 2 contains summary results for large test problems from $n = 240$ up to $n = 1000$, the time limit is 3 hours. Both Table 1, and Table 2 contain information presented in the folowing 6 columns:

Table 1: Summary table of 193 instances (small test problems for $n$=40 to 200)

| | ♯ Opt | ♯ Feas not opt | ♯ Feas equal to opt | ♯ Inf | Total Time | Average Time |
|---|---|---|---|---|---|---|
| Tol-ADVRP | 122 | 27 | 21 | 23 | 141.93 | 1.16 |
| CPLEX-ADVRP | 175 | 2 | 0 | 16 | 102598.38 | 586.27 |
| MSBB-ADVRP | 145 | 8 | 29 | 11 | 213.45 | 1.47 |

1. ♯ Opt - How many times each program find an optimal solution;

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2. ♯ Feas not opt - How many times each program produce feasible solution but not equal to the optimal after arriving to stopping criteria;

3. ♯ Feas equal to opt - How many times in each program the best feasible solution equal to the optimal solution has been found;

4. ♯ Inf - How many times any feasible solution is not found when the program reach the stopping criteria;

5. Total Time - The total time spent for all experiments which find the optimal solution;

6. Average Time - The average time for each experiment.

Table 2: Summary table of 75 instances (large test problems for $n$=240 to 1000)

| | ♯ Opt | ♯ Feas not opt | ♯ Feas equal to opt | ♯ Inf | Total Time | Average Time |
|---|---|---|---|---|---|---|
| Tol-ADVRP | 63 | 1 | 0 | 11 | 2934.36 | 46.57 |
| CPLEX-ADVRP | 3 | 3 | 0 | 69 | 12028.05 | 4009.35 |
| MSBB-ADVRP | 74 | 0 | 0 | 1 | 998.83 | 13.49 |

We carried out 193 experiments with relatively small size instances. The percentage in finding the optimal solution for the three methods compared are: 63% for `Tol-ADVRP`, 91% for `CPLEX-ADVRP`, and 75% for `MSBB-ADVRP`. It appears that CPLEX is more effective but less efficient for solving small size problem instances.

When we carried out 75 large size instances, the percentage of optimal solutions found is: 84% for `Tol-ADVRP` , 4% for `CPLEX-ADVRP` , and 99% for `MSBB-ADVRP`. Therefore, Multi-start method is the most effective and efficient for solving large test problems. It is interesting to note that success rate of `MSBB-ADVRP` is larger for large problem instances than for small ones.

## 5 Conclusions and Future work

We consider ADVRP and suggest two exact algorithms for solving it. They are based on Assignment Problem relaxation, as first time proposed by Laborte [6]. In order to rebuilt feasibility, we branch by using tolerance criterion. We found that our method is fast but it has difficulty which is the memory consumption. Therefore we introduce the new method based on randomness in choosing the next node in branch and bound tree. Computational experiments show that with our multi-start approach we are able to solve exactly 99% large instances up to 1000 customers, and 75% small instances. Comparing Multi-start with CPLEX according to time issue we get that our method Multi-start on average needs 1.47 seconds for small instances and 13.49 seconds for large ones, while CPLEX needs on average 586.27 seconds for small instances and 4009.35 seconds for large ones.

In summary, for small test problems the results of experiments emphasize that using Multi-start gives good quality of incumbent in reasonable computational time, while using CPLEX is useful if we are interested in finding an optimal solution even if that takes long computational times. For large test problems, the results of experiments emphasize that using Multistart

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branch and bound finds optimal solution in shorter computational times than both tolerance and CPLEX.

We are working currently to improve the running times of the algorithm by developing heuristic such as Variable Neighborhood Search [9], and to use it as an upper bound. Moreover we are trying to improve the lower bounding procedure.

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# Incidence Graphs of Bipartite $G$-Graphs

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**Abstract**

Defining graphs from groups is a widely studied area motivated, for example, by communication networks. The most popular graphs defined by a group are Cayley graphs. $G$-graphs correspond to an alternative construction. After recalling the main properties of these graphs and their motivation, we propose a characterization result. Then, we show that the incidence graph of a symmetric bipartite $G$-graph is also $G$-graph. We give an example of mesh of $d$-ary trees.

## 1 Introduction

A recent studied family of graphs constructed from groups are $G$-graphs. These graphs, introduced in [1], have also, like Cayley graphs, highly-regular properties. In particular, because the algorithm for constructing the $G$-graphs is simple, it appears as a useful tool to construct new symmetric and semi-symmetric graphs [2]. Moreover, thanks to $G$-graphs, some upper bounds in the cage graphs problem were improved ([3], see also [7]). One interesting direction is studying the $G$-graphs properties. In the following sections, we characterize these graphs and study their incidence graphs as follows:

- Section 2 - we give the definitions we need;
- Section 3 - we define $G$-graphs and give a characterization theorem;
- Section 4 - we study the incidence graphs of $G$-graphs;
- Section 5 - we give an example of Mesh of d-ary trees.

## 2 Definitions

### 2.1 Graphs Definitions

In this paper, graphs are undirected without loops. Specifically, we define an undirected graph $\Gamma = (V;E)$ by the vertex set $V$ and the edge set $E$. For a vertex $x \in V$ we call the *neighborhood* of $x$, the set $\Gamma(x)$ of vertices in $V$ adjacent to $x$.

**Intersection graph.**An *Intersection graph*, is an undirected graph obtained from a family of sets $S_i$, $i = 0,1,2,...$ by creating a vertex $v_i$ for each set $S_i$, and connecting two vertices by an $p$-edge whenever the corresponding two sets $S_i$ and $S_j$ have a nonempty intersection, and $\left|S_i \cap S_j\right| = p \geq 1$.

**Equitable partition.** Given a graph $\Gamma = (V;E)$ and a partition of its vertex set $\pi = \cup_{1 \leq i \leq r} C_i$, we say that $\pi$ is an *equitable partition* if and only if for all $1 \leq i, j \leq r$ there exists $b_{ij}$ such that $\forall x \in C_i$ we have

$\left|\Gamma(x)\cap C_j\right| = b_{ij}$. The edges between $C_i$ and $C_j$ induce a semiregular bipartite graph. All vertices from $C_i$ have the same degree.

**Graph isomorphism.** Let $\Gamma_1 =(V_1;E_1)$ and $\Gamma_2 = (V_2;E_2)$ be two graphs. An *isomorphism* from $\Gamma_1$ to $\Gamma_2$ is a bijection $(f,f^{\#})$ where $f: V_1 \longrightarrow V_2$ and $f^{\#}: E_1 \longrightarrow E_2$ with $f^{\#}(xy) = f(x)f(y)$, such that for every $xy \in E_1$ we have $f^{\#}(xy) \in E_2$. $Aut(\Gamma)$ denotes the group of automorphisms of a graph $\Gamma$ (isomorphisms from $\Gamma$ to itself) under composition law.

**Orbit partition.** For a graph $\Gamma = (V;E)$, let $H \leq Aut(\Gamma)$ be a subgroup of its automorphisms group. We define an equivalence relation on $V$ regarding to $H$ as follows: for any $u$ and $v$ in $V$, $u$ is in relation with $v$ if and only if there exists $h \in H$ such as $h(u) = v$. An *orbit* is an equivalence class. An *orbit partition* is the partition of $V$ associated to this relation. Orbit partitions are equitable partitions [6].

**Incidence Graph.** The *Incidence graph* of a simple graph $\Gamma = (V;E)$ is the graph $I\Gamma = (IV = V \cup E; IE)$ where $IE = \{ab : a = xy \in E, b = x \in V\}$.

### 2.2 Group Definitions

If $(G,\cdot,e)$ is a group (where $e$ denotes the identity element of $G$), for every $g$ in $G$ we define the order of $g$, denoted by $o(g)$, as the smallest integer $k$ such that $g^k = e$. The set $\langle g\rangle = \{e,g,g^2,\ldots,g^{o(g)-1}\}$ with the corresponding operation form a subgroup of $G$ called the *cyclic group* of $g$.

**Left and Right Cosets.** If $H$ is a fixed subgroup of a group $G$, the subset $xH = \{xh|h \in H\}$ is called *left coset of H containing x*. Let us select an element from each left coset of $H$ and denote by $T$ the resulting set of *left coset representatives*. The set $T$ is called the *left transversal* of $H$ in $G$. The notions of *right coset* and *right transversal* are defined analogously.

**Semi-direct-product of groups.** We define a *(left) group action* of $G$ on a set $X$ as a binary function from $G\times X$ to $X$, $(g,x) \to g\cdot x$ satisfying conditions: $e\cdot x = x$ for every $x \in X$ and $g\cdot\left(g^{'}\cdot x\right) = \left(g\cdot g^{'}\right)\cdot x$ for all $g$, $g^{'} \in G$ and $x \in X$. The action is *transitive* if $\forall x,y \in X, \exists g \in G$ such that $g\cdot x = y$.
For $x \in X$ we define the *stabilizer subgroup* of $x$ as the set of all elements in $G$ that fix $x$: $Stab_G x = \{g : g\cdot x = x\}$.

Let $H$ and $Q$ be two groups and $\varphi : Q\times H \longrightarrow H$ an action of $Q$ on $H$. The *semi-direct product* of $H$ and $Q$ by $\varphi$, denoted $H \rtimes_{\varphi} Q$, is defined as the group with underlying set $\{(h,q) : h \in H, q \in Q\}$ and the operation: $(h,q)\cdot(h',q') = \left(h\cdot\varphi(q,h'), q\cdot q^{'}\right)$. This group has $|H|\times|Q|$ elements.

## 3 $G$ - Graphs characterization

### 3.1 $G$ - Graphs Definition

**Definition.** We consider G a finite group, with $e$ the neutral element. Let $S$ be a nonempty subset of $G$. The *right G-Graph*, $\Phi(G;S)$, is the intersection graph of the right cosets of the cyclic groups $\langle s\rangle$, $s$ in $S$. By this definition, we construct $\Phi(G;S) = (V;E)$ in the following way:

1. For $s \in S$ denote $(s)x = (x,sx,s^2x,\ldots,s^{o(s)-1}x)$ the cycle of permutation $x \mapsto sx$. Notice that $\langle s\rangle x$ is the support of cycle $(s)x$. The vertices of $\Phi(G;S)$ are $V = \bigcup_{s\in S} V_s$ where $V_s = \{(s)x : x \in G\}$.

2. As in all intersection graphs the set of edges $E$ represent the intersection of the sets corresponding to the vertices. For $(s)x,(t)y \in V$, there exists an $p$-edge between $(s)x$ and $(t)y$ if and only if $|\langle s\rangle x \cap \langle t\rangle y| = p \geq 1$.

We denote $S = \{s_1,s_2,\ldots,s_k\}$. Then $V = \bigcup_{i=1}^{k} V_{s_i}$.

**Lemma 1.** *$\Phi(G;S)$ is a $|S|$-partite graph and this partition is an equitable partition. Every vertex from a class $V_{s_i}$ of the partition has the degree $(|S|-1)o(s_i)$.*

*Proof.* $V_{s_1}, V_{s_2}, \ldots, V_{s_k}$ is a $\Phi(G;S)$ partition. Since a class of partition $V_{s_i}$ consists of all cosets of the cyclic group $\langle s_i \rangle$ and all cosets of a cyclic group have the same cardinality, which is $o(s_i)$, then all vertices (sets) of the same class of partition have the same cardinality. The union of all cosets of a cyclic group is the hole group $G$. So each element of $G$ appears only once in each class of partition. It follows that a vertex from $V_{s_i}$ has every element linked to each of the other $k-1 = |S|-1$ classes of partition. Since a vertex form class $V_{s_i}$ has $o(s_i)$ elements, then each vertex of $V_{s_i}$ is incident to $(|S|-1)o(s_i)$ edges. □

**Short list of classical graphs which are identified as $G$-graphs.**

1. Bipartite complete graphs $(G = C_n \times C_k, S = \{(1,0),(0,1)\})$
2. The 3-prism $(G = C_3 \times C_3, S = \{(1,0),(0,1)\})$
3. The cuboctahedral graph $(G = C_2 \times C_2 \times C_2, S = \{(1,0,0),(0,1,0),(0,0,1)\})$
4. The square ($G$ is the Klein's group, $G = \{e,a,b,ab\}$, and $S = \{a,b\}$)
5. The cube $(G = A_4, S = \{(1,2,3),(1,3,4)\})$
6. The $2 \times 2$ grid on a 3D torus $(G = Q_2, S = \{a,b\})$
7. The Heawood's graph $(\langle a,b \mid a^7 = b^3 = e, ab = baa \rangle, S = \{b,ba\})$
8. The Pappus graph $\left(G = \langle a,b,c \mid a^3 = b^3 = c^3 = e, ab = ba, ac = ca, bc = cba \rangle,\ S = \{b,c\}\right)$

## 3.2 $G$ - Graph Characterization

**Lemma 2.** *Let $\Phi_1(G;S) = (V_1;E_1)$ and $\Phi_2(G;S) = (V_2;E_2)$ be the right and left G-Graphs of G. These two graphs are isomorphic.*

*Proof.* For every $s \in S$ the sets of left and right cosets of the cyclic group $\langle s \rangle$ have the same cardinality. If $T_s$ is a left transversal of $\langle s \rangle$ in $G$, then $T_s^{-1}$ is a right transversal of $\langle s \rangle$ in $G$.
Consider application $f : V_1 \to V_2$ such that $f((s)x) = x^{-1}(s)$. It is easy to see that $f$ is a bijection. Moreover, if there is an edge between $(s_i)x$ and $(s_j)y$, then there exist $\alpha$ and $\beta$ such as $s_i^{\alpha}x = s_j^{\beta}y$, hence $(s_i^{\alpha}x)^{-1} = (s_j^{\beta}y)^{-1}$ and we have $x^{-1}s_i^{o(s_i)-\alpha} = y^{-1}s_j^{o(s_j)-\beta}$. It follows that there exists an edge between the vertices $x^{-1}(s_i)$ and $y^{-1}(s_j)$, hence $(f, f^{\#})$ is an isomorphism. □

Let $\Phi(G;S) = (V;E)$ be a $G$-graph. For any $g \in G$ we can associate the morphism $(\delta_g, \delta_g^{\#}) : \delta_g : V \longrightarrow V$, defined by $\delta_g((s)x) = (s)xg$ and $\delta_g^{\#} : E \longrightarrow E$, $\delta_g^{\#}((s)x(t)y) = \delta_g((s)x)\delta_g((t)y) = (s)xg(t)yg$. Using these notions we have the following Theorem.

**Theorem 1.** *Let $\Phi(G;S) = (V;E)$ be a G-graph.*

*(1) $(\delta_g, \delta_g^{\#})$ is an automorphism of $\Phi(G;S)$.*

*(2) $\delta_G = \{(\delta_g, \delta_g^{\#}), g \in G\}$ form a group under the composition law, and $V_{s_i}$ for an $s_i$ fixed is an orbit regarding to $\delta_G$. A G-graph is orbit partitionable.*

*(3) Settle $s_i$ and $s_j$ in S. For every $x,u \in V_{s_i}$, $y,v \in V_{s_j}$ with $xy,uv \in E$, there exists $g \in G$ such that $\delta_g(x) = u$ and $\delta_g^{\#}(xy) = uv$.*

*(4) For every $(s)x \in V$ with $s \in S$, $Stab_{\delta_G}(s)x$ is a cyclic group of order $o(s)$.*

*Proof.* (1) Let $s \in S$. $\delta_g|_{V_s}$ is a bijection because $V_s$ is finite and if $(s)xg = (s)yg$ then there exists $i$ such that $xg = s^i yg$ which leads to $(s)x = (s)y$. That is equivalent with $\delta_g(V_s) = V_s$, hence $\delta_g$ is stable on each set $V_s$ for all $s \in S$.
If $(s)x(t)y$ is in $E$, then there exist $i$ and $j$ such as $s^i x = t^j y$, which implies that $s^i xg = t^j yg$. Hence we have $\delta_g^{\#}((s)x(t)y) = (s)xg(t)yg \in E$. It follows that $(\delta_g, \delta_g^{\#})$ is in $\mathscr{A}ut(\Phi(G;S))$, for every $g \in G$.

(2) From above it is obvious that $\delta_G = \{(\delta_g, \delta_g^{\#}), g \in G\}$ form a group under the composition law. Moreover, if $x$ and $y$ are in $G$ we have $xy^{-1}$ also in $G$. For every $(s_i)x$ and $(s_i)y \in V_{s_i}$, there exists an automorphism $(\delta_{x^{-1}y}, \delta^{\#}_{x^{-1}y}) \in \delta_G$ such as $\delta_{x^{-1}y}((s_i)x) = (s_i)xx^{-1}y = (s_i)y$, hence $V_{s_i}$ is an orbit regarding to the group $\delta_G = \{(\delta_g, \delta_g^{\#}), g \in G\}$.

(3) For $x, u \in V_{s_i}$, from (2), there exists $g_1 \in G$ such as $\delta_{g_1}(x) = u$. Since $(\delta_{g_1}, \delta^{\#}_{g_1})$ is an automorphism of $\Phi(G;S)$ stable on $V_{s_j}$, we have $\delta_{g_1}(y) = v_2$ with $v_2 \in V_{s_j}$ and $uv_2 \in E$. Let $u = (s_i)a$ and $v = (s_j)b$. Since $uv$ is in $E$, there exist $q$ and $k$ such that $s_i^q a = s_j^k b$. This is equivalent with $b = s_j^{-k} s_i^q a$. Hence we can write $v = (s_j)b = (s_j)s_j^{-k}s_i^q a = (s_j)s_i^q a$. Similarly $v_2$ can be written: $v_2 = (s_j)s_i^r a$. Now, let $g_2 = a^{-1}s_i^{q-r}a \in G$. We have
$\delta_{g_2}(u) = \delta_{g_2}((s_i)a) = (s_i)ag_2 = (s_i)a(a^{-1}s_i^{q-r}a) = (s_i)s_i^{q-r}a = (s_i)a = u$.
We also have $\delta_{g_2}(v_2) = \delta_{g_2}((s_j)s_i^r a) = (s_j)s_i^r ag_2 = (s_j)s_i^r a(a^{-1}s_i^{q-r}a) = (s_j)s_i^r s_i^{q-r}a = (s_j)s_i^q a = v$.
Hence we have $\delta_{g_1}(x) = u$, $\delta_{g_1}(y) = v_2$ and $\delta_{g_2}(u) = u$, $\delta_{g_2}(v_2) = v$. It is obvious that if we consider $g = g_1 g_2$ we obtain $\delta_g(x) = u$ and $\delta_g(y) = v$.

(4) From definition, for every $(s)x \in V$ with $s \in S$, $Stab_{\delta_G}(s)x = \{(\delta_g, \delta_g^{\#}), \delta_g((s)x) = (s)x\}$. Then $\delta_g \in Stab_{\delta_G}(s)x$ only if $\delta_g((s)x) = (s)xg = (s)x$. Then there exist $q, r$ such that $s^q xg = s^r x$, hence $g = x^{-1}s^{r-q}x$. It follows that $Stab_{\delta_G}(s)x = \{\delta_{x^{-1}s^p x}, p < o(s)\}$. Then $Stab_{\delta_G}(s)x$ is generated by $\delta_{x^{-1}sx}$, hence it is a cyclic group and the order of $Stab_{\delta_G}(s)x$ is equal to $o(s)$.

□

**Theorem 2.** ***Characterization of a G-graph.*** *Let $\Gamma = (V;E)$ be a graph with $V = \bigcup_{i\in I} V_i$, $|I| = k$. Assume that there exists $H$ a subgroup of $\mathscr{A}ut(\Gamma)$ such that the following properties are true:*

*(1) The partition $V = \bigcup_{i\in I} V_i$ is an orbit partition regarding to $H$.*

*(2) The graph $\Gamma$ contains a clique with k vertices, $K = \{x_1, x_2, \ldots x_k\}$, with $x_i \in V_i$ for every $i \in \{1,2,3\ldots k\}$.*

*(3) $Stab_H x_i$, $i \in \{1,2,3\ldots k\}$ are cyclic groups, pairwise distinct*

*(4) $\frac{o(H)}{|Stab_H x_i|} = |V_i|$, for every $i \in \{1,2,3\ldots k\}$.*

*Then there exists S such that $\Phi(H;S)$ is isomorphic to $\Gamma$, hence $\Gamma$ is a G-graph. Conversely, if $\Gamma = \Phi(G;S)$ is a G-graph, such that $S = \{s_1, \ldots, s_k\}$ and $s_1, \ldots, s_k$ are independent, then there exists $H$ a subgroup of $\mathscr{A}ut(\Gamma)$, satisfying properties (1)-(4).*

*Proof.* Assume there exists $H$ a subgroup of $\mathscr{A}ut(\Gamma)$ with the properties (1), (2), (3), (4). From (2) there exists a clique in $\Gamma$: $\{x_1, x_2, x_3, \ldots x_k\}$, $x_i \in V_i$. By (3), for every $i \in \{1,2,3\ldots k\}$, $Stab_H x_i$ are cyclic groups. Let $\sigma_i$, be the generator of $Stab_H x_i$ and settle $S = \{\sigma_1, \sigma_2, \sigma_3, \ldots \sigma_k\}$. We denote by $q_i$ the order of $\sigma_i$. Now, we can consider the left $G$-graph $\Phi(H;S)$ (which, according to Lemma 2, is isomorphic to the corresponding right $G$-graph). The set of its vertices is: $W = \bigcup_{i\in I} W_i$, with $W_i = \{a(\sigma_i) = (\sigma_i, a\sigma_i, \ldots, a\sigma_i^{q_i-1}), a \in H\}$. Consequently, each $W_i$ have the order $\frac{o(H)}{|Stab_H x_i|}$.

We define the map $\phi : W \longmapsto V$ in the following way: for every $a(\sigma_i) \in W_i$ let $\phi(a(\sigma_i)) = a(x_i)$. $\phi$ is an application, since if we have $a(\sigma_i) = b(\sigma_i)$, then there exists $t$ such that $b = a\sigma_i^t$ and we obtain $b(x_i) = a(x_i)$.

Since $a \in H$ and from (1) is obvious that $a(x_i) \in V_i$, hence $\phi$ is a map. Moreover, for any $y$ in $V_i$ there exists $a$ in $H$ such that we have $a(x_i) = y$, so $y = \phi(a(\sigma_i))$. It follows that the map is surjective. Since for all $i \in \{1,2,3\ldots k\}$ we have $|W_i| = |V_i|$, the map $\phi : W \longmapsto V$ is a bijection.
If $a(\sigma_i)$ and $b(\sigma_j)$ are adjacent in $\Phi(H;S)$, then we have $a\sigma_i^l = b\sigma_j^{l'}$ $(l,l' \geq 0)$ and $b = a\sigma_i^l\sigma_j^{-l'}$, hence $\phi(b(\sigma_j)) = b(x_j) = a\sigma_i^l\sigma_j^{-l'}(x_j)$. Since $\sigma_j$ is in $Stab_H x_j$, we obtain $\phi(b(\sigma_j)) = a\sigma_i^l(x_j)$. We also have $\phi(a(\sigma_i)) = a(x_i)$ and since $\sigma_i$ is in $Stab_H x_i$, we obtain $\phi(a(\sigma_i)) = a\sigma_i^l(x_i)$. Then, since there is an edge between $x_i$ and $x_j$, then there is an edge between $\phi(a(\sigma_i))$ and $\phi(b(\sigma_j))$. It follows that the map $\phi : W \longmapsto V$ is an isomorphism.

Conversely, assume $\Gamma = \Phi(G;S) = (V;E)$ is a $G$-graph. Then $K = \{(s_1),\ldots,(s_k)\}$ is a clique and $V = \bigcup_{i=1}^k V_{s_i}$. Consider $H = \delta_G \leq Aut(\Gamma)$. Using Theorem 1 the result follows (we have $Stab_{\delta_G}(s_i) = \langle \delta_{s_i} \rangle$ and $|Stab_{\delta_G}(s_i)| = o(s_i) = \frac{o(G)}{|V_i|} = \frac{o(H)}{|V_i|}$). □

# 4 Incidence graphs of G-Graphs

Let $\Gamma = (V;E)$ be a simple connected graph with all vertices of degree at least 3, and $I\Gamma = (IV = V \cup E; IE)$ be its incidence graph. Let $\Psi : Aut(\Gamma) \longrightarrow Aut(I\Gamma)$, $\Psi((f,f^\#)) \mapsto (h,h^\#)$ such that: $h(x) = f(x)$ for all $x \in V$, $h(a) = f^\#(a)$ for all $a \in E$, and $h^\#(xa) = h(x)h(a)$.

**Lemma 3.** *$\Psi$ is an isomorphism.*

*Proof.* (a) *We prove that $\Psi$ is an application.*
Since $f$ and $f^\#$ are bijections and $V \cap E = \emptyset$, it is obvious that $h : IV \longrightarrow IV$ is a vertex bijection. Settle $e = xa \in IE$. We have $a = xy \in E$ and $h^\#(e) = h(x)h(a) = f(x)f^\#(a) = f(x)(f(x)f(y)) \in IE$. It follows that $(h,h^\#) \in Aut(I\Gamma)$. Let $(f,f^\#) \in Aut(\Gamma)$. Then there exists only one automorphism $(h,h^\#)$ such that $\Psi(f,f^\#) = (h,h^\#)$, hence $\Psi(f,f^\#)$ is an application.

(b) *We prove that $\Psi$ is an isomorphism.*
Let $(f_1,f_1^\#)$ and $(f_2,f_2^\#) \in Aut(\Gamma)$. We have $\Psi((f_1,f_1^\#)) \circ \Psi((f_2,f_2^\#)) = (h_1,h_1^\#) \circ (h_2,h_2^\#) = (h_1 \circ h_2, h_1^\# \circ h_2^\#)$. It follows that $\Psi((f_1,f_1^\#)) \circ \Psi((f_2,f_2^\#)) = \Psi((f_1 \circ f_2, f_1^\# \circ f_2^\#))$, hence $\Psi$ is a morphism. Let $(f_1,f_1^\#)$ and $(f_2,f_2^\#)$ in $Aut(\Gamma)$ such as $\Psi((f_1,f_1^\#)) = \Psi((f_2,f_2^\#))$. Then $(h_1,h_1^\#) = (h_2,h_2^\#)$, hence $h_1 = h_2$ and $h_1^\# = h_2^\#$. So for all $x \in V$, $h_1(x) = f_1(x) = f_2(x) = h_2(x)$ and for all $a \in E$, $h_1(a) = f_1^\#(a) = f_2^\#(a) = h_2(a)$. Moreover $V \cap E = \emptyset$, hence $f_1 = f_2$ and $f_1^\# = f_2^\#$, so $\Psi$ is injective. Settle $(h,h^\#)$ in $Aut(I\Gamma)$. Let $f = h|_V$. We can easily see that $f : V \longrightarrow V$ is a bijection, since $h$ is one. Similarly we can consider $f^\# = h|_E$. Then, for any $a = xy \in E$, we have $f^\#(xy) = f(x)f(y)$. It follows that $(f,f^\#)$ is an automorphism, hence $\Psi$ is a surjective.

In conclusion $\Psi$ is a automorphism. □

Let $\Gamma = \Phi(G;S) = (V;E)$ be a $G$-graph and $S = \{s_1,s_2\}$ a generating set for $G$ with $o(s_1) = o(s_2) = p \geq 3$. Recall the action : $\delta : G \times (V;E) \longrightarrow (V;E)$ defined by $\delta(g,(s_i)x) = \delta_g((s_i)x) = (s_i)xg$ for every $x \in V$, $s_i \in S$ and $\delta(g,a) = \delta_g^\#(a)$ for $a \in E$. Since $|S| = 2$, $\Phi(G;S)$ is bipartite; let $V = V_1 \cup V_2$.

Next we assume that $s_1$ and $s_2$ are independent and there exists $f$ an automorphism of $G$ that swaps the two elements contained in $S$: $f(s_1) = s_2$ and $f(s_2) = s_1$.

**Proposition 1.** *Automorphism $f$ has order 2.*

*Proof.* Let $a \in G$; then $a = \prod_{k=1}^{n} s_1^{\lambda_k} s_2^{\mu_k}$. We have $f^2(a) = f(f(a)) = f(f(\prod_{k=1}^{n} s_1^{\lambda_k} s_2^{\mu_k})) = f(\prod_{k=1}^{n} s_2^{\lambda_k} s_1^{\mu_k}) = \prod_{k=1}^{n} s_1^{\lambda_k} s_2^{\mu_k} = a$. It follows that the order of $f$ is two. □

In order to prove the main result of this section, we define an automorphism $(\tau;\tau^{\#}) : (V;E) \longrightarrow (V;E)$ such that $\tau((s_i)x) = (f(s_i))f(x)$, for $s_i \in S, x \in G$. It is obvious that $(\tau;\tau^{\#}) \in Aut(\Gamma)$. From the definition it is easy to see that the order of $\tau$ is two.

Let $Z_2$ be the group formed by the set $\{(id;id^{\#});(\tau;\tau^{\#})\}$ under the composition law. It is obvious that this group is isomorphic with $\mathbb{Z}/2\mathbb{Z}$.

Let $G^*$ be the group constructed in the following way: $G^* = \delta_G \rtimes \mathbb{Z}/2\mathbb{Z}$, with the action $k : \mathbb{Z}/2\mathbb{Z} \times \delta_G \longrightarrow \delta_G$ such as $k(0,(\delta_g,\delta_g^{\#})) = (\delta_g,\delta_g^{\#})$ and $k(1,(\delta_g,\delta_g^{\#})) = (\tau\circ\delta_g,\tau^{\#}\circ\delta_g^{\#})$.

**Lemma 4.** *$G^*$ is an internal semi-direct product.*

*Proof.* $G^* = \delta_G \rtimes \mathbb{Z}/2\mathbb{Z}$ is an internal semi-direct product if and only if [8]:

- $G^* = \delta_G \times \mathbb{Z}/2\mathbb{Z} = \delta_G Z_2$;
- $\delta_G$ is normal in $G^*$
- $\delta_G \cap Z_2 = \{(id;id^{\#})\}$.

Since $Z_2 = \{(id;id^{\#});(\tau;\tau^{\#})\}$ it is well known from [8] that $\delta_G Z_2 = \delta_G \times \mathbb{Z}/2\mathbb{Z}$ if and only if $\delta_G Z_2 = Z_2 \delta_G$.
Let $(\delta_g,\delta_g^{\#}) \in \delta_G$. We have $(\tau\circ\delta_g)((s_1)x) = \tau(\delta_g((s_1)x)) = \tau((s_1)xg) = (f(s_1))f(x)f(g) = (s_2)f(x)f(g) = \delta_{f(g)}((s_2)f(x)) = \delta_{f(g)}((f(s_1))f(x)) = \delta_{f(g)}(\tau((s_1)x)) = \delta_{g'}(\tau((s_1)x)) = (\delta_{g'}\circ\tau)((s_1)x)$. Hence $\delta_G Z_2 = Z_2 \delta_G$.
$\delta_G$ is normal in $G^*$ if and only if $\tau\circ\delta_G\circ\tau \subseteq \delta_G$.
For $g \in G$ we have $(\tau\circ\delta_g\circ\tau)((s_1)x) = \tau(\delta_g((f(s_1))f(x))) = \tau(\delta_g((s_2)f(x))) = \tau((s_2)f(x)g) = (f(s_2))f(f(x)g) = (s_1)f(f(x))f(g) = (s_1)xf(g) = \delta_{f(g)}((s_1)x) = \delta_{g_2}((s_1)x)$ which is in $\delta_G$.
It is obvious that as $\tau(V_1) = V_2$, by Theorem 1, $\tau$ is not in $\delta_G$. It follows that $\delta_G \cap Z_2 = \{(id,id^{\#})\}$. □

Let $H$ be the following group: $H = \Psi(G^*) = \Psi(\delta_G \rtimes \mathbb{Z}/2\mathbb{Z})$.

**Theorem 3.** *Let G be a group having a generating set $S = \{s_1,s_2\}$ with $o(s_1) = o(s_2) = p \geq 3$ and an automorphism $f$ that swaps the two elements contained in S, and $\Gamma = \Phi(G;S) = (V;E)$ be a G-graph. Let $I\Gamma = (IV = V \cup E; IE)$ be the incidence graph of $\Gamma$. Then $I\Gamma$ is a G-graph, $I\Gamma = \Phi(H;S')$ with $S' = \{s_1',s_2'\}$ and $o(s_1') = p$, $o(s_2') = 2$.*

*Proof.* We shall prove that the group $H = \Psi(G^*) \subseteq Aut(I\Gamma)$ satisfies all conditions from Theorem 2-*G-graph characterization*.

*Condition* 1 - *The partition $IV = V \cup E$, is an orbit partition regarding to H.*
Let $x,y \in V$ be two vertices of graph $\Gamma = \Phi(G;S)$. There are two possibilities:

- both $x$ and $y$ are in the same partition (for example) $V_1$. Since group $\delta_G$ acts transitively on this vertex partition, there exists $g \in V_1$ such as $\delta_g(x) = y$.
- $x$ and $y$ are in different partitions; assume $x \in V_1$ and $y \in V_2$. Since $x \in V_1$, then $x' = \tau(x)$ belongs to $V_2$ and there exists $(\delta_{g_2},\delta_{g_2}^{\#}) \in \delta_G$ such that $\delta_{g_2}(x') = y$.
  By compounding the two functions we obtain $\delta_{g_2}(\tau(x)) = y$. Hence $G^*$ acts transitively on all $V$.
  As $\Psi$ is an isomorphism, $H = \Psi(G^*)$ acts transitively on $V$.

It follows that $G^*$ acts transitively on $V$ and $G^*$ acts transitively on $E$ (by Theorem 1), and since $G^*$ is isomorphic with $H = \Psi(G^*)$, $H$ acts transitively on $V$ and on $E$. In conclusion $I\Gamma = (IV, IE)$ is an orbit partitionable graph regarding to $H$, with the partition $IV = V \cup E$.

*Condition* 2 - *The graph* $I\Gamma$ *contains a clique with* 2 *vertices,* $K = \{x_1, x_2\}$.
As $I\Gamma$ is an incidence graph, it is obvious that it contains a clique with 2 elements (such that $x_1 \in V$, $x_2 \in E$).

*Condition* 3 - $Stab_H x_i$, $i \in \{1,2\}$ *are cyclic groups, pairwise distinct.*
Let $x \in V$ be a vertex of $\Gamma$. The subgroup $Stab_{\delta_G} x$ is cyclic of order $p$. We have $Stab_H x = \Psi(Stab_{G^*} x) = Stab_{\Psi(G^*)} x = Stab_{G^*} x$. But as $G^*$ is isomorphic to $\delta_G Z_2$ and $\tau(V_1) = V_2$, $\tau(V_2) = V_1$, it follows that $Stab_{G^*} x = Stab_{\delta_G} x$. Then $Stab_H x = Stab_{\delta_G} x$ is a cyclic group and its order is equal to $p$.
Let $a = (s_1)x(s_2)y \in E$ be an edge. We have $Stab_H a = \Psi(Stab_{G^*} a) = Stab_{\Psi(G^*)} a = Stab_{G^*} a$. $G^*$ is isomorphic with $\delta_G Z_2$ and $Stab_{\delta_G} a = \{(id, id^{\#})\}$. Let $(v, v^{\#}) \in Stab_{G^*} a$, $v = \tau \circ \delta_g$ and $v^{\#} = \tau^{\#} \circ \delta_g^{\#}$. We have $\tau^{\#}(\delta_g^{\#}(a)) = a$. This leads to $\tau^{\#}(\delta_g^{\#}((s_1)x(s_2)y)) = \tau^{\#}((s_1)xg(s_2)yg) = (f(s_1))f(xg)(f(s_2))f(yg) = (s_2)f(xg)(s_1)f(yg) = (s_1)x(s_2)y$. Then $f(xg) = f(x)f(g) = y$ and $f(yg) = f(y)f(g) = x$. Because the order of $f$ is 2 we have: $xg = f(y)$ and $yg = f(x)$, hence $xgf(g) = f(y)f(g) = x$.
Consequently $gf(g) = e$ and $f(g) = g^{-1}$ and $g = f(g^{-1})$, hence $f(g)f(g^{-1}) = g^{-1}g = e$. Because $f$ is an automorphism of order 2 we have $f(g^{-1}) = f(g)$, hence $g^{-1} = g$ and it follows that $f(g) = g$. Moreover, as $f|_{V_1} = V_2$, $f|_{V_2} = V_1$ the only fix point of $f$ is $e$ the identity. In conclusion $g = e$, and we have: $(v, v^{\#}) = (\tau, \tau^{\#})$, hence $Stab_{G^*} a = \{(id, id^{\#}); (\tau, \tau^{\#})\}$. Consequently $Stab_{G^*} a$ is a cyclic group of order 2.
Conclusion: for all $x \in V$, $|Stab_H x| = p$ and for all $a \in E$, $|Stab_H a| = 2$.

*Condition* 4 - $\frac{o(H)}{|Stab_H x_1|} = |V|$, $\frac{o(H)}{|Stab_H x_2|} = |E|$.
As we proved in Condition 3, for all $x \in V$ we have $|Stab_H x| = p$ and for all $a \in E$, $|Stab_H a| = 2$. We can easily prove that $o(H) = 2o(G)$, since $H = \Psi(\delta_G \rtimes Z_2)$ and $\Psi$ is an isomorphism. Moreover, by Lemma 1, $\Gamma$ is $p$-regular. Hence $|V| = \frac{o(H)}{|Stab_H x|}$ and $|E| = \frac{|V| \cdot p}{2} = \frac{\frac{o(H)}{p} p}{2} = \frac{o(H)}{|Stab_H a|}$.

To conclude, we have shown that $H$ satisfy for $I\Gamma$ all the four conditions of *Theorem - G-graph characterization*. So $I\Gamma$ is a $G$-graph. □

## 5 Mesh of d-ary tree

In this paper, we are only interested in the mesh of $d$-ary trees in one dimension, $MT(d,1)$. Let $B$ an alphabet of $d$-letters. We denote by $|u|$ the length of the word $u$. The *mesh of d-ary trees* $MT(d,1)$ is the graph with the vertex set $V = \{(u,v); |u| = 1 \text{ or } |v| = 1\}$, and $[(u,v),(u',v')] \in E(MT(d,1))$ if and only if $|u| = 1$ ,$u = u'$ and $v = v'\lambda$ or $|v| = 1$, $v = v'$ and $u = u'\lambda$ with $\lambda \in B$. The key properties of meshes of $d$-tree graphs are as follows:

- Number of vertices $N_v = d(d+2)$; number of edges $N_e = 2d(\frac{d^2-1}{d-1} - 1)$; diameter $D = 4$;
- The mesh of $d$-ary trees is not a Cayley graph.
- The mesh of $d$-ary trees is not vertex-transitive.

These graphs are very important in interconnection networks.
In the sequel we consider $n, k \geq 2$. Let $\mathbb{Z}_n \times \mathbb{Z}_k$ be the product of two cyclic groups. Such a product is generated by two elements, $a$ and $b$, with $a^n = b^k = 1$. More precisely, $\mathbb{Z}_n \times \mathbb{Z}_k$ is the group with description $\langle a, b \mid a^n = 1, b^k = 1, ab = ba \rangle$. In [2] the following was shown:

**Proposition 2.** *For* $S = \{a, b\}$*, the graph* $\Phi(\mathbb{Z}_n \times \mathbb{Z}_k, S)$ *of the product of two cyclic groups, is the complete bipartite graph* $K_{n,k}$.

It follows that for $n = k$ and $S = \{s_1 = (1,0), s_2 = (0,1)\}$ the graph $\Phi(\mathbb{Z}_n \times \mathbb{Z}_n, S)$ is a complete bipartite graph with degree equal to $n$. Consider the action $\varphi : \mathbb{Z}_2 \times (\mathbb{Z}_n \times \mathbb{Z}_n) \longrightarrow \mathbb{Z}_n \times \mathbb{Z}_n$ such that $\varphi(0,(x,y)) = (x,y)$ and $\varphi(1,(x,y)) = (y,x)$. From theorem of *Incidence Graphs* the graph $\Phi((\mathbb{Z}_n \times \mathbb{Z}_n) \times_\varphi \mathbb{Z}_2, S)$ is a $G$-graph. The following result was established in [5]

**Theorem 4.** *The G-graph $\Phi((\mathbb{Z}_n \times \mathbb{Z}_n) \times_\varphi \mathbb{Z}_2, S)$ is isomorphic to $MT(n,1)$.*

Then we have the following result.

**Theorem 5.** *The 1-dimensional mesh of n-ary trees is the incidence graph of $K_{n,n}$.*

*Proof.* The result follows from Proposition 2 and Theorems 3, 4. □

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# Minimal doubly resolving sets of some generalized Petersen graphs

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**Abstract**

In this paper we consider the problem of determining the minimal cardinality of double resolving sets for a class of generalized Petersen graphs $GP(n,1)$. It is proved that the minimal cardinality is equal to 4 if $n$ is even and equal to 3 if $n$ is odd.

## 1 Introduction

The metric dimension problem, introduced independently by Slater [12] and Harary [5], has been widely investigated. It arises in many diverse areas including network discovery and verification [1], geographical routing protocols [11], the robot navigation, connected joints in graphs, chemistry, etc.

Given a simple connected undirected graph $G = (V,E)$, where $V = \{1,2,\ldots,n\}$, $|E| = m$, $d(u,v)$ denotes the distance between vertices $u$ and $v$, i.e. the length of a shortest $u-v$ path. A vertex $x$ of graph $G$ is said to resolve two vertices $u$ and $v$ of $G$ if $d(x,u) \neq d(x,v)$. A vertex set $S = \{x_1,x_2,\ldots,x_k\}$ of $G$ is a *resolving set* of $G$ if every two distinct vertices of $G$ are resolved by some vertex of $S$. Given a vertex $t$, the $k$-touple $r(t,S) = (d(t,x_1),d(t,x_2),\ldots,d(t,x_k))$ is called the vector of metric coordinates of $t$ with respect to $S$. A *metric basis* of $G$ is a resolving set of the minimum cardinality. The *metric dimension* of $G$, denoted by $\beta(G)$, is the cardinality of its metric basis.

It was proved that the problem of computing the metric dimension of an arbitrary graph is NP-hard [7]. In [8] a genetic algorithm (GA) is used to approximate $\beta(G)$ for different classes of graph instances, such as pseudo-boolean, crew scheduling, graph coloring, hypercubes and Hamming graphs. In [9] a GA approach is used to obtain upper bounds for the metric dimension of large hypercubes up to $2^{90}$ nodes.

The concept of a doubly resolving set of a graph $G$ has been recently introduced by Cáceres et al. [3], who proved that the metric dimension of the Cartesian product $G\Box G$ is tied in a strong sense to doubly resolving sets of $G$ with the minimum cardinality. Vertices $x,y$ of the graph $G$ ($n \geq 2$) are said to doubly resolve vertices $u,v$ of $G$ if $d(u,x) - d(u,y) \neq d(v,x) - d(v,y)$. A vertex set $D$ of $G$ is a *doubly resolving set* of $G$ if every two distinct vertices of $G$ are doubly resolved by some two vertices of $D$, i.e. if there are no two distinct vertices of $G$ with the same difference between their corresponding metric coordinates with respect to $D$. The minimal doubly resolving set problem consists of finding a doubly resolving set of $G$ with the minimum cardinality, denoted by $\psi(G)$.

In [9] it has been proved that the minimal doubly resolving set problem is NP-hard and a genetic algorithm for handling large-scale instances has been developed.

Note that if $x,y$ doubly resolve $u,v$ then $d(u,x) - d(v,x) \neq 0$ or $d(u,y) - d(v,y) \neq 0$, and hence $x$ or $y$ resolves $u,v$. Therefore, a doubly resolving set is also a resolving set and $\beta(G) \leq \psi(G)$.

*Example 1.* Consider graph $G_1$ of Figure 1. Set $S_1 = \{A,B,C\}$ is a resolving set of $G_1$ since the vectors of metric coordinates for the vertices of $G_1$ with respect to $S_1$ are: $r(A,S_1) = (0,1,1)$; $r(B,S_1) = (1,0,2)$; $r(C,S_1) = (1,2,0)$; $r(D,S_1) = (2,1,1)$; $r(E,S_1) = (1,2,1)$; $r(F,S_1) = (2,1,2)$. It is easy to check that $S_1$ is also a doubly resolving set.

Figure 1: Graph in Example 1

However, $S_1$ is not a minimal resolving set since $S_2 = \{A,C\}$ is also a resolving set with smaller cardinality. On the other hand, set $S_3 = \{A\}$ is not a resolving set since $d(B,A) = d(C,A) = 1$. Using a similar argument it is easy to check that none of singleton vertices forms a resolving set, and hence $\beta(G_1) = 2$.

Note that $S_2$ is not a double resolving set because $d(F,A) - d(E,A) = d(F,C) - d(E,C) = 1$. Similarly, we can show that none of the subsets of two vertices forms a doubly resolving set. Thus, $S_1$ is a minimal double resolving set and $\psi(G_1) = 3$. □

Generalized Petersen graph $GP(n,k)$ ($n \geq 3$; $1 \leq k < n/2$) has $2n$ vertices and $3n$ edges, with vertex set $V = \{u_i, v_i | 0 \leq i \leq n-1\}$ and edge set $E = \{\{u_i, u_{i+1}\}, \{u_i, v_i\}, \{v_i, v_{i+k}\}\} | 0 \leq i \leq n-1\}$, where vertex indices are taken modulo $n$.

Generalized Petersen graphs were first studied by Coxeter [4]. Various properties of $GP(n,k)$ have been recently theoretically investigated in areas: isomorphism classes [13], distance-balanced property [16], domination number [15], minimum vertex cover [2], Roman domination [14], strongly distance balanced property [10].

It is easy to see that $GP(n,1) \cong C_n \Box P_2$, where $\Box$ is *the Cartesian product* of graphs. In [3] it has been proved that $\beta(C_n \Box P_2) = \begin{cases} 2, & n \text{ is odd} \\ 3, & n \text{ is even} \end{cases}$, which gives the value for $\beta(GP(n,1))$. Moreover, it has been proved in [6] that $\beta(GP(n,2)) = 3$ for $n \geq 5$. In this paper we determine $\psi(GP(n,1))$.

## 2 The minimal doubly resolving sets for $GP(n,1)$

In this section we consider the problem of determining the minimal cardinality of doubly resolving sets for a class of generalized Petersen graphs $GP(n,1)$ (see Figure 2).

Figure 2: $GP(5,1)$

It has been proved that the metric dimension $\beta(GP(n,1))$ is equal to

$$\beta(GP(n,1)) = \begin{cases} 2, & n \text{ is odd} \\ 3, & n \text{ is even} \end{cases}.$$

As, according to definition of doubly resolving sets, $\psi(GP(n,1) \geq \beta(GP(n,1)$, then

$$\psi(GP(n,1)) \geq \begin{cases} 2, & n \text{ is odd} \\ 3, & n \text{ is even} \end{cases}.$$

The main result of this section is the following theorem:

**Theorem 1.** *For $n \geq 3$, $\psi(GP(n,1)) = 3$ if $n$ is odd and $\psi(GP(n,1)) = 4$ if $n$ is even.*

The proof of the theorem uses the next three lemmas.

| $n$ | $i$ | $S_i(u_0)$ |
|---|---|---|
| | 1 | $\{u_{\pm 1}, v_0\}$ |
| | $2 \le i < \lfloor n/2 \rfloor$ | $\{u_{\pm i}, v_{\pm(i-1)}\}$ |
| $2t$ | $t$ | $\{u_t, v_{\pm(t-1)}\}$ |
| | $t+1$ | $\{v_t\}$ |
| $2t+1$ | $t$ | $\{u_{\pm t}, v_{\pm(t-1)}\}$ |
| | $t+1$ | $\{v_{\pm t}\}$ |

Table 1: $S_i(u_0)$ for $GP(n,1)$

**Lemma 1.** *Let $S_i(u_0) = \{w \in V : d(u_0,w) = i\}$ be the set of vertices in $V$ at distance $i$ from $u_0$. For $GP(n,1)$ with $n \ge 6$ sets $S_i(u_0)$ are given explicitly in Table 1, where $u_{-k}$, $v_{-k}$ denote $u_{n-k}$, $v_{n-k}$, respectively.*

**Proof.** By definition of $S_i(u_0)$ it is easy to see that $S_i(u_0)$ is the set of all $v \in V$ such that $\{v,a\} \in E$ for some $a \in S_{i-1}(u_0)$ and $v \notin S_l(u_0)$, $l \le i-1$. Now, using the definition of $GP(n,1)$, it immediately follows that $S_1(u_0) = \{u_1, u_{n-1}, v_0\}$. Let us prove by induction that, for $2 \le i < \lfloor n/2 \rfloor = t$

$$S_i(u_0) = \{u_i, u_{n-i}, v_{i-1}, v_{n-(i-1)}\} \tag{1}$$

For $i = 2$, starting from $S_1(u_0)$, we obtain $S_2(u_0) = \{u_2, u_{n-2}, v_1, v_{n-1}\}$. Suppose that (1) holds for $2 \le i \le l < \lfloor n/2 \rfloor - 1$. Starting from $S_l(u_0)$ we obtain the following candidates for members of $S_{l+1}(u_0)$: $v_{l-2}$, $v_{n-(l-2)}$, $u_{l-1}$, $u_{n-(l-1)}$, $v_l$, $v_{n-l}$, $u_{l+1}$, $u_{n-(l+1)}$. The first four vertices belong to $S_{l-1}(u_0)$ and consequently cannot belong to $S_{l+1}(u_0)$. For example, $v_{l-2} = v_{(l-1)-1} \in S_{l-1}(u_0)$ and $v_{n-(l-2)} = v_{n-((l-1)-1)} \in S_{l-1}(u_0)$. The set of remaining four vertices can be expressed in the form (1) for $i = l+1$. For example, $v_l = v_{(l+1)-1} \in S_{l+1}(u_0)$.

For $i \ge \lfloor n/2 \rfloor$, $S_i(u_0)$ depends on $n$. If $n = 2t$, starting from $S_{t-1}(u_0)$, we obtain the following candidates for $S_t(u_0)$: $v_{t-3}$, $v_{n-(t-3)}$, $u_{t-2}$, $u_{n-(t-2)}$, $v_{t-1}$, $v_{n-(t-1)}$, $u_t$. The first four vertices obviously belong to $S_{t-2}(u_0)$ and can be neglected. The set of remaining three vertices represents set $S_t(u_0)$ for $n = 2t$ from Table 1.

Similarly, starting from $S_t(u_0)$ the candidates for $S_{t+1}(u_0)$ are: $v_{t-2}$, $v_{n-(t-2)}$, $u_{t-1}$, $u_{n-(t-1)}$, $v_t$. As $v_{t-2}$, $v_{n-(t-2)}$, $u_{t-1}$, $u_{n-(t-1)} \in S_{t-1}(u_0)$, set $S_{t+1}(u_0)$ contains only $v_t$.

It is easy to see that a similar construction applied to $S_{t+1}(u_0)$ leads to $S_{t+2}(u_0) = \emptyset$

The expressions in Table 1 for $S_t(u_0)$ and $S_{t+1}(u_0)$ for $n = 2t+1$ can be verified in a similar way. □

Sets $S_i(u_0)$, defined in Lemma 1, can be used in order to find the distance between two arbitrary vertices from $V$. Namely, due to the symmetry of $GP(n,1)$, it follows that:

$$d(u_i,u_j) = d(v_i,v_j) = d(u_0,u_{j-i}),\ d(u_i,v_j) = d(u_0,v_{j-i})$$

Consequently, if we know the distance $d(u_0,w)$ for every $w \in V$, then we can reconstruct the distance between every two vertices from $V$.

**Lemma 2.** $\psi(GP(n,1)) > 2$ *for* $n = 2t+1$ *and* $t \ge 3$.

**Proof.** As for $n = 2t+1$ $\psi(GP(n,1)) \ge 2$ it is sufficient to prove that every subset $D$ of $V$ with $|D| = 2$ is not a doubly resolving set, i.e. there exists a pair of vertices from $V$ which are not doubly resolved by two vertices from $D$. Due to the symmetry of $GP(n,1)$ we can suppose that $u_0 \in D$. In Table 2 five possible types of set $D$ are presented and for each of them the corresponding non-doubly resolved pair of vertices from $V$ is found.

Let us prove that vertices $u_0$, $v_0$ are not doubly resolved by two vertices from set $D = \{u_0, u_i\}$, $0 < i \leq n-1$, i.e. that $u_0$ and $v_0$ have the same difference between their corresponding metric coordinates with respect to $D$. These coordinates can be found using the distances calculated in Lemma 1 and given in Table 1 in the following way.

Obviously, $d(u_0,u_0) = 0$ and $d(u_0,v_0) = 1$. For $1 \leq i \leq t$, $d(u_i,u_0) = i$ and $d(u_i,v_0) = d(u_0,v_{-i}) = i+1$. When $t+1 \leq i \leq n-1 = 2t$, then $1 \leq n-i \leq t$ and, therefore, $d(u_i,u_0) = d(u_{n-(n-i)},u_0) = n-i$ and $d(u_i,v_0) = d(u_0,v_{n-i}) = n-i+1$. So, $d(u_0,u_0) - d(u_0,v_0) = d(u_i,u_0) - d(u_i,v_0) = -1$, and $\{u_0,u_i\}$ is not a doubly resolving set of $G$.

In a similar way we can verify non-doubly resolved pairs of vertices for all other possible types of set $D$ given in Table 2. □

| $D$ | Non-doubly resolved pair |
|---|---|
| $\{u_0,u_i\}$, $0 < i \leq n-1$ | $u_0$, $v_0$ |
| $\{u_0,v_i\}$, $0 \leq i < t$ | $u_0$, $u_{n-1}$ |
| $\{u_0,v_t\}$ | $u_1$, $v_0$ |
| $\{u_0,v_{t+1}\}$ | $u_{n-1}$, $v_0$ |
| $\{u_0,v_i\}$, $t+1 < i \leq n-1$ | $u_0$, $u_1$ |

Table 2: Non-doubly resolved pairs for $n = 2t+1$

**Lemma 3.** *$\psi(GP(n,1)) > 3$ for $n = 2t$ and $t \geq 3$.*

**Proof.** For $n = 2t$, $\psi(GP(n,1)) \geq 3$ and, therefore, we should prove that every subset $D$ of $V$ with $|D| = 3$ is not a doubly resolving set of $G$. We can suppose that $u_0 \in D$. In Table 3 we recognize ten possible types of set $D$ and for each of them a non-doubly resolved pair of vertices from $V$ is given.

Let us prove that vertices $u_0$ and $u_{n-1}$ are not doubly resolved by any two vertices from set $D = \{u_0,u_i,v_j\}$, $0 < i < t$, $0 \leq j < t$. Using distances from Table 1 it follows that: $d(u_0,u_0) = 0$, $d(u_0,u_{n-1}) = 1$, $d(u_i,u_0) = i$ and $d(u_i,u_{n-1}) = d(u_0,u_{n-(1+i)}) = 1+i$ for $0 < i < t$.

Also, $d(v_j,u_0) = j+1$ for $0 \leq j < t$. As $d(v_j,u_{n-1}) = d(v_{j-n+1},u_0)$ and $-(n-1) \leq j-n+1 \leq -t$ for $0 \leq j < t$, then $d(v_j,u_{n-1}) = d(v_{n+(j-n+1)},u_0) = d(v_{j+1},u_0) = j+2$ for $0 \leq j < t$.

Consequently, $d(u_0,u_0) - d(u_0,u_{n-1}) = d(u_i,u_0) - d(u_i,u_{n-1}) = d(v_j,u_0) - d(v_j,u_{n-1}) = -1$, i.e. $\{u_0,u_i,v_j\}$ is not a doubly resolving set of $G$.

Similarly, we can consider all other types of set $D$ from Table 3 and verify their corresponding non-doubly resolved pairs of vertices.□

Let us define set $D^*$ as

$$D^* = \begin{cases} \{u_0,u_1,u_t,v_0\}, & n = 2t \\ \{u_0,u_t,v_t\}, & n = 2t+1 \end{cases}.$$

In the next lemma we determine the vectors of metric coordinates with respect to set $D^*$ for all vertices of $GP(n,1)$.

**Lemma 4.** *The vectors of metric coordinates with respect to set $D^*$ for all vertices of $GP(n,1)$, $n \geq 6$, are given in Table 4.*

**Proof.** Using the symmetry of $GP(n,1)$ and results from Table 1, distances from any vertex of $GP(n,1)$ to vertices of set $D^*$, i.e. its vector of metric coordinates with respect to $D^*$, can be easily determined.

| $D$ | Non-doubly resolved pair |
|---|---|
| $\{u_0,u_i,u_j\}$, $0<i\neq j\leq n-1$ | $u_0$, $v_0$ |
| $\{u_0,u_i,v_j\}$, $0<i<t$, $0\leq j<t$ | $u_0$, $u_{n-1}$ |
| $\{u_0,u_i,v_j\}$, $t<i$, $j\leq n-1$ | $u_0$, $u_1$ |
| $\{u_0,u_i,v_j\}$, $0<i<t$, $t\leq j\leq n-1$ | $u_{n-1}$, $v_0$ |
| $\{u_0,u_i,v_j\}$, $t<i\leq n-1$, $0<j\leq t$ | $u_1$, $v_0$ |
| $\{u_0,u_t,v_j\}$, $0<j<t$ | $u_1$, $v_{n-1}$ |
| $\{u_0,u_t,v_j\}$, $t<j\leq n-1$ | $u_{n-1}$, $v_1$ |
| $\{u_0,u_t,v_t\}$ | $v_{t-1}$, $v_{t+1}$ |
| $\{u_0,u_t,v_0\}$ | $u_1$, $u_{n-1}$ |
| $\{u_0,u_i,v_0\}$, $t<i\leq n-1$ | $v_0$, $v_1$ |

Table 3: Non-doubly resolved pairs for $n=2t$

Let us show how we can find distances from vertex $v_{n-i+1}$ to all vertices of set $D^*$, for $2\leq i<t$ and $n=2t,\ 2t+1$.

As $d(u_0,v_{n-i+1})=d(u_0,v_{n-(i-1)})$ and $1\leq i-1<t-1$, then it follows from Table 1 that $d(u_0,v_{n-i+1})=i$ for all values of $n$. Also, according to this table, $d(u_1,v_{n-i+1})=d(u_0,v_{n-i})=i+1$ for all values of $n$.

As $d(u_t,v_{n-i+1})=d(u_0,v_{n-i+1-t})$, then, for $n=2t$, $d(u_t,v_{n-i+1})=d(u_0,v_{t-i+1})$. Since $2\leq t-i+1\leq t-1$, it follows that $d(u_t,v_{n-i+1})=t-i+2$. For $n=2t+1$, from $d(u_t,v_{n-i+1})=d(u_0,v_{t-i+2})$ and $3\leq t-i+2\leq t$, it follows $d(u_t,v_{n-i+1})=t-i+3$.

As $d(v_0,v_{n-i+1})=d(u_0,u_{n-i+1})=d(u_0,u_{n-(i-1)})$ for $1\leq i-1<t-1$, then, according to Table 1, $d(v_0,v_{n-i+1})=i-1$ for all $n$. Finally, as $d(v_t,v_{n-i+1})=d(u_t,u_{n-i+1})=d(u_0,u_{n-i+1-t})$, then, for $n=2t+1$, $d(v_t,v_{n-i+1})=d(u_0,u_{t-i+2})$. Since $3\leq t-i+2\leq t$, then, directly from Table 1, $d(v_t,v_{n-i+1})=t-i+2$.

In this way we have obtained the vector of metric coordinates for $v_{n-i+1}$ with respect to $D^*$, presented in Table 4. In a similar way we can find the vectors of metric coordinates for all remaining vertices. □

| $i$ | $S_i(u_0)$ | $n=2t$<br>$D^*=\{u_0,u_1,u_t,v_0\}$ | $n=2t+1$<br>$D^*=\{u_0,u_t,v_t\}$ |
|---|---|---|---|
| 0 | $u_0$ | $(0,1,t,1)$ | $(0,t,t+1)$ |
| 1 | $u_1$ | $(1,0,t-1,2)$ | $(1,t-1,t)$ |
| | $u_{n-1}$ | $(1,2,t-1,2)$ | $(1,t,t+1)$ |
| | $v_0$ | $(1,2,t+1,0)$ | $(1,t+1,t)$ |
| $2\leq i<t$ | $u_i$ | $(i,i-1,t-i,i+1)$ | $(i,t-i,t-i+1)$ |
| | $u_{n-i}$ | $(i,i+1,t-i,i+1)$ | $(i,t-i+1,t-i+2)$ |
| | $v_{i-1}$ | $(i,i-1,t-i+2,i-1)$ | $(i,t-i+2,t-i+1)$ |
| | $v_{n-i+1}$ | $(i,i+1,t-i+2,i)$ | $(i,t-i+3,t-i+2)$ |
| $t$ | $u_t$ | $(t,t-1,0,t+1)$ | $(t,0,t+1)$ |
| | $u_{n-t}$ | ——— | $(t,1,2)$ |
| | $v_{t-1}$ | $(t,t-1,2,t-1)$ | $(t,2,1)$ |
| | $v_{n-t+1}$ | $(t,t+1,2,t-1)$ | $(t,3,2)$ |
| $t+1$ | $v_t$ | $(t+1,t,1,t)$ | $(t+1,1,0)$ |
| | $v_{n-t}$ | ——— | $(t+1,2,1)$ |

Table 4: Vectors of metric coordinates

**Proof of Theorem 1.** Starting from Table 4, it is easy to verify that, for $n \geq 6$, $D^*$ is a doubly resolving set of $GP(n,1)$, i.e. there do not exist two vertices $x, y \in V$ such that the vector $r(x,D^*) - r(y,D^*)$ has the same coordinates. Really, since $u_0 \in D^*$, then the first metric coordinate of a vertex from $S_i(v_0)$ with respect to $D^*$ is equal to $i$. It is easy to check that for each $i \in \{1,\ldots,t+1\}$ there do not exist two vertices $x, y \in S_i(u_0)$ such that all coordinates of the vector $r(x,D^*) - r(y,D^*)$ are equal to 0. Also, it can be easily seen that, for each $i, j \in \{1,\ldots,t+1\}, i \neq j$, there do not exist two vertices $x \in S_i(u_0)$ and $y \in S_j(u_0)$ such that all coordinates of the vector $r(x,D^*) - r(y,D^*)$ are equal to $i - j$.

Since $D^*$ is a doubly resolving set for $n \geq 6$, and $|D^*| = 4$ for $n = 2t$ and $|D^*| = 3$ for $n = 2t+1$, then, according to Lemmas 2 and 3, $D^*$ is a minimal doubly resolving set of $GP(n,1)$ and, consequently, for $n \geq 6$, $GP(n,1) = 3$ if $n$ is odd and $GP(n,1) = 4$ if $n$ is even.

In order to determine $\psi(GP(n,1))$ for $n = 3,4,5$ we used a total enumeration technique. In all these cases set $D^*$ remains the minimal doubly resolving set of $GP(n,1)$. Therefore Theorem 1 holds also for $n = 3,4,5$.□

# 3 Conclusions

In this paper we theoretically determine the minimal cardinality $\psi(GP(n,1))$ of doubly resolving sets for the class $GP(n,1)$ of generalized Petersen graphs. The future work will be focused on determining the strong metric dimension of $GP(n,1)$.

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# Generalization of a multi-period knapsack problem

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**Abstract**

This paper looks at a generalized version of the multi-period knapsack problem introduced by Faaland in 1981. This problem is inspired by a specific real-life situationcorresponds to the case where a set of hardware items is to be managed and their replacement dates determined, given a budget over a time horizon comprising a set of periods. The particular characteristic of this problem is the possibility of carrying forward any unused budget from one period to the next, which corresponds to the multi-periodicity aspect in the model. We call it the multi-period renewal equipment problem (MPR). We begin with establishing links to the Faaland's multiperiod knapsack problem. Next, some complexity results are presented. In particular, we show that the MPR problem is strongly NP-hard when the number of periods is unbounded. Next, two heuristics for solving the MPR problem and some experimental results are provided.

## 1 Introduction

The *Knapsack Problem* (KP) has a central place among the integer programming models with binary variables. In the standard knapsack problem the quantity $\sum_{i\in N} p_i x_i$ is to be maximized subject to the constraint $\sum_{i\in N} w_i x_i \leq b$, where $x_i \in \{0,1\}$, $N = \{1,2,\ldots,n\}$, $p_i$ is the *value* or *profit* of item $i$, $w_i$ is the *weight* of item $i$ and $b$ is the knapsack *capacity*, all assumed to be non-negative. In addition to the standard problem, a number of different variants of the problem have been put forward and investigated by researchers over the last decades. This paper looks at one of these variants that we have called the **Multi-Period Renewal equipment problem** (MPR), which generalizes the multiperiod knapsack problem introduced by Faaland in 1981. It is inspired by a specific real-life situation where a set of hardware items is to be managed and their replacement dates determined, given a budget over a time horizon comprising a set of periods. The particular characteristic of this variant of the problem is the possibility of carrying forward any unused budget from one period to the next, corresponding to the multi-periodicity aspect in the statement of the problem. The contribution of this paper is a new knapsack model originating from a real industrial context, together with a complexity study and two efficient heuristics for solving it.

**Paper organization.** The paper is organized as follows. In Section 2 we present the mathematical model and links to the multiperiod knapsack problem of Faaland. In Section 3 we investigate the computational complexity of the MPR problem. In Section 5 we propose two heuristics for the MPR problem and provide a comparative experimental study of them.

## 2 Context and formulation

This problem has been encountered when elaborating a maintenance strategy of a set of equipments under a contractual obligation to spend a given amount of money ($B$) on their maintenance and renewal during a given time horizon $M$. An equipment is assumed to be replaced at most once over $M$. In line with an internal budgeting policy the company allots an annual budget $b_j$ to such expenditure, that is $\sum_{j\in M} b_j = B$.

Given that the entire budget has to be used up, the company carries any unused budget at year $j$ over to the following years. For each piece of equipment the replacement cost is taken to be constant over time, and the profit attributable to the replacement is calculated according to a formula based on such elements as the probability of failure, the expected lifetime of the equipment, its importance in the industrial process, etc. Hence, the profit change along with time and this change corresponds to a certain deterioration process [1]. Thus the related cost, profit and budget coefficients are assumed to be known with certainty, in contrast to conventional renewal theory which relies on probability theory, (see for instance Cox [2]).

Before formulating the mathematical model of the MPR problem, let us give the notation used throughout the paper:
Let $N$ be a set of $n$ equipments, and $M$ a horizon of $m$ periods. $x_{i,j}$ is the assignment decision variable, that is to say $x_{i,j} = 1$ if equipment $i$ is replaced in period $j$, and 0 otherwise. $p_{i,j}$ is the *profit* obtained when replacing equipment $i$ at period $j$; $w_i$ is the *replacement cost*, of equipment $i$ (it remains unchanged over periods), and $b_j$ gives the *budget* alloted to period $j$ and $B_j = \sum_{t=1}^{j} b_t$ gives the cumulative budget. From now on, in line with the notation commonly used for knapsack models, we shall use the term *item* for equipment, *weight* instead of replacement cost, and *capacity* for budget.

All these data are assumed to take non-negative values. Our problem can be mathematically formulated as follows:

$$(P) \qquad \max \sum_{j\in M}\sum_{i\in N} p_{i,j}x_{i,j} \tag{1}$$

$$\sum_{t=1}^{j}\sum_{i\in N} w_i x_{i,t} \leq \sum_{t=1}^{j} b_t = B_j, \qquad \forall j \in M, \tag{2}$$

$$\sum_{j\in M} x_{i,j} \leq 1, \qquad \forall i \in N, \tag{3}$$

$$x_{i,j} \in \{0,1\}, \quad \forall i \in N, j \in M. \tag{4}$$

In the above model the sum of the weights of all items chosen from period 1 to period $j$ cannot exceed the cumulated capacity $B_j$ for all $j \in M$ (2). Each item $i$ can be assigned to at most one period $j$ (3). The multi-period aspect lies in the fact that each constraint involves the current period and all preceding ones. The total profit is to be maximized (1). As far as we know, we were the first to model the problem in [3].

**Links with Faaland's multiperiod knapsack problem.** Let us recall first the *Multi-Period Knapsack problem* (MPK) introduced by Faaland in 1981 [4]. Faaland considers a set $N$ of items and a set $M$ of periods. To each period $j \in M$ there corresponds a subset $N_j = \{j \in 1,...,m\}$ of items that can be assigned to this period. Note that $\bigcup_{j=1}^{m} N_j = N$ and $N_k \cap N_j = \emptyset$ for each pair $(j,k)|j \neq k$ of items. For each item $i$, a *profit* $p_i$ and a *weight* $w_i$ are given. The cumulative weight of all items chosen from period 1 to period $j$ cannot exceed the *capacity* $B_j$ associated with period $j$. The total profit has to be maximized by selecting items in their associated periods. The decision variables in this problem are unbounded: an item $i$ may be chosen more than once in period $j$ such that $i \in N_j$. Faaland proposed a polynomial algorithm to solve exactly the continuous relaxation of MPK, and in so doing to compute an upper bound of MPK. He also proposed a *branch and bound* algorithm using this upper bound.

The **binary version** of this problem, in which an item is chosen at most once in its associated period,

is called BMPK. Given the above notation, BMPK can be formulated as follows:

$$\max \sum_{j \in M} \sum_{i \in N_j} p_i x_i \tag{5}$$

$$\sum_{t=1}^{j} \sum_{i \in N_t} w_i x_i \leq B_j, \quad \forall j \in M, \tag{6}$$

$$x_i \in \{0,1\}, \quad \forall i \in N. \tag{7}$$

Thus the weight of any chosen item will impact subsequent periods, given that in each period the cumulative weight is considered. The overall profit is maximized by choosing items in each period (5), without violating the cumulative capacity constraints (6). It now becomes apparent that BMPK and MPR have some similarities. What is different is that items in BMPK can be only assigned to a single (i.e. its associated) period. As we shall show, BMPK can be seen to be a special case of MPR (see Section 3.1), in which an item may be chosen on at most one occasion.

# 3 Complexity

The formulation $P$ (1-4) describes the general case of *MPR*. We focus particularly on complexity issues related to our problem and discuss in detail the following special cases with respect to their computational complexity:

- ($P^1$): $p_{i,j}$ values are all 0 except for one period, for all $i \in N$;
- ($P^2$): $p_{i,j}$ values are *non-increasing* during the time horizon, for all $i \in N$.

As we will see, solving $P^1$ leads to a solution to the BMPK problem discussed in Section 2.1. We show in particular that $P^1$ and BMPK can be solved in pseudo-polynomial time. Finally, we show that $P^2$, (and MPR) are strongly NP-hard when the number of periods is unbounded.

## 3.1 Complexity of $P^1$ and BMPK

Let us first look at BMPK and compare it with $P^1$. Consider an instance of $P^1$ in which for each $i$, $p_{i,j}$ are all 0 except for one period. Now consider an optimal solution of this instance. In this solution, any item $i$ assigned to period $j$ for which $p_{i,j} = 0$ can be removed from the solution without decreasing the profit. Note that the obtained solution is also an optimal solution for the instance of BMPK, which is obtained in the following way:

- each item $i$ has the same weight $w_i$ and each period $j$ has the same capacity $B_j$ in both instances.
- in the BMPK instance, each item $i$ belongs to set $N_j$ such that $j$ is the only period for which $p_{i,j} \neq 0$ in $P^1$ instance.

Hence, for any optimal solution for $P^1$ an optimal solution for the corresponding BMPK problem can be deduced. Obviously, there is a bijection relation between instances of BMPK and these of $P^1$. Furthermore, the following proposition is straightforward.

**Proposition 1.** *Any optimal solution for BMPK is also optimal for the corresponding problem $P^1$.*

Let us now consider the complexity of BMPK.

**Proposition 2.** *The BMPK problem (and hence the $P^1$) can be solved in pseudo-polynomial time $O(nB_m)$.*

*Proof.* This problem can be solved through a dynamic programming schema slightly adapted from the classical schema used for the standard knapsack problem.

First, let $A_i(C)$ denote the maximum profit that can be obtained when considering only the first $i$ items and keeping the total cumulative weight lower than or equal to $C$. There is one restriction in our dynamic programming, namely that the items are introduced in the order determined by sets $N_j$ $(1 \leq j \leq m)$: that is to say that items are indexed from 1 to $n$ such that if $i \in N_j$ and $k \in N_l$ with $j < l$ then $i < k$. Initially, $A_0(C) = 0$ for all $C$ in $[0, B_m]$. Then, the following recursive formula is applied for any $i$ and $C \in [0, B_m]$, where $j$ is the period in which $i$ has to be assigned, (i.e., $i \in N_j$):

- $A_i(C) = A_{i-1}(C)$, for $C < w_i$;
- $A_i(C) = \max\{A_{i-1}(C), A_{i-1}(C - w_i) + p_i\}$, for $w_i \leq C \leq B_j$;
- $A_i(C) = A_i(B_j)$, for $B_j < C \leq B_m$.

Hence the computational complexity is in $O(nB_m)$ time. □

## 3.2 Complexity of $P^2$

We consider in this paragraph the special case $P^2$, where $p_{i,j}$ decrease throughout time horizon for any item $i$, $i \in N$. We show that $P^2$ is strongly NP-complete when the number of periods is unbounded. To show this we prove that the 3-Partition problem can be reduced to a special case of $P^2$.

**The $P^2$ decision problem.** Any solution of $P^2$ can be presented as an assignment of items to $m$ periods. Let $N_t = \{i | x_{i,t} = 1\}$ be the subset of items assigned to period $t$. The capacity constraints are $\sum_{t=1}^{j} \sum_{i \in N_t} w_i \leq B_j$, for all $j \in M$. Then, an instance can be defined as follows:
*Instance.* We have a finite set of items $\{1, 2, ..., n\}$, positive integers $V$, $m$, and an increasing sequence $B_1, B_2, \ldots, B_m$. For each item $i$, some positive weight $w_i$ and a set of positive profits $p_{i,1}, p_{i,2}, \ldots, p_{i,m}$ are given.
*Question.* Do $m$ disjoint subsets $N_1, N_2, ..., N_m$ of items exist such that $\sum_{t=1}^{m} \sum_{i \in N_t} p_{i,t} \geq V$ and for all $j \in \{1, \ldots, m\}$ we have $\sum_{t=1}^{j} \sum_{i \in N_t} w_{i,t} \leq B_j$?

**The 3-Partition decision problem.** 3-Partition is known to be a strongly NP-hard problem ([6]). The decision problem can be defined as follows:
*Instance.* We have a finite set A = $\{a_1, a_2, \ldots, a_{3q}\}$ of $3q$ numbers, a positive integer bound $B$ such that for each $a_i \in A$, $B/4 < a_i < B/2$ and $\sum_{i=1}^{3q} a_i = qB$.
*Question.* Can A be partitioned into $q$ disjoint sets $A_1, A_2, \ldots, A_q$ such that for any $t$, $1 \leq t \leq q$, we have $|A_t| = 3$ and $\sum_{a_i \in A_t} a_i = B$?

**Theorem 1.** *The $P^2$ decision problem is NP-complete in the strong sense.*

*Proof.* First, the $P^2$ decision problem is clearly in NP, because a nondeterministic algorithm needs only to guess the subsets $N_t$ of items of $N$ and check in polynomial time that the cumulative profit for all items in all these subsets is less than or equal to $V$.
An instance of $P^2$ can be built from an instance of 3-Partition in polynomial time. Given an arbitrary instance I of 3-Partition, an instance $f(I)$ of $P^2$ can be defined as follows: the number of periods $m = q$; the number of items $n = 3q$; $w_i = a_i$, for all $i \in N$; profit $p_{i,t} = a_i/t$ for all $i \in N$ and $t \in M$; capacity $B_j = jB$; $V = B\sum_{t=1}^{m} 1/t$. Note here that $p_{i,t}$ values are then non-increasing for $t \in \{1, ..., M\}$.
Let us first suppose that the answer to I is *yes*, which means that A can be partitioned into $q$ disjoint subsets $A_1, A_2, \ldots, A_q$ such that for all $1 \leq t \leq q$, $|A_t| = 3$ and $\sum_{a_i \in A_t} a_i = B$. Then, for $f(I)$, let $N_t = \{i | a_i \in A_t\}$, for all $t \in M$. It can easily be shown that:

1. For $1 \leq j \leq m$, $\sum_{t=1}^{j} \sum_{i \in N_t} w_i = \sum_{t=1}^{j} \sum_{i \in N_t} a_i = \sum_{t=1}^{j} B = jB \leq B_j$. This ensures that the assignment is valid;

2. The total profit $\sum_{t=1}^{m} \sum_{i \in N_t} p_{i,t} = \sum_{t=1}^{m} \sum_{i \in N_t} a_i/t = \sum_{t=1}^{m} B/t = B \sum_{t=1}^{m} 1/t \geq V$.

This shows that the answer to $f(I)$ is also *yes*.

Inversely, let us suppose that the answer to $f(I)$ is *yes*, meaning that there exists an assignment of items such that the total profit $\sum_{t=1}^{m} \sum_{i \in N_t} p_{i,t} \geq V$. Given that $\sum_{t=1}^{j} \sum_{i \in N_t} w_i \leq B_j = jB$ for all $j \in M$, it can be proved that $\sum_{t=1}^{j} \sum_{i \in N_t} p_{i,t} \leq B \sum_{t=1}^{j} 1/t$ for any $j \in M$. This value is the optimum objective value obtained for the corresponding standard LP problem; that is, we allow variables $x_{it} \in [0,1]$. Thus, we have for $j = m$, $\sum_{t=1}^{m} \sum_{i \in N_t} p_{i,t} \leq B \sum_{t=1}^{m} 1/t = V$. The maximum total profit is only reached when $\sum_{i \in N_t} w_i = B$ for all $t \in M$, and this maximum profit is equal to $\sum_{t \in M} B/t$. Therefore, $\sum_{t=1}^{m} \sum_{i \in N_t} p_{i,t} = V$. Furthermore, since $B/4 < w_i < B/2$ and $\sum_{i \in N_t} w_i = B$ for all $t \in M$, there can only be three items assigned in each period $t$. Hence, the answer to I is also *yes*.

It can be concluded that the $P^2$ decision problem is NP-complete in the strong sense, which means that the original $P^2$ problem is strongly NP-hard. □

We can now deduce the complexity for the problem MPR.

**Theorem 2.** *The MPR problem is strongly NP-hard.*

**A fondamental property.** In this section, we show a property stating that any instance $I$ of the MPR problem can be reduced to an instance of the $P^2$ problem. Let us begin with proposing a preprocessing procedure, named in the following with $\mu$, which converts the $p_{i,j}$ values so that they are non-increasing over the time horizon. This may be achieved as follows, the $p'_{i,j}$ values representing the profit values in the new instance $\mu(I)$: Obviously, the following result holds:

**Algorithm 1:** Procedure profit transformation $\mu$

```
Data: an instance I
for i ∈ N do
    p'_{i,m} ← p_{i,m};
    for j ← m − 1 to 1 do
        p'_{i,j} ← max(p_{i,j}, p'_{i,j+1})
```

**Proposition 3.** *Any optimal solution for instance I is also optimal for instance $\mu(I)$, and from any optimal solution for instance $\mu(I)$ an optimal solution can be constructed for instance I.*

As a consequence, since $\mu(I)$ is also an instance of $P^2$, any algorithm solving problem $P^2$ can be used to solve problem $P$. In other words, $P^2$ corresponds to "hard" cases of MPR.

# 4 Resolution methods for MPR

In this section we present methods specifically designed for solving the $P^2$ problem. Notice also that this solution strategy was initially intended to be integrated in a simulation tool, and computational efficiency in terms of CPU time was therefore an important requirement. In this section, two heuristic methods, called Back&Forth and CombMT&D, are described.

**The Back&Forth heuristic.** This heuristic is inspired by the Dantzig heuristic for MPR. It is implemented in two steps. The main idea behind this heuristic is to assign temporarily the *best* items in the first step and to improve the assignments in the second. In contrast to the Dantzig heuristic, an item is not assigned to the earliest possible period, but it is assigned temporarily to a later period where the profit is not significantly diminished. In the second step the assignment is moved forward if capacity constraints allow. During the algorithm, $B'_j$ gives the remaining capacity values, with respect to assigned items. Initially, $B'_j = B_j$, for all $j \in \{1,\ldots,m\}$. For each unassigned item $i$, we denote as $epp_i$ the earliest possible period to which $i$ can be assigned, according to the current capacity constraints $B'_j, j \in M$, i.e. $epp_i = \text{argmin}_j\{w_i \leq B'_t, \forall t \in \{j,\ldots,m\} | j \in M\}$. Otherwise, if $w_i > B'_m$ ($i$ cannot be assigned to any of the periods), set $epp_i = m+1$. Heuristic Back&Forth proceeds as follows:

1. Let $\bar{N}$ be the collection of the unassigned items. Find the item $z = \text{argmax}_i\{p_{i,epp_i}/w_i | i \in \bar{N} \wedge epp_i \leq m\}$. Next, determine the period $t_z (t_z \geq epp_z)$ in which item $z$ will be temporarily assigned. $t_z$ corresponds to the largest period where the reduced profit $(p_{z,epp_z} - p_{z,t_z})$ does not exceed a certain value $\lambda(p_{z,epp_z} - p_{z,m}), 0 \leq \lambda \leq 1$. Remove $z$ from $\bar{N}$ and update all concerned $B'_j$. Update $epp_i$, for all $i \in \bar{N}$. Repeat this step until the assignment is no longer possible;

2. Let $N_0 = N \setminus \bar{N}$ be the collection of items assigned in the first step. Find item $x = \text{argmax}_i\{(p_{i,epp_i} - p_{i,S_i})/w_i | i \in N_0 \wedge epp_i \neq S_i\}$, where $epp_i$ is the earliest period in which item $i$ can be placed and $S_i$ is the current period in which item $i$ is assigned. Assign item $i$ to period $epp_i$ and update all concerned $B'_j$. Update $epp_i$, for all $i \in N_0$. Repeat this step until no further items can be brought forward.

**The CombMT&D heuristic.** This heuristic is a combination of the Martello&Toth ([5]) and Dantzig heuristics (proposed for the unbounded knapascak problem in [7]). It is implemented in two steps. In the first step, items are iteratively assigned to periods until $m-1$, respecting a particular order combining the Dantzig and Martello&Toth heuristics. In the second step, the unassigned items are assigned iteratively to the last period $m$ according to the Dantzig heuristic. Each assignment satisfies the capacity constraints. Heuristic CombMT&D proceeds as follows:

1. Let $\bar{N}$ be the collection of unassigned items. Find item $z = \text{argmax}_i\{(p_{i,epp_i}/w_i)^\alpha (p_{i,epp_i} - p_{i,epp_i+1})/w_i | i \in \bar{N} \wedge epp_i \leq m-1\}$ and assign item $z$ to period $epp_z$. Remove $z$ from $\bar{N}$ and update all concerned $B'_j$. Update $epp_i$, for all $i \in \bar{N}$. Repeat this step until the assignment is no longer possible;

2. Consider the set $\bar{N}$ of remaining unassigned items. Find item $x = \text{argmax}_i\{(p_{i,m}/w_i) | i \in \bar{N} \wedge epp_i \leq m\}$ and assign it to period $m$. Update $B'_m$ et $epp_i$, for all $i \in \bar{N}$. Repeat this step until the assignment is no longer possible.

We note that the complexity time of each of them is $O(n^2 m)$.

# 5 Computational experiments

In order to evaluate the effectiveness of the above heuristics, we tested instances of both practical and theoretical types. The main difference between practical and theoretical types is that for practical instances the profits for an item $i$ correspond to a function which attains its maximum value at a certain period $r_i$ and then progressively decreases over the subsequent periods. Therefore, in accordance with Proposition 3, item $i$ will not be assigned before $r_i$. This reduces the number of periods to which it is possible to assign item $i$, thus making instances easier to solve, and this is confirmed in Tables 1 by the

small gap values of this practical type compared with other types on instances of identical or similar sizes. Otherwise, for theoretical instances, in order to test a more general case and evaluate the robustness of each heuristic, the maximum profit for all items is deemed to occur in the first period, and the manner in which profits decrease over subsequent periods is random.

Four types of instances are tested. Type 1 are practical ones. For each item $i$, the highest profit is attained at a certain period $r_i$. After period $r_i$, profits decrease in accordance with certain rules (see the details below). The preprocessing procedure is applied to transform an instance of this type to an instance of $P^2$(profits in non-increasing order from the first period). The size of tested instances for this type are $n = [500, 1000, 5000, 10000]$ and $m = [15, 20, 25]$. Type 2, type 3 and type 4 are theoretical types. For type 2 and type 3, profits $p_{i,j}$ decrease randomly with a certain distribution for each item $i$. For type 4, profits $p_{i,j}$ decrease linearly from the first period with different slopes for each item $i$. The size of tested instances for the last three types are $n = [50, 100, 500, 1000]$ and $m = [10, 30, 50]$. In addition, capacity values considered for each type of instance are generated as *similar capacities*, following [5]. Notice that in order to stick to the application, the practical instances have different values for $m$ and $n$.

### 5.1 Experimental results

All tests were run on a PC with a 2 GHz microprocessor and 2 GB of Ram. To evaluate the performances of our heuristics, we have measured the gap with the optimal solution (or the best integer solution) provided by the MIP. We have run the algorithm of CPLEX 11 with a time limit fixed to 600 seconds to obtain optimal solutions or near optimal solutions of the instances. There are 54.3% of instances which are not solved to optimality at the end of the time limit. However, the results of MIP are generally very close to the optimal solutions. Based on the informations provided by CPLEX, the average final gap of MIP over all the tested instances is 0.04%. The results of the heuristics are then evaluated by the best integer solution obtained by the MIP algorithm. While it is obvious that the results of the heuristics are not really comparable with these results, note that if we let to CPLEX the same time which is taken in the worst case by the heuristics, CPLEX does not provide better solutions in most of cases.

The parameters $\lambda$ and $\alpha$ used in our two heuristics have been chosen empirically. We have set $\lambda$ to 0.25 and $\alpha$ to 3. In Tables 1 we provide the average relative distance ("gap") between the result of each heuristic and the best integer solution of MIP. The gap is computed as *(the best integer solution of MIP - the value of the heuristic)/(the best integer solution of MIP)*. The CPU time in the tables are in *milliseconds*. The 0.0 values for *time* means that the CPU time is less than 0.1 millisecond. *BF* represents the Back&Forth heuristic and *Comb* represents the CombMT&D heuristic.

We tested all methods on 480 instances (120 instances for each type) (for each type and for each size $(n,m)$, 10 instances are generated). Table 1 provides the gap values and the CPU time for instances of each size and of each type. The Back&Forth heuristic performs well in general. It obtains the best results for instances of type 1 but does a little bit less better for other instances. The CombMT&D heuristic obtains the best results for types 2, 3 and 4 while for type 1 it performs significantly less well than the Back&Forth. In terms of execution time, heuristic Back&Forth takes almost twice more time than the other heuristic. This observation is valid for all tested instances.

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| type | m | Algo | n=500 | | n=1000 | | n=5000 | | n=10000 | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | gap | time | gap | time | gap | time | gap | time |
| | 15 | *BF* | 0.03 | 21.9 | 0.02 | 88.2 | 0.01 | 2327.0 | 0.00 | 9402.0 |
| | | *Comb* | 0.37 | 13.9 | 0.28 | 53.0 | 0.18 | 1314.8 | 0.19 | 6208.0 |
| | 20 | *BF* | 0.06 | 25.5 | 0.03 | 100.3 | 0.01 | 2494.4 | 0.01 | 10911.1 |
| | | *Comb* | 0.28 | 14.7 | 0.29 | 56.3 | 0.20 | 1363.5 | 0.21 | 6626.4 |
| | 25 | *BF* | 0.06 | 28.2 | 0.04 | 113.2 | 0.02 | 2885.7 | 0.01 | 12732.6 |
| | | *Comb* | 0.50 | 14.0 | 0.42 | 55.7 | 0.29 | 1391.5 | 0.22 | 7244.4 |
| 1 | | | n=50 | | n=100 | | n=500 | | n=1000 | |
| | 10 | *BF* | 3.47 | 0.4 | 3.46 | 0.9 | 3.26 | 17.7 | 3.44 | 0.5 |
| | | *Comb* | 3.15 | 0.2 | 2.50 | 0.6 | 1.96 | 13.7 | 1.89 | 55.3 |
| | 30 | *BF* | 3.07 | 0.5 | 2.90 | 1.7 | 3.33 | 34.5 | 3.58 | 134.3 |
| | | *Comb* | 4.09 | 0.0 | 3.24 | 0.6 | 2.18 | 13.6 | 1.95 | 54.1 |
| 2 | 50 | *BF* | 3.09 | 1.0 | 3.72 | 2.4 | 3.05 | 49.3 | 3.18 | 188.2 |
| | | *Comb* | 3.90 | 0.3 | 3.48 | 0.5 | 2.18 | 13.7 | 1.94 | 50.5 |
| | 10 | *BF* | 4.81 | 0.2 | 3.77 | 0.8 | 4.06 | 18.5 | 4.32 | 73.5 |
| | | *Comb* | 3.91 | 0.3 | 3.76 | 0.6 | 2.90 | 14.2 | 2.81 | 55.0 |
| | 30 | *BF* | 4.83 | 0.5 | 4.14 | 1.4 | 3.88 | 34.6 | 3.74 | 135.9 |
| | | *Comb* | 4.88 | 0.2 | 3.78 | 0.6 | 3.48 | 13.9 | 3.24 | 52.6 |
| 3 | 50 | *BF* | 5.45 | 0.8 | 4.02 | 2.5 | 3.23 | 48.2 | 3.40 | 186.5 |
| | | *Comb* | 5.01 | 0.2 | 4.26 | 0.6 | 3.57 | 13.3 | 3.29 | 49.0 |
| | 10 | *BF* | 2.35 | 0.3 | 2.42 | 0.8 | 2.14 | 19.3 | 2.00 | 75.8 |
| | | *Comb* | 2.01 | 0.2 | 1.71 | 0.4 | 1.47 | 14.5 | 1.38 | 62.7 |
| | 30 | *BF* | 1.99 | 0.6 | 2.18 | 2.0 | 2.06 | 39.9 | 2.05 | 155.7 |
| | | *Comb* | 1.97 | 0.5 | 1.83 | 0.7 | 1.66 | 14.9 | 1.61 | 60.0 |
| 4 | 50 | *BF* | 2.18 | 1.0 | 2.18 | 3.2 | 2.09 | 60.2 | 1.99 | 231.6 |
| | | *Comb* | 2.22 | 0.2 | 1.99 | 0.9 | 1.72 | 15.4 | 1.64 | 60.2 |

Table 1: Gap and CPU time (ms)


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# MPI Parallelization of Bee Colony Optimization*

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**Abstract**

The Bee Colony Optimization (BCO) algorithm is a meta-heuristic that belongs to the class of biologically inspired stochastic swarm optimization methods, based on the foraging habits of bees in nature. In this paper we propose two synchronous parallelization strategies for a distributed memory multiprocessor architecture under the MPI communication library. The presented experimental results addressing the problem of static scheduling independent tasks on identical machines show that our parallel BCO algorithm provides excellent performance when a coarse-grained parallelization strategy is used. On the other hand, a fine-grained strategy is not suitable for this kind of target multiprocessor architecture.

**Keywords:** Parallelization strategy; MPI communication library; meta-heuristics; scheduling problems.

## 1 Introduction

The Bee Colony Optimization (BCO) algorithm is a meta-heuristic that belongs to the class of biologically inspired stochastic swarm optimization methods, based on the foraging habits of honey bees. It has successfully been applied to various combinatorial optimization problems. The basic idea behind BCO is to create a multi-agent system (colony of artificial bees) capable of successfully solving difficult combinatorial optimization problems. The artificial bee colony behaves somewhat similarly to bee colonies found in nature, but differences between the two kinds of colonies are also evident. BCO is a meta-heuristic which belongs to the class of constructive methods. It was designed as a method which builds solutions from scratch within the execution steps, unlike the local search-based meta-heuristics which perform iterative improvements of the current best solution. To the best knowledge of the authors, no works have been reported on parallelization strategies of BCO, despite its inherent parallelism. Therefore, the main goal of this work is to study the potential strategies for the parallelization of BCO. We propose two synchronous parallelization strategies of BCO on distributed memory IBM HPC Linux Cluster Server+16 $\times$ 2 Dual Core Intel Processors on 2.33GHz/1333MHz with 4MB RAM, Ethernet 3rd Party e1350 SMC 8848M Switch Bundle. Our implementations use C as the programming language and utilize MPI communication library.

The rest of this paper is organized as follows. Section 2 contains a brief description of the BCO algorithm. Parallelization strategies and implementation details are given in Section 3. Experimental evaluation is described in Section 4, while Section 5 concludes the paper.

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## 2 Bee Colony Optimization

Lučić and Teodorović [6, 7] were among the first to use the basic principles of collective bee intelligence in solving combinatorial optimization problems. BCO is a population based algorithm in which a population of *artificial bees* (consisting of $B$ individuals) searches for the optimal solution. Every artificial bee generates solutions to the problem within the construction steps, over multiple iterations until some predefined stopping criterion is met. Each step of the BCO algorithm is composed of two alternating phases: the *forward pass* and the *backward pass*, which are repeated until all solutions (one for each bee) are completed. During the forward pass, every bee is exploring the search space. It applies a predefined number of moves ($NC$), which construct a partial solution, yielding to a new (partial or complete) solution. Having obtained the new partial solutions, the bees start the second phase, the so-called backward pass.

At the beginning of the backward pass, all bees share information about their solutions. In nature, bees would perform a dancing ritual, which would inform other bees about the amount of food they have found, and the proximity of the patch to the hive. In the optimization search algorithm, the values of objective functions are compared. Each bee decides with a certain probability whether it will stay loyal to its solution or not. The bees with better solutions have a higher chance to keep and advertise them. The bees that are *loyal* to their partial solutions are called *recruiters*. Once a solution is abandoned, the bee becomes *uncommitted* and has to select one of the advertised solutions. This decision is made probabilistically, in such a way that the better advertised solutions have a bigger opportunity to be chosen for further exploration.

The two phases of the search algorithm, the forward and backward passes, alternate in order to generate all of the required solutions (one for each bee). When all solutions are completed the best among them is used to update the global best solution, and an iteration of BCO is completed. BCO iterations are repeated until a stopping condition is met. The possible stopping conditions can include the maximum total number of iterations, the maximum total number of iterations without improvement of the objective function, and the maximum allowed CPU time.Once the stopping condition is met, the global best solution is reported.

The BCO algorithm parameters whose values need to be set prior to the algorithm execution are: the number of bees $B$ and the number of constructive moves during one forward pass ($NC$). The following is the pseudocode of the BCO algorithm:

*Do*

1. Initialization: an empty solution is assigned to each bee.
2. *For* $(s = 0; s < NC; s++)$ //count moves
   (a) *For* $(b = 0; b < B; b++)$ //**forward pass**
   1) Evaluate all possible moves;
   2) Choose one move using the roulette wheel.
   //**backward pass**
   (b) *For* $(b = 0; b < B; b++)$
   Evaluate (partial/complete) solution for bee $b$;
   (c) *For* $(b = 0; b < B; b++)$
   Loyalty decision using the roulette wheel for bee $b$;
   (d) *For* $(b = 0; b < B; b++)$
   *If* ($b$ is uncommitted), choose a recruiter by the roulette wheel.
3. Evaluate all solutions and find the best one.

*while* stopping criteria is not satisfied.

# 3 Parallelization Strategies

The main goal of parallelization is to speedup the computations needed to solve a particular problem by engaging several processors and dividing the total amount of work between them. For stochastic algorithms this goal may be defined in one of the following two ways: 1) accelerate the search for the same quality solution or 2) improve the solution quality by allowing more processors to run the same amount of (CPU or wall-clock) time as a single one does. When meta-heuristics are considered, a combination of gains may be obtained: parallel execution can enable an efficient search of different regions of the solution space, yielding an improvement of the final solution quality within a smaller amount of execution time.

A significant amount of work has already been done on the parallelization of meta-heuristics. The approach can be twofold. The theoretical aspects of parallelization could be considered, and the practical applications of parallel meta-heuristics to different optimization problems proposed. The survey paper [3] summarizes these works and proposes an adequate taxonomy.

The well-known performance measures for parallel programs are *speed-up* $S_q$ and *efficiency* $E_q$ [1] and we also use them to evaluate and justify our approach. They are defined as follows:

$$S_q = \frac{T_{seq}^{best}}{T_q}, \quad E_q = \frac{S_q}{q} = \frac{T_{seq}^{best}}{qT_q}.$$

Here, $T_{seq}^{best}$ denotes the execution time of best known sequential algorithm on a single processor, while $T_q$ represents the execution time of the parallel algorithm on $q$ processors.

When meta-heuristics are considered, the performance of parallelization strategy is also influenced by the quality of the final solution. Namely, meta-heuristics represent stochastic search procedures (and BCO is not an exception) which may not result in an identical solution even after repeated sequential executions. On the other hand, parallelization may assure the extension of the search space, which could result in either an improvement or a degradation of the final solution's quality. Therefore, the quality of the final solution should also be considered as a parameter of parallelization strategy performance.

## 3.1 Parallelization of BCO

The BCO algorithm is created as a multi-agent system which naturally provides a good basis for the parallelization on different levels. High-level parallelization assumes a coarse granulation of tasks and can be applied to iterations of BCO. Smaller parts of the BCO algorithm (the forward and backward passes within a single iteration) also contain a lot of independent executions, and are suitable for low-level parallelization. In this work we consider both strategies in a synchronous way.

Coarse-grained parallelization in its simplest form represents the independent execution of BCO on different processors. It could be obtained by the division of the stopping criterion among processors. For example, if the stopping criterion is allowed CPU time (given as a *runtime* value in seconds), we could run BCO in parallel on $q$ processors for $runtime/q$ seconds. A similar rule can be introduced in the case when the stopping criterion is the allowed number of iterations. In both cases, each processor independently performs a sequential variant of BCO, but with a reduced value of the stopping criterion. We named this variant of parallelized BCO the distributed BCO (DBCO). Other way to implement the coarse-grained parallelization strategy could be the following: Instead of the stopping criterion, we could divide the number of bees. Namely, if the sequential execution uses $B$ bees for the search, our parallel variant executing on $q$ processors is using only $B/q$ bees. This way results in a sequential BCO on each processor, but

with a smaller number of bees. This is also a distributed BCO, that we will refer to as BBCO since the bees are distributed among processors. Independent runs on different processors also allow us to change the search parameters, and we can therefore assure diversification of the search process.

Each artificial bee acts as an individual agent during the forward pass when partial solutions are generated. The generation of a partial solution is independent from the rest of the computations. This leads us to the fine-level parallelization. Within the concrete implementation, we have the following scenario: the forward pass is executed independently on each processor, while the backward pass requires a tight coordination between processors. For the corresponding computations within the backward pass it is necessary to have the information about all generated partial solutions. Nevertheless, those computations could also be spread among all processors and accompanied by the required communication. Our implementation of this strategy will be called FBCO.

The implementation of the fine-grained parallelization strategy required us to define the relation between the number of processors $q$ and the number of bees $B$. Namely, each processor is responsible for $B/q$ bees and these two numbers should be divisible. More details about the implementation and experimental evaluation of our approach to the parallelization of BCO are given in the next section.

# 4 Experimental Evaluation

The proposed parallelization strategies are tested on a scheduling problem. In our previous paper [5] the implementation of BCO for scheduling independent tasks to identical machines is proposed. This implementation represents an excellent starting point for developing parallelization strategies of the BCO method due to the simplicity of the treated problem. Therefore, we use it as a pattern in spite of the fact that the sequential version performed well.

For the experimental evaluation various problem instances, the instances that have been used in [5], are used. This allows us to easily compare the sequential and parallel versions of BCO, and measure the performance of various parallelization strategies.

We have chosen the representative subset of test examples, namely the hard test instances from [4] with *a priori* known optimal solutions.

Our target architecture for parallelized BCO is a homogeneous and completely connected network of processors. We distinguish the processor that communicates with the user and name it *master*. It is usually marked as processor 0. The other $q-1$ processors are called *working processors* or *slaves*. Their marks are processor 1 up to processor $q-1$. The parallel versions of BCO execute on all $q$ processors, i.e. the computations are assigned to the master as well. In our experiments we used a different number of processors, ranging from 2 to 12.

To calculate speedup end efficiency, we assume that BCO from [5] is the best sequential algorithm. To assure fairness of obtained results, we compared the parallel versions of BCO with the original sequential version executed on a single processor of our parallel architecture (instead of the parallel version executed for $q = 1$).

## 4.1 Distributed BCO (DBCO)

First, we tested our coarse-grained parallelization strategy, named distributed BCO (DBCO). Table 1 contains the scheduling results for test instance with (*a priori*) known optimal solution consisting of 100 tasks while the number of machines is changed. For all examples, within DBCO we set $B = 5$, $NC = 10$ and the stopping criterion is selected to be 1000 iterations.

Table 1: DBCO Scheduling results - test problems with known optimal solutions from [4]

| example | $m$ | $q$ | OPT | DBCO | DBCO CPU time | $S_q$ | $E_q$ |
|---|---|---|---|---|---|---|---|
| Iogra100 | 2 | 1 | 800 | 800 | 4.60 | 1.00 | 1.00 |
| | | 2 | | 800 | 2.55 | 1.80 | 0.90 |
| | | 3 | | 800 | 1.70 | 2.70 | 0.90 |
| | | 4 | | 800 | 1.27 | 2.60 | 0.90 |
| | | 5 | | 800 | 1.01 | 4.55 | 0.91 |
| Iogra100 | 4 | 1 | 800 | 800 | 4.79 | 1.00 | 1.00 |
| | | 2 | | 800 | 2.54 | 1.89 | 0.94 |
| | | 3 | | 800 | 1.72 | 2.78 | 0.93 |
| | | 4 | | 800 | 1.28 | 3.74 | 0.93 |
| | | 5 | | 800 | 1.03 | 4.65 | 0.93 |
| Iogra100 | 6 | 1 | 800 | 801 | 4.78 | 1.00 | 1.00 |
| | | 2 | | 801 | 2.58 | 1.85 | 0.93 |
| | | 3 | | 802 | 1.73 | 2.76 | 0.92 |
| | | 4 | | 801 | 1.29 | 3.71 | 0.93 |
| | | 5 | | 801 | 1.04 | 4.60 | 0.92 |
| Iogra100 | 8 | 1 | 800 | 803 | 4.80 | 1.00 | 1.00 |
| | | 2 | | 803 | 2.58 | 1.86 | 0.93 |
| | | 3 | | 804 | 1.71 | 2.81 | 0.94 |
| | | 4 | | 804 | 1.28 | 3.75 | 0.94 |
| | | 5 | | 805 | 1.03 | 4.66 | 0.93 |
| Iogra100 | 9 | 1 | 800 | 807 | 5.03 | 1.00 | 1.00 |
| | | 2 | | 806 | 2.59 | 1.94 | 0.97 |
| | | 3 | | 809 | 1.74 | 2.89 | 0.96 |
| | | 4 | | 808 | 1.33 | 3.78 | 0.95 |
| | | 5 | | 808 | 1.05 | 4.79 | 0.96 |
| Iogra100 | 12 | 1 | 800 | 811 | 4.97 | 1.00 | 1.00 |
| | | 2 | | 814 | 2.62 | 1.9 | 0.95 |
| | | 3 | | 813 | 1.76 | 2.82 | 0.94 |
| | | 4 | | 813 | 1.32 | 3.76 | 0.94 |
| | | 5 | | 814 | 1.07 | 4.78 | 0.96 |
| Iogra100 | 16 | 1 | 800 | 819 | 5.15 | 1.00 | 1.00 |
| | | 2 | | 821 | 2.62 | 1.96 | 0.98 |
| | | 3 | | 822 | 1.83 | 2.81 | 0.94 |
| | | 4 | | 819 | 1.38 | 3.73 | 0.93 |
| | | 5 | | 828 | 1.05 | 4.90 | 0.98 |

In the first column of Table 1 the name of the example is given. The second column contains the number $m$ of machines within each example. The number of parallel processors $q$ executing DBCO is given in the third column of our table. The optimal schedule length represents the content of column four, while lengths of schedules obtained by DBCO for different $q$ are placed in the fifth column. Column six contains the CPU time required by DBCO to complete 1000

iterations (actually the CPU time required by $q$ processor to complete $1000/q$ iterations). The corresponding speedup $S_q$ and efficiency $E_q$ are given in the last two columns. It is important to note that the CPU time required by DBCO to complete all necessary computations is actually the CPU time of the processor that is the last one to finish its work, i.e. it is equal to the maximum of all processors' running times. The resulting tables include the best obtained schedule length, as well as the worst (longest) required CPU time. To illustrate the results on each particular processor, Table 2 presents the running times and obtained schedule lengths for each processor when scheduling Iogra100 onto 6 machines. As can be seen from this table, both the running time and the solution quality may differ from processor to processor. The smallest schedule length and the largest execution time characterize distributed execution of BCO.

Table 2: DBCO detailed results for Iogra100 scheduled on 6 machines

| $q$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| CPU time | 4.48 | 2.26 | 1.53 | 1.14 | **1.04** |
| | | **2.58** | 1.69 | **1.29** | 0.92 |
| | | | **1.73** | 1.27 | 1.04 |
| | | | | 1.22 | 1.02 |
| | | | | | 1.01 |
| DBCO | 801 | 804 | 803 | 805 | 803 |
| | | **801** | **802** | **801** | 805 |
| | | | 804 | 803 | 804 |
| | | | | 803 | **801** |
| | | | | | 805 |

Based on the results presented in Table 1 we can conclude that for those examples DBCO achieves an almost linear speedup with efficiency above 90%. Regarding the solution quality, we noticed that it is preserved for easier examples. Sometimes we noticed a slight improvement in the solution quality, while for larger examples (scheduling on larger number of machines) the parallelization may cause some degradation due to the reduction of the stopping criterion - (2.7% at most).

## 4.2 Bees distribution within BCO (BBCO)

The second way to implement the coarse-grained parallelization strategy (by dividing the total number of bees among the processors) we named BBCO. We tested this implementation on Iogra100 assuming that $B = 12$, $NC = 10$ and the stopping criterion is set to 1000 iterations. The number of processors $q$ takes the following values: 2, 3, 4, 6, 12. This actually means that sequential BCO works with 12 bees, BBCO on $q = 2$ processors executes 1000 iterations with 6 bees, and so on. Finally, on $q = 12$ processors, BBCO is searching for schedules with one bee per processor. The results for the larger test instances are given in Table 3.

This type of coarse-grained parallelization strategy assures an excellent speedup and efficiency, due to the reduction of computations assigned to each processor. This resulted in a superlinear speedup and efficiency greatly above 1.00. Nevertheless, the difficulty of a given scheduling problem can not always be conquered and a small degradation of the solution quality may occur. On the other hand, as Table 3 shows, sometimes improvements are also evident.

Table 3: BBCO Scheduling results - test problems with known optimal solutions from [4]

| example | $m$ | $q$ | OPT | BBCO | BBCO CPU time | $S_q$ | $E_q$ |
|---|---|---|---|---|---|---|---|
| Iogra100 | 9 | 1 | 800 | 807 | 11.88 | 1.00 | 1.00 |
| | | 2 | | 807 | 5.82 | 2.04 | 1.02 |
| | | 3 | | 806 | 3.54 | 3.56 | 1.12 |
| | | 4 | | 806 | 2.18 | 5.42 | 1.36 |
| | | 6 | | 806 | 0.80 | 14.85 | 2.48 |
| | | 12 | | 806 | 0.25 | 47.52 | 3.96 |
| Iogra100 | 12 | 1 | 800 | 811 | 12.01 | 1.00 | 1.00 |
| | | 2 | | 813 | 5.88 | 2.04 | 1.02 |
| | | 3 | | 813 | 3.59 | 3.35 | 1.12 |
| | | 4 | | 814 | 2.19 | 5.48 | 1.37 |
| | | 6 | | 817 | 0.86 | 13.97 | 2.33 |
| | | 12 | | 817 | 0.27 | 44.48 | 3.71 |
| Iogra100 | 16 | 1 | 800 | 819 | 12.18 | 1.00 | 1.00 |
| | | 2 | | 821 | 5.89 | 2.07 | 1.03 |
| | | 3 | | 822 | 3.59 | 3.39 | 1.13 |
| | | 4 | | 825 | 2.22 | 5.49 | 1.37 |
| | | 6 | | 825 | 0.91 | 13.38 | 2.23 |
| | | 12 | | 823 | 0.26 | 46.84 | 3.90 |

### 4.3 Fine-grained BCO (FBCO)

The implementation of the fine-grained parallelization strategy, is named FBCO. Due to the intensive communication between processors, we needed to minimize the amount of data exchanged during the search. Once again we used Iogra100 examples assuming $B = 12$, $NC = 10$ and 1000 iterations as the stopping criterion. The number of processors $q$ takes the following values: 2, 3, 4, 6, 12. Results obtained by FBCO are given in Table 4.

As can be seen from Table 4, this parallelization strategy slows the computations. This is due to the communication delays which are caused by the intensive data exchange between processors. This strategy is obviously more suitable for shared-memory multiprocessor systems. In addition, asynchronous parallelization strategies are definitely the most interesting challenge for future work.

## 5 Conclusion

Two synchronous parallelization strategies for Bee Colony Optimization (BCO) meta-heuristic are proposed in this paper. They are implemented on completely connected homogeneous multiprocessor system, in which processors communicate by exchanging messages. The first strategy is based on independent execution of portions of the BCO algorithm on different processors and exchanging the best solution at the end. It is implemented in two ways, both of which proved to be very efficient. For the first variant the obtained speedup is almost linear, and quality of the solution is not degraded significantly (below 3% with respect to the sequential result). The sec-

Table 4: FBCO Scheduling results - test problems with known optimal solutions from [4]

| example | $m$ | $q$ | OPT | FBCO | FBCO CPU time | $S_q$ | $E_q$ |
|---|---|---|---|---|---|---|---|
| Iogra100 | 4 | 1 | 800 | 800 | 11.51 | 1.00 | 1.00 |
| | | 2 | | 800 | 14.02 | 0.82 | 0.41 |
| | | 3 | | 800 | 14.99 | 0.77 | 0.26 |
| | | 4 | | 800 | 15.67 | 0.73 | 0.18 |
| | | 6 | | 800 | 16.42 | 0.70 | 0.12 |
| | | 12 | | 800 | 18.28 | 0.63 | 0.05 |
| Iogra100 | 16 | 1 | 800 | 819 | 12.18 | 1.00 | 1.00 |
| | | 2 | | 821 | 14.30 | 0.85 | 0.43 |
| | | 3 | | 822 | 14.94 | 0.82 | 0.27 |
| | | 4 | | 823 | 15.75 | 0.77 | 0.19 |
| | | 6 | | 825 | 16.46 | 0.74 | 0.12 |
| | | 12 | | 825 | 18.39 | 0.66 | 0.06 |

ond way performed even better, resulting in a superlinar speedup and a negligible degradation of the quality of the solution. The fine-grained parallelization strategy represents parallelization at a lower level, and is actually based on cooperative work between several processors. Since it requires an intensive exchange of data between the processors during the search, it is more suitable for shared-memory multiprocessor systems. Therefore, our implementation of this strategy resulted in a small increase of the parallel BCO's execution time. This may be improved by a reduction of message exchange frequency or by developing an architecture-dependent parallelization strategy that would minimize the number of messages travelling through the system. The latest concepts represent interesting ideas for further research on this topic.

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# Optimization over Stochastic Integer efficient Set

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**Abstract**

The problem of optimization over an integer efficient set of a Multiple objective Integer Linear Stochastic Programming (MOILSP) problem is being studied in this paper. Once the problem is converted into a deterministic one by adapting the 2-levels recourse approach, a new pivoting technique is applied to generate an optimal efficient solution without having to enumerate all of them. This method combines two techniques, the L-Shaped method and the combined method developed in [8].

**Keywords** Multi-objective Programming, stochastic Programming, 2-levels recourse model, efficient solution.

## 1 Introduction

In this paper we particularly focus in solving multiple objective stochastic linear programming (MOSLP) problem in presence of integer decision variables. We have propose a new technique that combines L-Shaped method [8] and a method that consists of optimizing an arbitrary linear function over the whole set of discrete deterministic efficient solutions of multiple objective Integer linear programming problem (MOILP) without having to enumerate explicitly all non dominated solutions. Few methods are proposed in the literature and most of them are using the techniques of goal programming (interactive) and the stochastic programming. In this context, we cite some methods : PROTRADE [6], PROMISE [11] in the continuous case and the discrete, STRANGE-MOMIX Method [10] and search and cut technique([1]).

We follow the notations used by Peter Kall and Janos Mayer [8]. Given a Multiple Objective Integer Stochastic Linear Programming problem (MOISLP)

$$(P_{SC}) \begin{cases} \text{“ min ”} Z_k & = C_k x; \quad k = 1..K \\ s.t. & Ax = b \\ & Tx = h \\ & x \in \mathbb{N} \end{cases} \tag{1}$$

where $x$ is the decision vector variable of dimension $(n \times 1)$. $C_k, T, h$ are random matrices of dimensions $(K \times n), (\ell \times n)$ and $(\ell \times 1)$ respectively, with a known joint probability distribution which is not influenced by the choice of the decision $x$ and defined on a probability space $(\Omega, \Xi, P)$. We assume that $A$, $b$ are fixed known integer data (deterministic) of dimensions $(m \times n)$ and $(m \times 1)$ respectively.
The main problem that we are studying is given by :

$$(P_E(SC)) \begin{cases} \min \phi(x) = dx \\ s.t.\ x \in E(P_{SC}) \end{cases} \tag{2}$$

$d$ is a random line vector of dimension $n$ and $E(P_{SC})$ is the set solution of the problem $(P_{SC})$. Our main purpose is to solve the problem $(P_{SC})$ without having to enumerate all the elements of $E(P_{SC})$. In the next section we introduce the associate deterministic problem, we show some basic results concerning the L-shaped decomposition method. In section 3, we give some important results that deal with optimization over efficient set. Section 4, presents the proposed method illustrated by a didactic example and finally, we conclude by suggesting some other issues.

## 2 Passage to deterministic equivalent problem of MOSILP

A mono-criteria stochastic linear programming problem does not have meaning since some of the parameters are not well known at the moment the decision maker has to take decision about the values of the variables, the sense of the optimization can not be preserved. In addition when many criteria are to be taken all together at the same time the task becomes extremely difficult. We suppose some probability space $(\Omega, \Xi, P)$ that defines the probabilistic aspect of random parameters in our problem. The decision on $x$ has to be taken before the realization of the random variables is known. But consequently, after the observation of the random variables realization, it may turn out that $Tx \neq h$, i.e. $x$ lies out of the admissibility region. In this case, it may be necessary to compensate for the deficiency, i.e. for $h - Tx$, after its observation. This is can be done by the introduction of recourse defining the constraints $Wy = h - Tx, y \geq 0$. We suppose a fixed recourse matrix $W$ and the recourse costs is taken as linear by $q'y$. Obviously we want to achieve this compensation with minimal costs. Hence the recourse problem is defined by

$$(P_1) \begin{cases} Q(x;T,h) = & \min q'y \\ s.t. & Wy = h - Tx \\ & y \geq 0 \end{cases} \tag{3}$$

The expectation value of the $k^{th}$ criterion is $\widetilde{Z}_k = E(Z_k + Q(x,\xi)); k = 1, \cdots, K$ and the deterministic multiple objective integer linear programming problem and the main problem can be stated as follows :

$$(P) \begin{cases} \text{“min”}\widetilde{Z}_k = & \mathbb{E}(Z_k + Q(x;T,h)) \quad k = 1,..,K \\ s.t. & Ax = b \\ & x \in \mathbb{N} \end{cases} \tag{4}$$

$$(P_E) \begin{cases} \min \widetilde{\phi} = & \mathbb{E}(d)\, x \\ s.t. & x \in E(P_{SC}) \end{cases} \tag{5}$$

The relaxed problem can be stated as follows :

$$(P_R) \begin{cases} \min \widetilde{\phi} = & \mathbb{E}(d)\, x \\ s.t. & x \in D = \{x \in \mathbb{R}^n | Ax = b, x \in \mathbb{N}\} \end{cases} \tag{6}$$

We suppose that the penalties $q^r = q(\xi^r)$ of the constraint violations are given. Here a recourse function $Q(x, \xi^r)$ is added to each criterion $Z_{kr}$ and the corresponding penalty is given by

$$Q(x, \xi^r) = \min_z \left\{ (q^r)^t z | W(\xi^r) z = h(\xi^r) - T(\xi^r) x; z \geq 0 \right\}. \tag{7}$$

# 3 Theoretical results

In this section we present two sets of most important results. The first set concerns the feasibility and optimality tests as was introduced in L-shapped technique (see[**?**, 8]) and the second set contains some basic definitions and results in multiple objective integer linear programming theory and optimization over efficient solutions set.

## 3.1 Feasibility Test

We use the dual problem (8) of (7) to test wether a given solution $x^0$ will yield feasible second-stage problems for all possible realizations of $\xi$.

$$(F_{test}) \begin{cases} \max & \pi^t \left(h(\xi^r) - T(\xi^r)x\right) \\ s.t. & \pi^t W \leq (q^r)^t \\ & \pi \in \mathbb{R} \end{cases} \tag{8}$$

According to Farkas's lemma $\left\{ \ z|Wz = h\left(\xi^r\right) - T\left(\xi^r\right)x^0, \ z \geq 0 \ \right\} \neq \varnothing$ if and only if

$$u^t W \leq 0 \text{ implies } u^t \left[h\left(\xi^r\right) - T\left(\xi^r\right)x^0\right] \leq 0$$

.
Therefore, $Q(x^0, \xi^r)$ is infeasible if and only if $P = \{\pi|\pi^t W \leq (q^r)^t\}$ has an extreme ray $u$ such that $u^t \left[h\left(\xi^r\right) - T\left(\xi^r\right)x^0\right] > 0$.
To check out for feasibility of the second stage-problems, we solve the following problem

$$(F_{Dual}) \begin{cases} \max & u^t \left(h(\xi^r) - T(\xi^r)x^0\right) \\ s.t. & u^t W \leq 0 \\ & \|u\|_1 \leq 1 \\ & u \in \mathbb{R} \end{cases} \tag{9}$$

The constraints $\|u\|_1 \leq 1$ bounds the value of $u$.
In case where $u_r^t \left[h\left(\xi^r\right) - T\left(\xi^r\right)x^0\right] > 0$ for some $\xi^r$; $r \in \{1, \cdots, R\}$ and $u_r$ is an optimal solution of problem (9); We add feasibility cut

$$u_r^t \left[h\left(\xi^r\right) - T\left(\xi^r\right)x^0\right] \leq 0 \tag{10}$$

## 3.2 Optimality Test

Resolution of the problem $(P_2')$ permit to test the optimality of a given solution

$$(P_2') \begin{cases} \text{“ min ”}\widetilde{\phi} = & E(\phi) + \theta; \\ s.t. & x \in D = S \cap \mathbb{Z} \\ & \theta \geq Q(x) \end{cases} \tag{11}$$

Where, $S = \left\{x \in \mathbb{R}^n | Ax = b, u_r^t T\left(\xi^r\right)x \geq u_r^t h\left(\xi^r\right), r = 1, \cdots, R, x \geq 0\right\}$,
$S = \{x \in \mathbb{R}|\widetilde{A}x = \widetilde{b}, x \geq 0\}$ is a non-empty, compact polyhedron in $\mathbb{R}^n$ and
$Q(x) = E(Q(x,\xi)) = \sum_{r=1}^{R} p^r Q(x, \xi^r) = \sum_{r=1}^{R} p^r (q^r)^t z^r$.
We solve problem $(P_2')$ and we obtain a feasible solution. Efficiency and non-dominance are defined as follows :

**Definition 1.** *A point $\overline{x} \in S$ is an efficient solution for $(P_2')$ if and only if there is no $x \in S$ such that $\widetilde{z}_i(x) \leq \widetilde{z}_i(\overline{x})$ for all $i \in \Im = \{1, 2, \cdots, K\}$ and $\widetilde{z}_i(x) < \widetilde{z}_i(\overline{x})$ for at least one $i \in \Im$. Otherwise, $\overline{x}$ is not efficient and the corresponding vector $(\widetilde{z}_1(\overline{x}), \widetilde{z}_2(\overline{x}), ..., \widetilde{z}_p(\overline{x}))$ is said to be dominated.*

The next theorem provides another characterization of an efficient solution that is integrated as a test-procedure in our method.

**Theorem 1.** *Let $x^0$ be an arbitrary element of the region $D$; $x^0 \in E(P)$ if and only if the optimal value of the objective function $\Theta(\psi, x)$ is null in the following integer linear programming problem:*

$$\left(P\left(x^0\right)\right) \quad \left\{ \begin{array}{ll} \max & \Theta(\psi_1, \ldots, \psi_K, x_1, \ldots, x_n) = \sum_{i=1}^{K} \psi_i \\ s.t. & \left\{ \begin{array}{l} \sum_{j=1}^{n} \widetilde{c}_j^i x_j + \psi_i = \sum_{j=1}^{n} \widetilde{c}_j^i x_j^0 \quad \forall i \in \{1, \ldots, K\} \\ x = (x_1, \ldots, x_n) \in D \\ \psi_i : \text{ are real nonnegative integer variables} \\ \text{for all } i \in \{1, \ldots, K\} \\ \widetilde{c}_j^i : \text{ is the } j^{th} \text{ component of row vector} \\ \qquad \widetilde{c}^i \text{ in problem } (P_2') \end{array} \right. \end{array} \right. \tag{12}$$

The proof of the theorem is omitted; it can be found in [1]. Its utilization guarantees that the feasible solution is either efficient or otherwise, provides an efficient solution for problem $(P_2')$.

Once an efficient solution is generated and added to the current list where the expected objective $E(\phi(x))$ is evaluated, new constraints over the feasible set $D$ of the relaxed problem $(P_R)$ are imposed, discarding from further consideration not only efficient solutions generated previously, but also any other feasible solutions with dominated objectives vectors. The algorithm terminates when the current feasible space becomes empty.

Assuming that all coefficients of matrix $C$ are integers. Afterwards, at iteration $k$, using Sylva and Crema's idea, see ([3]), the feasible set $D$ is reduced gradually by eliminating all dominated solutions by $C\hat{x}^k$. The resolution of the following problem enables us to perform this elimination.

$$(P_R^k) \equiv \min\{dx, x \in D - \bigcup_{s=1}^{k} D_s\}.$$

Where $D_s = \{x, x \in \mathbb{Z}_+^n, Cx \leq Cx^s\}$ and $\{Cx^s\}_{s=1}^k$ are non dominated criteria solutions of $(P)$ obtained at iterations $1, 2, ..., k$ respectively.

$$D - \bigcup_{s=1}^{k} D_s = \left\{ \begin{array}{l} c^i x \leq (c^i x^s - 1) y_i^s + M_i(1 - y_i^s), \\ \text{i=1,2,...,p; s=1,2,...,k ;} \\ y_i^s \in \{0, 1\};\ i = 1, 2, ..., p; s = 1, 2, ..., k \\ \sum_{i=1}^{p} y_i^s \geq 1; \\ x \in D \end{array} \right\} \tag{13}$$

$M_i$ is an upper bound for any feasible value of the $i^{th}$ objective function. The associate variables $y_i^s$, $i = 1, ..., p$ of $\hat{x}^s$ and additional constraints are added to impose an improvement on at least objective function. Note that when $y_i^s = 0$, the constraint is not restrictive and when $y_i^s = 1$, a strict improvement is forced in the $i^{th}$ objective function evaluated at $\hat{x}^s$.

# 4 The method

The technical presentation of the proposed method is outlined in the following algorithm :

**Algorithm 1:** Optimizing a Linear Function over Integer Efficient Set

**Input Data**
$\downarrow K, m, m_1, n$ : The dimensions of the problem;
$\downarrow S, Pr$ : Number of scenarios and their probability vector;
$\downarrow A_{(m\times n)}$, $b_{(m\times 1)}$: Deterministic constraints parameters;
$\downarrow T_{(m_1\times n)}(sc), h_{(m_1\times 1)}(sc)\ \forall sc \in \{1,\cdots,S\}$: Stochastic constraints parameters;
$\downarrow d_{(1\times n)}(sc)$, $\forall sc \in \{1,\cdots,S\}$: Stochastic main criterion vector;
$\downarrow Cr_{(p\times n)}(sc)$, $\forall sc \in \{1,\cdots,S\}$: Stochastic criteria matrix;
$\downarrow W_{(m_1\times n)}$ : a fixed recourse matrix; $q_{(1\times n_1)}$ : the penalties of the constraint violations;
**Output**
$\uparrow X_{opt}$: optimal solution of the problem $(P_E)$;
$\uparrow \mathbb{E}\left(\phi_{opt}\right)$: optimal value of criterion $\phi$;
**Initialization** $\mathbb{E}\left(\phi_{opt}\right) \leftarrow +\infty$, $l \leftarrow 0$, research $\leftarrow true$, $\theta \leftarrow -\infty$;
Solve the deterministic relaxed problem $(P_R)$;
$\uparrow x_0, \uparrow \mathbb{E}(z_0) =$ Pb_relaxed $(\downarrow d, \downarrow A, \downarrow b)$;
**if** *the problem does not have a feasible solution* **then**
  problem $(P)$ is not feasible;
**else**
  **while** $research \neq true$ **do**
    **Feasibility and Optimality Test**
    **for** $sc \leftarrow 1$ **to** $R$ **do**
      $Feasibility \leftarrow false$;
      **while** $Feasibility \neq true$ **do**
        *Solve problem* $(F_{dual})$ (9);
        $\beta \leftarrow u^t\left[h(\xi^r) - T(\xi^r)\,x^\ell\right]$;
        **if** $\beta > 0$ **then**
          $Feasibility \leftarrow false$ ;
          update constraints by adding a feasibility cut (10) ;
          $D^\ell$ is the current admissible region;
          $x^\ell$ is an optimal solution if there existe;
        **else**
          $Feasibility \leftarrow true$;
    $Q \leftarrow 0$
    **for** $sc \leftarrow 1$ **to** $R$ **do**
      *Solve problem* $(F_{test})$ *(8 );*
      $Q \leftarrow Q + p \times Q\left(x^\ell; sc\right)$;
    **while** $\theta < Q$ **do**
      $D^\ell = D^\ell \cap \{$Optimality constraint$\}$;
      Solve $(P_R)$; $x^\ell$ is an optimal solution with a penalty value $\theta$;
    **EFFICIENCY TEST**
    Solve $\left(P\left(x^\ell\right)\right)$; $\Theta$ is the optimal solution criteria;
    **if** $\Theta \neq 0$ **then**
      $x^\ell$ is not efficient;
      $\overline{x}^\ell$ an optimal solution of $\left(P\left(x^\ell\right)\right)$ is efficient;
      **Feasibility and Optimality Test for the solution** $\overline{x}^\ell$;
      $X_{opt} = \overline{x}^\ell$, $\phi_{opt} = \widetilde{\phi}(\overline{x}^\ell)$;
      $\ell \leftarrow \ell + 1$;
    **else**
      $x^\ell$ is an efficient solution;
      $X_{opt} = x^\ell$, $\phi_{opt} = \widetilde{\phi}(x^\ell)$;
      $\ell \leftarrow \ell + 1$;
    *Solve problem* $(P_R^k) \equiv \min\{dx, x \in D - \bigcup_{s=1}^{k} D_s\}$;
    **if** $D_{oml} = \emptyset$ ***Or*** $\phi(x^l) > \phi_{opt}$ **then**
      $X_{opt}$ is an optimal efficient solution with value $\phi_{opt}$;
      $research \leftarrow false$;
    **else**
      $research \leftarrow true$;

# 5 Didactic Example

Consider the following multiple objective integer linear programming stochastic problem

**Scenario 1**

$$(P_{sc1})\begin{cases} \text{“min”}Z_1 = -9x_1 + 4x_2 \\ \text{“min”}Z_2 = 3x_1 - 5x_2 \\ \text{“min”}Z_3 = 8x_1 - 11x_2 \\ s.t. \quad x_1 \le 5 \\ \qquad x_2 \le 7 \\ \qquad x_1 + x_2 \le 10 \\ \qquad x_1 + 2x_2 \le 3 \\ \qquad -2x_1 + x_2 \le 5 \\ \qquad x_1 \, , \, x_2 \in \mathbb{N} \end{cases}$$

$$(PE_{sc1})\left\{ \min \phi_1(x) = 4x_1 - 3x_2 \right.$$

$q_1' = (1\ 0\ 6\ 2)\,; \quad P_1 = \dfrac{1}{2}$

**Scenario 2**

$$(P_{sc2})\begin{cases} \text{“min”}Z_1 = 3x_1 - 2x_2 \\ \text{“min”}Z_2 = 7x_1 + 1x_2 \\ \text{“min”}Z_3 = -4x_1 + 9x_2 \\ s.t. \quad x_1 \le 5 \\ \qquad x_2 \le 7 \\ \qquad x_1 + x_2 \le 10 \\ \qquad x_1 \le 6 \\ \qquad 3x_1 + 4x_2 \le 1 \\ \qquad x_1 \, , \, x_2 \in \mathbb{N} \end{cases}$$

$$(PE_{sc2})\left\{ \min \phi_2(x) = -2x_1 + 9x_2 \right.$$

$q_2' = (5\ 3\ 2\ 1)\,; \quad P_2 = \dfrac{1}{2}$

The recourse matrix and the deterministic constraints matrices are given for both scenarios by

$$W = \begin{pmatrix} -2 & -1 & 2 & 1 \\ 3 & 2 & -5 & -6 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 5 \\ 7 \\ 10 \end{pmatrix}$$

The stochastic constraints matrices are given for both scenarios by

$$T^1 = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \qquad h^1 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}; \qquad T^2 = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix} \qquad h^2 = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

The deterministic multiple objective integer linear programming problem

$$(P)\begin{cases} \text{“min”}E(Z_1) = -3x_1 + 1x_2 \\ \text{“min”}E(Z_2) = 5x_1 - 2x_2 \\ \text{“min”}E(Z_3) = 2x_1 - x_2 \\ s.t. \quad x_1 \le 5 \\ \qquad x_2 \le 7 \\ \qquad x_1 + x_2 \le 10 \\ \qquad x_1 \, , \, x_2 \in \mathbb{N} \end{cases} \tag{14}$$

The main deterministic relaxed problem is defined by

$$(P_R)\begin{cases} \min E(\phi) = x_1 + 3x_2 \\ s.t. \quad x_1 \le 5 \\ \qquad x_2 \le 7 \\ \qquad x_1 + x_2 \le 10 \\ \qquad x_1 \, , \, x_2 \in \mathbb{N} \end{cases} \tag{15}$$

**step0:** $l = 0$, $\phi_{opt} = +\infty$, $H^0 = D$, $M_1 = 7$, $M_2 = 25$, $M_3 = 10$

**Step 1:** The relaxed problem $(P_R)$ is solved. the minimum is at $x^0$=$(0\ 0)^t$

**Step 2: feasibility test**

The resolution of problem $(F_{Dual})$ for both scenarios gives a maximum at $\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$ for the first scenario and a maximum at $\begin{pmatrix} \frac{5}{7} \\ \frac{2}{7} \end{pmatrix}$ for the second scenario.

$u_1^t\left[h^1 - T^1x^0\right] = \left(\frac{2}{3}\ \frac{1}{3}\right)\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \frac{11}{3}$ indicating that $x^0$ is not feasible for both scenarios.

We construct the feasible constraint for the first scenario, $u_1^t\left[h^1 - T^1x^0\right] > 0$, $5x_2 \geq 11$; we add this cut to the first problem$(P_R)$ and we get a new integer solution $x^1 = (0\ 3)^t$; we test its feasibility : for $(Pu_1)$, the maximum is at:$\begin{pmatrix} u_1^1 \\ u_2^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

For the second scenario, the maximum is at: $\begin{pmatrix} u_1^2 \\ u_2^2 \end{pmatrix} = \begin{pmatrix} \frac{5}{7} \\ \frac{2}{7} \end{pmatrix}$. $u_2^t\left[h^2 - T^2x^1\right] = \frac{8}{7} > 0$; the feasibility cut is $11x_1 + 8x_2 \geq 32$. Adding this cut to the problem $(P_{R1})$ we obtain a new solution $x^2 = (1\ 3)^t$. We get $u_1 = u_2 = 0$. The solution is feasible for both first and second scenario.
The optimality test of $x^2 = (1\ 3)^t$: the resolution of $(P(\pi_1))$ and $(P(\pi_2)$ gives maximums at $\begin{pmatrix} \pi_1^1 \\ \pi_2^1 \end{pmatrix} = \begin{pmatrix} -1 \\ -0.5 \end{pmatrix}$ and $\begin{pmatrix} \pi_1^2 \\ \pi_2^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ respectively. $Q\left(x, sc1\right) = \pi_1\left[h^1 - T^1x^2\right] = 2$, $Q\left(x, sc2\right) = \pi_2\left[h^2 - T^2x^2\right] = 5$, thus, $Q(x) = P_1Q\left(x, sc1\right) + P_2Q\left(x, sc2\right) = \frac{1}{2}(2) + \frac{1}{2}(5) = \frac{7}{2}$.
$\theta = -\infty < Q(x)$, we introduce the optimality cut of the forme: $\theta \geq \sum_{r=1}^{2} p^r\pi^t[h(sc^r) - T(sc^r)(x)]$
The main deterministic problem with the constraints $2x_1 - 5x_2 + 4\theta \geq 1$ is being solved $\theta = Q(x) = 3.5$. Then $x^2 = (1\ 3)^t$ is the optimal feasible solution. We test the efficiency of this solution, we obtain $\Theta = 0$ and $x^2 = (1\ 3)^t$ is an efficient solution of $(P(x^2))$; $\phi_{opt} = +\infty, \phi(x^2) = 20 < \phi_{opt}, X_{opt} = (1\ 3)^t, \phi_{opt} = 20$
**Step 3** we put $\ell := \ell + 1$ and solve problem $(P_R)$ adding the following constraints

$$\begin{array}{rcrcrcl}
 & & 5x_2 & & & \geq & 11 \\
11x_1 & + & 8x_2 & & & \geq & 32 \\
-3x_1 & + & x_2 & + & 8y_1^1 & \leq & 7 \\
5x_1 & - & 2x_2 & + & 27y_2^1 & \leq & 25 \\
2x_1 & - & x_2 & + & 12y_2^1 & \leq & 10 \\
 & & & & y_1^1 & \leq & 1 \\
 & & & & y_2^1 & \leq & 1 \\
 & & & & y_3^1 & \leq & 1 \\
y_1^1 & + & y_2^1 & + & y_3^1 & \geq & 1 \\
x_1 & , & x_2 & , & y_1^1 & , & y_2^1 \quad , \quad y_3^1 \quad \in \quad \mathbb{N}
\end{array}$$

An optimal solution is $x^3 = (2\ 3)^t$, $\phi(x^3) = 11$, $\phi(x^3) < \phi_{opt}$ and we return again to step 2.
**Iteration 2** $x^3 = (2\ 3)^t$ is admissible for both scenarios and optimal with $Q\left(x, sc1\right) = 2, Q\left(x, sc2\right) = 4$, $Q(x) = P_1Q\left(x, sc1\right) + P_2Q\left(x, sc2\right) = \frac{1}{2}(2) + \frac{1}{2}(4) = 3$. $\theta = Q(x) = 3$. Then $x^3 = (2\ 3)^t$ is the optimal feasible solution. According to the efficiency test, $x^3$ is not efficient for problem $(P)$, $\Theta \neq 0$, the solution $x^4 = (3\ 6)^t$ is an efficient one, we test again its feasibility and optimality as above, we obtain $\begin{pmatrix} u_1^1 \\ u_2^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} u_1^2 \\ u_2^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
The solution is feasible for first and second scenario. The optimality test of $x^4 = (3\ 6)^t$ gives $\begin{pmatrix} \pi_1^1 \\ \pi_2^1 \end{pmatrix} = \begin{pmatrix} -1 \\ -0.5 \end{pmatrix}$ and $\begin{pmatrix} \pi_1^2 \\ \pi_2^2 \end{pmatrix} = \begin{pmatrix} -3.67 \\ -0.78 \end{pmatrix}$
$Q\left(x, sc1\right) = \pi_1\left[h^1 - T^1x^4\right] = 9.5$; $Q\left(x, sc2\right) = \pi_2\left[h^2 - T^2x^4\right] = 13.95$
$Q(x) = P_1Q\left(x, sc1\right) + P_2Q\left(x, sc2\right) = \frac{1}{2}(9.5) + \frac{1}{2}(13.95) = 11.725$

$\theta = Q(x)$, then $x^4 = (3\ 6)^t$ is the optimal solution with $\theta = 11.725$; $\phi(x^4) = 21, \phi_{opt} < \phi^(x^4)$. The algorithm terminates with $x_{opt} = (1\ 3)^t$ and $\phi_{opt} = 20$.

# 6 Conclusion

We have presented an exact method that optimizes a linear function over an integer efficient solutions set in stochastic environment. We achieve this objective by combining two techniques : one uses L-shapped method and the second explores progressively the admissible region going through only efficient solutions that ameliorates the main linear criterion; the domain is being reduced consequently until it becomes empty. The problem with deterministic parameters is known to be very hard, resolving it under uncertainty becomes harder. As far as we know, the problem has not been yet studied in the literature, we suggest development of new benchmarks in order to make a rational experimental study. Concerning the complexity, as we are obliged to transform the stochastic problem into deterministic one, the problem remains very hard as was stated in deterministic case by N.C. Guyen ([9]).

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# Optimal Life Insurance, Consumption and Investment Decisions

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**Abstract**

We describe an extension to Merton's famous continuous time model of optimal consumption and investment, in the spirit of previous works by Pliska and Ye, to allow for a wage earner to have a random lifetime and to use a portion of the income to purchase life insurance in order to provide for his estate, while investing his savings in a financial market composed by one risk-free security and an arbitrary number of risky securities whose diffusive term is given by a multi-dimensional Brownian motion. The wage earner's problem is then to find the optimal consumption, investment, and insurance purchase decisions in order to maximize expected utility of his family consumption, of the size of the estate in the event of premature death, and of his total wealth at the time of retirement. Dynamic programming methods are used to obtain explicit solutions for the case of discounted constant relative risk aversion utility functions and some economically relevant results are obtained.

## 1 Introduction

We consider the problem faced by a wage earner having to make decisions continuously about three strategies: consumption, investment and life insurance purchase during a given interval of time $[0, \min\{T, \tau\}]$, where $T$ is a fixed point in the future that we will consider to be the retirement time of the wage earner and $\tau$ is a random variable representing the wage earner's time of death. We assume that the wage earner receives his income at a continuous rate $i(t)$ and that this income is terminated when the wage earner dies or retires, whichever happens first. One of our key assumptions is that the wage earner's lifetime $\tau$ is a random variable and, therefore, the wage earner needs to buy life insurance to protect his family for the eventuality of premature death. The life insurance depends on a premium insurance rate $p(t)$ such that if the insured pays $p(t) \cdot \delta t$ and dies during the ensuing short time interval of length $\delta t$ then the insurance company will pay one dollar to the insured's estate (so this is like term insurance with an infinitesimal term). We also assume that the wage earner wants to maximize the satisfaction obtained from a consumption process with rate $c(t)$. In addition to consumption and purchase of a life insurance policy, we assume that the wage earner invests the full amount of his savings in a financial market consisting of one risk-free security and a fixed number $N \geq 1$ of risky securities with diffusive terms driven by $M$-dimensional Brownian motion.

The wage earner is then faced with the problem of finding strategies that maximize the utility of (i) his family consumption for all $t \leq \min\{T, \tau\}$; (ii) his wealth at retirement date $T$ if he lives that long;

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and (iii) the value of his estate in the event of premature death. Various quantitative models have been proposed to model and analyze this kind of problem, at least problems having at least one of these three objectives. This literature is perhaps highlighted by Yaari [10] who considered the problem of optimal financial planning decisions for an individual with an uncertain lifetime as well as by Merton [5, 6] who emphasized optimal consumption and investment decisions but did not consider life insurance. These two approaches were combined by Richard [9], who considered a life-cycle life insurance and consumption-investment problem in a continuous time model. Later, Pliska and Ye [7, 8] introduced a continuous-time model that combined the more realistic features of all those in the existing literature and extended the model proposed previously by Richard. While Richard assumed that the lifetime of the wage earner is limited by some fixed number, the model introduced by Pliska and Ye had the key feature that the duration of life is a random variable which takes values in the interval $]0,\infty[$ and is independent of the stochastic process defining the underlying financial market. Moreover, Pliska and Ye made the following refinements to the theory: (i) the planning horizon $T$ is now seen as the moment when the wage earner retires, contrary to Richard's interpretation as maximum life size; and (ii) the utility of the wage earner's wealth at the planning horizon $T$ is taken into account as well as the utility of lifetime consumption and the utility of the bequest in the event of premature death.

Whereas Pliska and Ye's financial market involved only one security that was risky, in the present work we study the extension where there is an arbitrary (but finite) number of risky securities. The existence of these extra risky securities gives greater freedom for the wage earner to manage the interaction between his life insurance policies and the portfolio containing his savings invested in the financial market. Some examples of these interactions are described below.

Following Pliska and Ye, we use the model of uncertain life found in reliability theory, commonly used for industrial life-testing and actuarial science, to model the uncertain time of death for the wage earner. This enables us to replace Richard's assumption that lifetimes are bounded with the assumption that lifetimes take values in the interval $]0,\infty[$. We then set up the wage earner's objective functional depending on a random horizon $\min\{T,\tau\}$ and transform it to an equivalent problem having a fixed planning horizon, that is, the wage earner who faces unpredictable death acts as if he will live until some time $T$, but with a subjective rate of time preferences equal to his "force of mortality" for his consumption and terminal wealth. This transformation to a fixed planning horizon enables us to state the dynamic programming principle and derive an associated Hamilton-Jacobi-Bellman (HJB) equation. We use the HJB equation to derive the optimal feedback control, that is, optimal insurance, portfolio and consumption strategies. Furthermore, we obtain explicit solutions for the family of discounted Constant Relative Risk Aversion (CRRA) utilities and examine the economic implications of such solutions.

In the case of discounted CRRA utilities our results generalize those obtained previously by Pliska and Ye. For instance, we obtain: (i) an economically reasonable description for the optimal expenditure for insurance as a decreasing function of the wage earner's overall wealth and a unimodal function of age, reaching a maximum at an intermediate age; (ii) a more controversial conclusion that possibly an optimal solution calls for the wage earner to sell a life insurance policy on his own life in his career. Nonetheless, the extra risky securities in our model introduce novel features to the wage earner's portfolio and insurance management interaction such as: (i) a young wage earner with small wealth has an optimal portfolio with larger values of volatility and higher expected returns, with the possibility of having short positions in lower yielding securities; and (ii) a wage earner who can buy life insurance policies will choose a more conservative portfolio than a wage earner who is without the opportunity to buy life insurance, the distinction being clearer for young wage earners with low wealth. Full details of the analysis reviewed here will be provided in a forthcoming paper [1, 2].

## 2 The optimal control problem

The wage earner is then faced with the problem of finding strategies that maximize the utility of:

**(i)** his family consumption for all $t \leq \min\{T, \tau\}$;

**(ii)** his wealth at retirement date $T$ if he lives that long;

**(iii)** the value of his estate in the event of premature death.

This problem can be formulated by means of optimal control theory: the wage earner goal is to maximize some cost functional subject to (i) the (stochastic) dynamics of the state variable, i.e. the dynamics of the wealth process $X(t)$; (ii) constraints on the control variables, i.e. the consumption process $c(t)$, the premium insurance rate $p(t)$ and the portfolio process $\theta(t)$; and (iii) boundary conditions on the state variables.

Let us denote by $\mathscr{A}(x)$ the set of all admissible decision strategies, i.e. all admissible choices for the control variables $\nu = (c, p, \theta) \in \mathbb{R}^{N+2}$. The wage earner's problem can then be restated as follows: find a strategy $\nu = (c, p, \theta) \in \mathscr{A}(x)$ which maximizes the expected utility

$$V(x) = \sup_{\nu \in \mathscr{A}(x)} E_{0,x}\left[\int_0^{T\wedge\tau} U(c(s),s)\,\mathrm{d}s + B(Z(\tau),\tau)I_{\{\tau\leq T\}} + W(X(T))I_{\{\tau>T\}}\right], \tag{1}$$

where $T \wedge \tau = \min\{T, \tau\}$, $I_A$ denotes the indicator function of event $A$, $U(c,\cdot)$ is the utility function describing the wage earner's family preferences regarding consumption in the time interval $[0, \min\{T, \tau\}]$, $B(Z,\cdot)$ is the utility function for the size of the wage earners's legacy in case $\tau < T$ and $W(X)$ is the utility function for the terminal wealth at time $t = T$.

## 3 Dynamic programming principle

Let us denote by $\mathscr{A}(t,x)$ the set of admissible decision strategies $\nu = (c, p, \theta)$ for the dynamics of the wealth process with boundary condition $X(t) = x$. For any $\nu \in \mathscr{A}(t,x)$ we define

$$J(t,x;\nu) = E_{t,x}\left[\int_t^{T\wedge\tau} U(c(s),s)\,\mathrm{d}s + B(Z(\tau),\tau)I_{\{\tau\leq T\}} + W(X(T))I_{\{\tau>T\}} \;\middle|\; \tau > t, \mathscr{F}_t\right]$$

and note that the optimal control problem (1) can be restated in dynamic programming form as

$$V(t,x) = \sup_{\nu\in\mathscr{A}(t,x)} J(t,x;\nu)\,.$$

**Lemma 1** (Dynamic programming principle)**.** *For $0 \leq t < s < T$, the maximum expected utility $V(t,x)$ satisfies the recursive relation below*

$$V(t,x) = \sup_{\nu\in\mathscr{A}(t,x)} E\left[\exp\left(-\int_t^s \lambda(v)\mathrm{d}v\right)V(s,X(s)) + \int_t^s \overline{F}(u,t)U(c(u),u) + f(u,t)B(Z(u),u)\,\mathrm{d}u \;\middle|\; \mathscr{F}_t\right].$$

## 4 The family of discounted CRRA utilities

In this section we describe the special case where the wage earner has the same discounted CRRA utility functions for the consumption of his family, the size of his legacy and his terminal wealth.

Assume that $\gamma < 1$, $\gamma \neq 0$ and $\rho > 0$ and let

$$U(c,t) = \mathrm{e}^{-\rho t}\frac{c^\gamma}{\gamma}\,, B(Z,t) = \mathrm{e}^{-\rho t}\frac{Z^\gamma}{\gamma}\,, W(X) = \mathrm{e}^{-\rho T}\frac{X^\gamma}{\gamma}\,. \tag{2}$$

### 4.1 The optimal strategies

**Proposition 1.** *The optimal strategies in the case of discounted constant relative risk aversion utility functions are given by*

$$\begin{aligned} c^*(t,x) &= \frac{1}{e(t)}(x+b(t)) \\ p^*(t,x) &= \eta(t)\left((D(t)-1)x+D(t)b(t)\right) \\ \theta^*(t,x) &= \frac{1}{x(1-\gamma)}(x+b(t))(\sigma\sigma^T)^{-1}\alpha(t)\ , \end{aligned}$$

*where*

$$\begin{aligned} b(t) &= \int_t^T i(s)\exp\left(-\int_t^s r(v)+\eta(v)\,\mathrm{d}v\right)\mathrm{d}s \\ e(t) &= \exp\left(-\int_t^T H(v)\,\mathrm{d}v\right)+\int_t^T \exp\left(-\int_t^s H(v)\,\mathrm{d}v\right)K(s)\,\mathrm{d}s \\ H(t) &= \frac{\lambda(t)+\rho}{1-\gamma}-\gamma\frac{\Sigma(t)}{(1-\gamma)^2}-\frac{\gamma}{1-\gamma}(r(t)+\eta(t)) \\ D(t) &= \frac{1}{e(t)}\left(\frac{\lambda(t)}{\eta(t)}\right)^{1/(1-\gamma)} \\ K(t) &= \frac{(\lambda(t))^{1/(1-\gamma)}}{(\eta(t))^{\gamma/(1-\gamma)}}+1 \\ \Sigma(t) &= \sum_{n=1}^N\left((\mu_n(t)-r(t))\sum_{i=1}^N\xi_{ni}(\mu_i(t)-r(t))\right)-\frac{1}{2}\sum_{m=1}^M\left(\sum_{n=1}^N\sigma_{nm}(t)\left(\sum_{i=1}^N\xi_{ni}(\mu_i(t)-r(t))\right)\right)^2 . \end{aligned}$$

From the explicit knowledge of the optimal strategies, several economically relevant conclusions can be obtained, such as:

- under realistic restrictions on the parameter space, the current wealth of the wage earner has a negative effect on his life insurance purchase; however, the future income of the wage earner has a positive effect on his life insurance purchase;
- the interest rate $r$ and the risk-aversion parameter $\gamma$ have a negative effect on the wage earner life insurance purchase, while the utility discount rate $\rho$ and the hazard rate $\lambda$ have a positive effect on the wage earner life insurance purchase.

### 4.2 The interaction between life insurance purchase and portfolio management

The extra risky securities in our model introduce novel features to the wage earner's portfolio and insurance management interaction. The following result compares the portfolio management of a wage earner who can buy life insurance policies with that of a wage earner who is without the opportunity to buy life insurance.

**Theorem 1.** *Suppose that for every $n\in\{1,...,N\}$ we have that $\mu_n(t)>r(t)$ and let $(\tilde{c},\tilde{\theta})$ be the optimal strategies for the standard consumption-investment control problem obtained from* (1) *by setting $p(t)\equiv 0$ and $B(Z,t)\equiv 0$. Then, we have that*

$$\tilde{\theta}_n(t)>\theta_n(t)\ , \qquad n=1,...,N\ ,$$

*that is, a wage earner who can buy life insurance policies will choose a more conservative portfolio than a wage earner who is without the opportunity to buy life insurance.*

# 5 Conclusion

We have introduced a model for optimal insurance purchase, consumption and investment for a wage earner with an uncertain lifetime with an underlying financial market consisting of one risk-free security and a fixed number of risky securities with diffusive terms driven by multidimensional Brownian motion.

When we restrict ourselves to the case where the wage earner has the same discounted CRRA utility functions for the consumption of his family, the size of his legacy and his terminal wealth, we obtain explicit optimal strategies and describe new properties of these optimal strategies. Namely, we obtain economically relevant conclusions such as: (i) a young wage earner with smaller wealth has an optimal portfolio with larger values of volatility and higher expected returns; and (ii) a wage earner who can buy life insurance policies will choose a more conservative portfolio than a similar wage earner who is without the opportunity to buy life insurance.

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# Vehicle-routing with load balancing for work planning

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**Abstract**

A vehicle routing (VR) model is presented, which was employed to plan the work of traveling agents who provide a certain service – artificial insemination of cows. An additional requirement for the work planning task is to balance the work load among the agents, which requires modifying the VR model and the solution scheme. The results demonstrate the value of using mathematical optimization for work planning tasks – improvements in the order of 20% and more were obtained in key performance measures: service level, as measured by arrival time to the last customer, traveling distances, traveling times and working time (traveling plus service). Moreover, these improvements were obtained in parallel to better balancing the work load among the agents – the standard deviation of the number of services provided by an agent is reduced by 18%, but the standard deviation of the duration of the working day is reduced by **58**%!

These criteria, particularly the minimization of a sum or an average, while attempting a uniform distribution of the individual elements of the sum, often stand in conflict, as illustrated herein. Consequently, the achievement of both is neither intuitive nor trivial, certainly not in the significant extent reported.

**KEYWORDS**

Vehicle routing; Traveling salesman; Production-line balancing, OR in agriculture; Cows; Artificial insemination.

## 1. INTRODUTION

A vehicle routing (VR) Problem; e.g., Toth and Vigo (2002), arises when demand exceeds the capacity of a single vehicle and must be divided, either among several vehicles or to several tours of the same vehicle. Consequently, the solution involves the allocation of customers to vehicles or tours and the determination of the desired route for each tour. The routing component for each tour is a traveling salesman problem (TSP) (Laporte, 1992), Lawler et al. (1990). That is, the TSP can be seen as a special case of the VRP in which vehicle capacity is larger than the demand (Clark and Wright, 1962).

In this presentation, the demand is for service and the is divided among several vehicles. The goal is to plan the work of the agents who ride the vehicles and provide the service, and a requirement is added that the workload will be balanced among the agents. Accordingly, the capacity limits are not parameters,but the demand is to be divided most equally among the agents. Moreover, workload is not measured by a single measure but by two – in addition to demand, time is considered, which comprises both service and travel time. Service time is correlated to the demand but not completely, and again, time limits are not parameters but total time is to be divided most evenly among the agents. Furthermore, total time constitutes the objective criterion, too – it is to be minimized. Minimization of a sum and uniform distribution of its components do not necessarily go together, and may result in conflicts. Due to the additional requirement for workload balancing, existing solution methods; e.g., (Baldacci and Mingozzi, 2009), (Potvin, 2009) have to be modified. The solution scheme developed in this study is presented following the presentation of the model.

This study was initiated in order to develop tools for work planning for the cows-artificial-insemination (AI) service in Israel. Time is crucial in insemination: insemination of a cow within 6 to 17 h of entering

estrus increases the chance of conception and after 17 hours, the conception ratio declines rapidly (Maatje et al., 1997). In the US, reproductive efficiency plays an important role in the economic viability of the Dairy industry. De Vries (2006) reported that 1 percentage point increased pregnancy rate (PR) is valued between \$22 and \$35/cow per year when PR varied from 15 to 19%. Most cow inseminations performed today in developed countries are artificial, in Israel, over 95%: ~ 350,000/yr inseminations are artificial. In artificial insemination, sperm is obtained from bulls in central locations under laboratory conditions and is injected into the reproductive systems of cows in estrus in various farms. AI has many advantages compared to natural services such as the elimination of venereal diseases, more accurate dry-off dates, reduced incidence of dystocia, increased safety for farm employees (Vishwanah, 2003) and greater genetic improvement resulting in daughters that are more productive and profitable (Norman and Powell, 1992).

In Europe, especially in the eastern countries, a large variety in farming systems exists – farms of up to 10 cows, 10-50 cows, 50-150 cows, and over 150 cows. In such "many but small" farms targeted breeding is particularly important since with the growth of those farms genetic level is becoming critical. Uncontrolled herds in rural areas are abundant in the new EU countries and often, cows are bred in the hills or sub-hilly rural regions. The accessibility of these farms is often difficult (Eben-Chaime et al., 2011). In situations like these, the development of planning tools is essential and priceless for any AI company (governmental, farmers' organization owned or a private company) who aims to reach a significant number of farms.

Both the model and the solution scheme were applied on real data of the Israeli AI service and the results are impressive.

## 2. THE MODEL

The model was adapted from Toth and Vigo (2002) with additional constraints.

**Parameters**

C – the set of customers. n – the number of customers.

D – the set of agents' homes. m – the number of agents.

Agents' homes are the depots in the model. In case more than one agent lives in the same place, then following Lim and Wang (2005), a dummy depot is created for each of them to form a multi-depot model, with a single vehicle in each depot.

$e_{ij}$ – the traveling time from customer i to customer j.

$q_i$ – daily average of the number of service units required by customer i.

Q – daily average of the number of services provided by an agent, $Q = \sum_{i \in C} q_i \Big/ m$.

$t_i$ – average daily service time for customer i.

T – average daily working time of an agent – service plus traveling.

**Decision variable**

$x_{ijk} = \begin{cases} 1, & \text{Route. if agent k visit customer i after customer j .} \\ 0, & \text{Otherwise} \end{cases}$

$y_{ik} = \begin{cases} 1, & \text{Customer allocation. if customer i is visited by agent k.} \\ 0, & \text{Otherwise} \end{cases}$

**The model**

$$min \sum_{i,j \in C \cup D} \sum_{k \in D} e_{ij} x_{ijk} \qquad (1)$$

**s.t**

$$\sum_{k \in D} y_{ik} = 1 \qquad \forall i \in C \cup D \qquad (2)$$

$$\sum_{i,j \in C \cup D} e_{ij} x_{ijk} + \sum_{i \in C} t_i y_{ik} \le T + \varepsilon_1 \qquad \forall k \in D \qquad (3)$$

$$\sum_{i,j\in C\cup D} e_{ij}x_{ijk} + \sum_{i\in C} t_i y_{ik} \geq T - \varepsilon_2 \qquad \forall k \in D \qquad (4)$$

$$\sum_{i\in C} q_i y_{ik} \leq Q + \varepsilon_3 \qquad k = 1,....,m \qquad (5)$$

$$\sum_{j\in C\cup D} x_{ijk} = \sum_{j\in C\cup D} x_{jik} = y_{ik} \qquad \forall i \in D\cup C \qquad \forall k \in D \qquad (6)$$

$$\sum_{i,j\in s} x_{ijk} \leq |S| - 1 \qquad \forall S \subseteq \{2,.....n\} \qquad \forall k \in D \qquad (7)$$

$$y_{ik} \in \{0,1\} \qquad \forall i \in D\cup C \qquad \forall k \in D \qquad (8)$$

$$x_{ijk} \in \{0,1\} \qquad \forall i,j \in D\cup C \qquad \forall k \in D \qquad (9)$$

The objective function (1) – minimizes total daily travel time of all agents.

The constraints (2) ensure that all customers are serviced. By constraints (2), the sum, $\Sigma_{i,k}\, t_i\, y_{ik}$, is a constant, independent of the variables $y_{ik}$. Hence, (1) also minimizes the average, among agents, of the total working time, $T = \left(\sum_{i,j,k} e_{ij}x_{ijk} + \sum_{i,k} t_i y_{ik}\right) \Big/ m$.

Constraints (3) and (4) are the addition of this work!

Together, (3), (4) and (5) enforce balanced work load among the agents. Equations (3) and (4) balance the total working time – including travels, among the agents – the smaller is $\varepsilon_1 + \varepsilon_2$,. Equations (5) constrain the maximal value of the daily average number of services provided by an agent. While Constraints (5) resemble the capacity constraints of the VRP, they differ in that $\varepsilon_3$, as well as $\varepsilon_1$, $\varepsilon_2$, and T are not parameters. The smaller are $\varepsilon$'s, the better is the balance. The exclusion of travel times in (5) makes significant differences (see Section 6). Constraints (6) are twofold: first, conservation of flow – the number of arrivals to a customer equals the number of departures; second – a customer can be served only by the agent to which it is allocated. Constraints (7) are TSP constraints that prevent sub-tours. Constraints (8) and (9) – the decision variables are binary – 0 or 1.

## 3. THE SOLUTION SCHEME

The solution scheme is a cluster-first-route-second (e.g., Potvin, 2009) scheme and comprises four steps. The first two are clustering steps: (1) initial allocation of customers to agents; (2) local improvement of the clusters by neighborhood search to reduce traveling distances; next, an (3) additional neighborhood search to improve the workload balance among the agents; and finally (4) determination of the optimal route for each agent within his selected area.

Four heuristic algorithms for initial allocation of customers to agents were considered: (1) minimal-added-travel-time algorithm; (2) max-min algorithm; (3) nearest-depot algorithm; and (4) center-of-mass (CM) algorithm. The first three do parallel assignment: the urgency for each customer is calculated, giving consideration to all depots at the same time, and the customer with the highest urgency is allocated first. The CM algorithm differs in that customer allocation is determined according to work area: one depot – agent home, is selected each time and customers are added to this depot, until the total working time or the average number of services provided reaches the upper limit. With any algorithm, the allocation of customers to depots satisfies all model constraints: after each allocation of a customer to a depot, the total working time – the sum of travel times and service times – is recalculated.

***The minimal-added-traveling-time algorithm***

This algorithm was developed by of Lim and Wang (2005).

1. Define the agent's home as the depot for each route.
2. Initialize a route for each agent.
3. Find the customer with the minimal addition to the traveling time of one of the routes.
4. Add the customer to the route.
5. Repeat steps 3 and 4 until all customers are allocated.

***The max-min algorithm***

This algorithm is based partly on the saving algorithm of Clark and Wright (1962) and partly on the weight-based line-balancing algorithms of Nahmias (2001).

1. Initialize a route for each agent.
2. Calculate for each customer the traveling time from each depot, and find the nearest depot to the customer.
3. Sort the customers according to traveling time: the customer with the **longest traveling time at the top** of the list.
4. Allocate the customer at the top of the list – the remotest customer to its nearest depot.
5. Repeat steps 3 and 4 until all customers are allocated.

***The nearest-depot algorithm***

This is the first stage of a two-stage algorithm commonly used for MDVR problems, in which customers are allocated to the nearest depots. The second stage – routing is the 4$^{th}$ step of our scheme.

1. Initialize a route for each agent;
2. Calculate the traveling time from each depot to each customer, and finds the nearest depot to the customer;
3. Sort the customers according to traveling time: the customer with the **shortest traveling time at the top** of the list;
4. Allocates the customer at the top of the list to its nearest depot.
5. Repeat steps 3 and 4 until all customers are allocated.

***The CM algorithm***

This algorithm constructs clusters of customers, starting from the peripheral areas – the north and the south – and working towards the center. Each cluster –work area is linked to the nearest depot – agent home. Customers are added to the cluster in order of their closeness to its center of mass, as defined by traveling distance, while no constraint is broken. After each customer addition, the center of mass is recalculated. The algorithm:

1. Sort the customers according to x-y coordinates: the northernmost customer at the top of the list;
2. Choose a customer from the unallocated pool, alternately from the top and the bottom of the customer list;
   2.1. Initialize a work area: allocate the customer to its nearest depot;
   2.2. Calculate the center of mass of the work area;
   2.3. Add the customer nearest to the center of mass of the work area;
   2.4. As long it is possible to add customers to the work area without violating model's constraints, return to step 2.2;
3. Repeat step 2, until all customers are allocated.

## 4. RESULTS

In Table 1 current performances and the outcomes of both clustering steps are compared. The contribution of employing mathematical optimization for work planning is apparent – total and average distances and times are reduced in all but the total traveling distance with Max-min initial allocation. Also apparent is the contribution of the second step, which was necessary for the Max-min initial allocation to improve the current traveling distances. Of particular importance is the service level, which is improved by all algorithms – arrival time to the last cow is reduced.

However, the reductions in total values and averages are obtained on account of the variances – standard deviations of both load measures are bigger – and often significantly bigger than current values. Then again, the variances are larger despite the decrease in the maximal values of both day duration and the number of services!! Apparently, the larger variances are due to smaller minimal values. To re-balance the work load, Step 3 was added, whose contribution is presented in Table 2.

Notice that the average number of services - cows treated, is not changed as this number is part of the input to the model, not a decision variable.

These results demonstrate the conflicts between goals. First, total and average value increased by Step 3 for the standard deviations to decrease. In addition, the nearest depot algorithm that results in the smaller

Table 1. The contribution of both clustering steps

| | Current | Minimal added traveling time | | Max-min | | Nearest depot | | Center of mass | |
|---|---|---|---|---|---|---|---|---|---|
| | | Step 1 | Step 2 | Step 1 | Step 2 | Step 1 | Step 2 | Step 1 | Step 2 |
| Traveling distance of all agents (km) | 4408.5 | 4087.6 | 3785.1 | 5034 | 4117.6 | 3853.4 | 3492.2 | 4037.6 | 3662.4 |
| Average traveling time of all agents (h) | 92.2 | 83.5 | 76.4 | 103.4 | 82.6 | 80.6 | 72.1 | 83 | 74.2 |
| Longest agent day (h) | 8 | 7.1 | 7.2 | 7.2 | 7.2 | 7.6 | 7.6 | 6.2 | 6.2 |
| Shortest agent day (h) | 4 | 0.2 | 0.2 | 3.4 | 2.7 | 1.3 | 1.7 | 2.6 | 3.7 |
| Average duration of agent day (h) | 6.1 | 5.8 | 5.5 | 6.4 | 5.7 | 5.7 | 5.4 | 5.7 | 5.4 |
| Standard deviation (SD) of day duration (h) | 1.2 | 1.4 | 1.5 | 0.9 | 1.2 | 1.7 | 1.7 | 0.7 | 0.6 |
| Service level – arrival time to the last customer (h) | 7.6 | 7 | 6.7 | 6.9 | 6.7 | 7.2 | 7.3 | 5.9 | 5.9 |
| Maximal number of services by an agent | 50 | 44 | 43.8 | 43 | 42.9 | 48.8 | 48.9 | 42.5 | 42.4 |
| Minimum number of services by an agent | 25.9 | 2.2 | 2.2 | 16.9 | 19.7 | 9 | 14.5 | 18.9 | 23.4 |
| Average number of services by an agent | 37.6 | 37.6 | 37.6 | 37.6 | 37.6 | 37.6 | 37.6 | 37.6 | 37.6 |
| SD of the number of services by an agent | 6.1 | 10.7 | 10 | 6.1 | 5.8 | 11.6 | 11.3 | 6.3 | 4.9 |

Table 2. Work-load balancing – second neighbourhood search

| | Current situation | Minimal added traveling time | | Max-min | | Nearest depot | | Center of mass | |
|---|---|---|---|---|---|---|---|---|---|
| | | Step 2 | Step 3 | Step 2 | Step 3 | Step 2 | Step 3 | Step 2 | Step 3 |
| Traveling distance of all agents (km) | 4408.5 | 3785.1 | 3988.2 | 4117.6 | 4322.6 | 3492.2 | 3631.4 | 3662.4 | 3682.3 |
| Average traveling time of all agents (h) | 92.2 | 76.4 | 80.4 | 82.6 | 86.9 | 72.1 | 75.1 | 74.2 | 74.7 |
| Longest agent day (h) | 8 | 7.2 | 7.2 | 7.2 | 7.2 | 7.6 | 7.6 | 6.2 | 6.2 |
| Shortest agent day (h) | 4 | 0.2 | 3.8 | 2.7 | 4.7 | 1.7 | 1.7 | 3.7 | 4.5 |
| Average duration of agent day (h) | 6.1 | 5.5 | 5.7 | 5.7 | 5.9 | 5.4 | 5.5 | 5.4 | 5.5 |
| Standard deviation (SD) of agent day duration (h) | 1.2 | 1.5 | 0.9 | 1.2 | 0.8 | 1.7 | 1.4 | 0.6 | 0.5 |
| Service level – arrival time to the last customer (h) | 7.6 | 6.7 | 6.7 | 6.7 | 6.7 | 7.3 | 6.9 | 5.9 | 5.9 |
| Maximal number of services by an agent | 50 | 43.8 | 43.9 | 42.9 | 42.9 | 48.9 | 48.9 | 42.4 | 42.4 |
| Minimum number of services by an agent | 25.9 | 2.2 | 14.6 | 19.7 | 19.7 | 14.5 | 15.3 | 23.4 | 23.4 |
| Average number of services by an agent | 37.6 | 37.6 | 37.6 | 37.6 | 37.6 | 37.6 | 37.6 | 37.6 | 37.6 |
| SD of the number of services by an agent | 6.1 | 10 | 8.1 | 5.8 | 5.5 | 11.3 | 9.9 | 4.9 | 5 |

total traveling distance, yield also the larger standard deviations. The center of mass algorithm seems to provide the best combination – smallest standard deviation, earliest arrival to the last cow and shorter days while the total traveling time is less than 2% of the shortest.

The final step is routing. Its results are presented in Table 3. Total traveling distance, average traveling time and average day duration are further reduced, while all other measures are not changed.

Table 3. Allocating customers to agents' areas – four steps

| | Current situation | Minimal added traveling time | | Max-min | | Nearest depot | | Center of mass | |
|---|---|---|---|---|---|---|---|---|---|
| | | Step 3 | Step 4 | Step 3 | Step 4 | Step 3 | Step 4 | Step 3 | Step 4 |
| Traveling distance of all agents (km) | 4408.5 | 3988.2 | 3845.4 | 4322.6 | 4280.8 | 3631.4 | 3527.7 | 3682.3 | 3606.7 |
| Average traveling time of all agents (h) | 92.2 | 80.4 | 75.8 | 86.9 | 84.8 | 75.1 | 72 | 74.7 | 72.4 |
| Longest agent day (h) | 8 | 7.2 | 7.2 | 7.2 | 7.2 | 7.6 | 7.6 | 6.2 | 6.2 |
| Shortest agent day (h) | 4 | 3.8 | 3.8 | 4.7 | 4.7 | 1.7 | 1.7 | 4.5 | 4.4 |
| Average duration of agent day (h) | 6.1 | 5.7 | 5.5 | 5.9 | 5.8 | 5.5 | 5.4 | 5.5 | 5.4 |
| Standard deviation (SD) of agent day duration (h) | 1.2 | 0.9 | 0.9 | 0.8 | 0.8 | 1.4 | 1.4 | 0.5 | 0.5 |
| Service level – arrival time to the last customer (h) | 7.6 | 6.7 | 6.7 | 6.7 | 6.7 | 6.9 | 6.9 | 5.9 | 5.9 |
| Maximal number of services by an agent | 50 | 43.9 | 43.9 | 42.9 | 42.9 | 48.9 | 48.9 | 42.4 | 42.4 |
| Minimum number of services by an agent | 25.9 | 14.6 | 14.6 | 19.7 | 19.7 | 15.3 | 15.3 | 23.4 | 23.4 |
| Average number of services by an agent | 37.6 | 37.6 | 37.6 | 37.6 | 37.6 | 37.6 | 37.6 | 37.6 | 37.6 |
| SD of the number of services by an agent | 6.1 | 8.1 | 8.1 | 5.5 | 5.5 | 9.9 | 9.9 | 5 | 5 |

To summarize, consider the center of mass algorithm. The service level is improved by more than 22% - arrival time to the last cow decreased to 5.9 hours from the current value of 7.6 hours. Similar improvements were obtained in the traveling distance (-18%), average traveling time (-21) and longest day duration (22.5%). The decrease in the average day duration is smaller since the shortest day is longer than its current length to enable work load balancing.

The improvement in work load balance is most significant – 18% reduction in the standard deviation of the number of services provided by each agent and **58**% reduction in the standard deviation of day duration – the distribution of day durations among the agents is now much more homogenous, between 4.4 to 6.2 hours rather than 4 to 8 hours. The difference between the changes of the two standard deviations reflects the lack of homogeneity in the geographical distribution of both agents – inseminators and clients – farms.

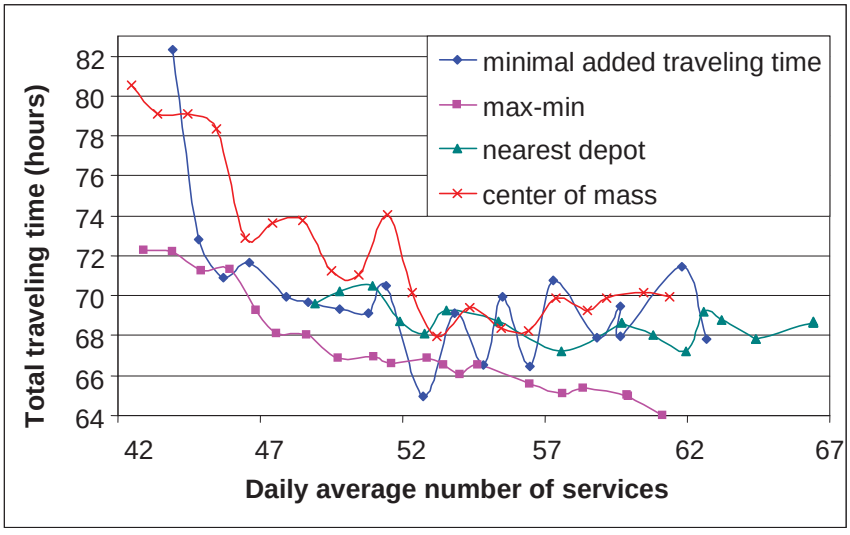


Figure 1: Total traveling time against daily average number of services.

## 5. DISCUSSION

As noted the minimization of total/average distances and times often stands in conflict with the distribution of their values uniformly. This conflict is further illustrated in Figure 1, where the effects of the upper limit on the daily average of the number of services, on the total traveling time, as determined by the various algorithms, for a maximal working day of 7.7 h, are portrayed. Relaxing the constraint on the daily number of services has a diminishing effect: increasing the limit from 42 to about 52 services per day significantly reduces the total traveling time (of all agents) for all the initial-allocation algorithms, but from 52 hours and on the effect is mixed.

## 6. CONCLUSSIONS

In this study, a vehicle routing model was employed to plan the work of traveling agents who provide a certain service – artificial insemination of cows. An additional requirement for the work planning task is to balance the work load among the agents, which requires modifying the VR model and the solution scheme.

The results demonstrate the value of using mathematical optimization for work planning tasks – improvements in the order of 20% and more were obtained in key performance measures: service level, as measured by arrival time to the last customer, traveling distances, traveling times and working time (traveling plus service). Moreover, these improvements were obtained in parallel to better balancing the work load among the agents – the standard deviation of the number of services provided by an agent is reduced by 18%, but the standard deviation of the duration of the working day is reduced by **58**%!

These criteria often stand in conflict. Particularly the minimization of a sum or an average conflicts an attempt for a uniform distribution of the individual elements of the sum, as illustrated herein. Consequently, the achievement of both is neither intuitive nor trivial, certainly not in the significant extent reported above.

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# Experimental analysis of selective uncrossing as a modification to the *2-Opt* technique for solving the Euclidean traveling salesman problem

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**Abstract**

This paper proposes the use of a variant of the classic 2-Opt technique based on the work of Leeuwen and Schoone (1980). This variant constructs a Hamiltonian circuit using different construction techniques, only eliminating and exchanging the pairs of edges that intersect, under three criteria. The methodology uses elements of computational geometry to make the search for the most promising edges more efficient. The experimental results show that the proposed variant is faster by several orders of magnitude with a slight difference in tour value, in respect of the classic 2-Opt technique.

**KEYWORDS**

Euclidean Traveling Salesman Problem, 2-Opt Heuristic, local search

## 1. INTRODUCTION

The Traveling Salesman Problem (TSP) is one of the problems that has been most studied and addressed by different fields of investigation. There is a vast amount of literature covering this problem from an experimental and/or theoretical perspective. The texts of Lawler (1985) as well as Gutin and Punnen (2002) provide an excellent historical context, together with giving the bases and methodologies for solving this problem. The Euclidian traveling salesman problem (ETSP) is a special case of the metric traveling salesman problem, in which the nodes to be visited are found on the Euclidian plane. This special case is, strictly speaking, known to be *NP-hard* (Papadimitriou, 1977), but permits approximation approaches that are based on dynamic programming, as demonstrated by Arora (1998) and Mitchell (1996). The symmetric TSP considers that traveling from point *A* to point *B* implies the same cost as traveling from *B* to *A*.

To solve this problem there are two basic methodologies: exact implicit or partial enumeration techniques that generally have a high computational cost, and heuristic solution techniques, whose strategies are based on common sense, but cannot guarantee that the best path is taken at each step; however, they offer approximate solutions (local optima) in solution times that are several orders of magnitude smaller than those obtained using exact techniques.

Branch and cut methods (Dantzig *et al.*,1954; Applegate *et al.*,1998; Applegate *et al.*, 2003) constitute the favorite methodology for exact solution, as they use strategies in which branches that do not offer an optimum solution can be discarded, so they are less expensive in terms of time than other exact techniques such as, for example, dynamic programming (Bellman,1962) or cutting planes (Schiebel, W.*et. al*, 1975; Miliotis, P., 1978; Fleischmann, B., 1985).

There are several metaheuristic and heuristic techniques that solve the Euclidian traveling salesman problem. The difference between these techniques is that heuristic techniques contain strategies that only solve the problem in question, whereas metaheuristics are strategies that can be applied to other problems. This project analyzes the heuristic techniques that solve the Euclidian traveling salesman problem. These techniques can be divided into two classes: construction techniques and improvement or

local search techniques. Construction techniques generate a Hamiltonian circuit incrementally and stop as soon as they find a valid circuit[1] or tour. These methods include the nearest neighbor heuristic and heuristics that use different insertion criteria (Rosenkrantz *et. al.*, 1977). Other algorithms in this category start from a subcircuit followed by a vertex insertion process. The most representative algorithms in this category are: arbitrary insertion (Puri and Gu, 1993), convex hull insertion (Sosic and Gu, 1990), largest angle insertion (Norback and Love, 1977) and (Norback and Love, 1979) and insertion with time difference (Or, 1976). In general, insertion techniques offer a better quality of initial circuit than the nearest neighbor technique because, in the case of the latter, few nodes remain at the end of the construction of the circuit that have not been visited, therefore the circuit has to be closed necessarily with high-cost edges. Insertion techniques that start with a subcircuit are more efficient in computation time than the ones that start from a single node and, in particular, the one that starts from the convex hull. The nearest neighbor and the insertion techniques, which will be used in this project, are the most commonly used.

Rosenkrantz *et al*. (1977) observed that some algorithms guarantee a circuit that is $\alpha$ times longer than the optimum length, at most. To date, the algorithm with the best polynomial approximation time is still Christofides' algorithm (1976), which finds a circuit $\alpha$= 1.5 times the length of the optimum circuit.

Local search heuristics basically use two principal kinds of mechanisms for circuit improvement: *edge* exchange and *chain* exchange. The *2-Opt* technique is a special case of the *r-Opt* edge exchange mechanism. In a given iteration, *r-Opt* removes *r* edges from the current circuit and finds the best reconnection of the remaining *r* chains. The complexity of an *r-Opt* iteration is $O(n^r)$. Croes (1952) was the first person to propose a systematic *2-Opt* method; and cites the theoretical bases for this local search technique, reporting for the first time success with neighborhood-based iterative methods when applied to combinatory optimization problems. The concept proposed by Croes was generalized by Lin and Kernighan (1973), who proposed a technique for exchanging *r* edges, specifying the best option for *r* in each iteration. Very sophisticated implementations of this method have been developed (Johnson and McGeoch, 1997; Helsgaun, 2000; Johnson et.al., 2001). And studies such as that of (Walter, 1989) show that this is a polynomially bounded technique. These techniques are probably the best heuristics for the TSP.

Despite the computational superiority of *r-Opt* implementations, simple exchange techniques such as *2-Opt* and *3-Opt* are still popular because of their easy of use and excellent performance considering the calculation time required. These techniques are often used as subroutines within improvement heuristics that solve, for example, the vehicle routing problem (M. L. Fisher, 1994).

To assess the performance of the proposed *Opt 2* modification, we used Euclidean test instances of the TSPLIB that the University of Heidelberg freely provides on its page (http://www.iwr.uni-heidelberg.de/groups/comopt/software/TSPLIB95). In a Euclidean test instance, the $V$ vertices correspond to points in the Euclidean distance space (most of dimension 2) and the distance function represents a metric of the Euclidean distance.

Leeuwen and Schoone (1980) analyzed a variant of the *2-Opt* local search technique for solving the Euclidean traveling salesman problem and mathematically demonstrated that if, in a Hamiltonian circuit, only the edges that intersect each other are iteratively replaced and the circuit reconnected again, the resulting circuit, in the worst case scenario, has complexity $O(n^3)$, meaning a substantial improvement in comparison to the exponential complexity that the classic 2-Opt technique can have.

This paper consists of six sections together with the conclusions. The second section describes the Euclidian traveling salesman problem and the third, the classic 2-Opt technique, as well as our proposed variant of this technique. The fourth section explains the conditions of the experiment, the fifth details the proposed methodology, while the sixth section explains the experimental results, followed by the conclusions.

## 2. DEFINITION OF THE EUCLIDIAN TRAVELING SALESMAN PROBLEM

Englert *et al*. (2008) provide a formal definition of the symmetric Euclidian traveling salesman problem. One starts from a set $V = \{v_1,...,v_n\}$ of *vertices* (depending on the context, they can also be called *points*) and for each pair of different vertices there is a distance function d: $V$ x $V \rightarrow \mathbb{R}_{\geq 0}$ that associates a distance $d(v_i,v_j)=d(v_j,v_i)$ with each pair $\{v_i,v_j\}$ of vertices. The aim is to find a circuit of minimum length that visits each vertex exactly once and returns to the initial vertex at the end of the run.

[1] In this text, all the circuits refer to Hamiltonian circuits.

A pair ($V$,d) from a nonempty set $V$ and a function d: $V$ x $V \rightarrow Ä_{\geq 0}$ is called a *metric space* if, for all the $x, y, z \in V$ the following properties are satisfied:

a) $d(x, y) = 0$ if and only if $x = y$ (*reflexivity*),
b) $d(x, y) = d(y, x)$ (*symmetry*),
c) $d(x, z) \leq d(x, y) + d(y, z)$ (*triangle inequality)*

If ($V$,d) is a metric space, then d is called a *metric* over $V$. An instance of the TSP with vertices $V$ and a distance function d is a *metric instance* of the TSP.

A well-known class of metrics on $Ä^d$ is the class of $L_p$ metrics. For each $p \in Ñ$, the distance $d_p(x, y)$ of two points $x \in Ä^d$ and $y \in Ä^d$ in respect of the metric $L_p$ is given by $d_p(x,y) = \sqrt[p]{|x_1 - y_1|^p + ... + |x_d - y_d|^p}$. The metric $L_1$ is often called the *Manhattan metric* and $L_2$ is known as the *Euclidian metric*. For $p \rightarrow \infty$, the metric $L_p$ converges on the metric $L_\infty$ defined by the distance $d_¥(x, y) = \max\{|x_1 - y_1|, ..., |x_d - y_d|\}$. An instance of the TSP ($V$,d) with $V \subseteq Ä^d$ where d is equal to $d_p$ restricted to $V$ is called an $L_p$ instance. The terms *Manhattan instance* and *Euclidian instance* are also used to denote the instances $L_1$ and $L_2$ respectively. When $p$ is clear in the context, d is written instead of $d_p$. We only examined Euclidian instances in this paper.

# 3. THE 2-OPT TECHNIQUE

## 3.1 Description of the 2-Opt technique

The *2-Opt* technique is one of the most basic, successful and commonly-used local search techniques of the edge exchange type for solving the symmetric TSP. The *2-Opt* technique starts with an initial circuit that is arbitrary or else has been built earlier using a construction technique and this circuit is successively improved by exchanging two of the edges contained in the circuit for another two edges. To be more precise, in each step of the *2-Opt* algorithm two edges $e_1 = \{u_1, u_2\}$ and $e_2 = \{v_1, v_2\}$ are chosen from the circuit, in such a way that $u_1$, $u_2$, $v_1$, $v_2$ are distinct and appear in the circuit in this order, then the algorithm replaces these edges with edges $\{u_1, v_1\}$ and $\{u_2, v_2\}$, resulting in a shorter circuit. The algorithm ends on a local optimum where no more improvements can be achieved.

It is a verifiable fact that if two edges $e_1$ and $e_2$ are successive edges, a feasible circuit cannot be constructed by means of the exchange that has been described. Therefore, the only feasible movements are the replacement of zero edges or of two non-successive edges $e_1$ and $e_2$. It is obvious that a single edge cannot be exchanged. The above implies that the size of the neighborhood of a circuit $\tau$ has $1+ n(n-3)/2 = \Theta(n^2)$ possible circuits or neighborhoods in total. Note that if the distance matrix is not symmetric, the direction of the circuit would have to be selected, increasing the size of the neighborhood by a factor of two $2+n(n-3)$. Therefore, in the worst case scenario, in a *2-Opt* iteration there is a complexity of $O(n^2)$.

The 2-Opt technique can, in some instances, achieve surprising results in both time and quality of the tour in respect of the optimum length, in a subquadratic number of steps (Johnson and McGeoch, 1997). However, there have been instances constructed for which an exponential number is required (Englert *et al.*, 2008).

There are experimental studies (Bentley, 1990; Babin, 2005) that show how significant improvements can be achieved in the response time of the 2-Opt technique. There have also been papers that show very efficient implementations of hardware (Mavroidis, 2007). However, theoretical analyses are still limited, (Englert *et al*., 2008).

There are two aspects that have to be analyzed when implementing the *2-Opt* local search technique. In the first place one must corroborate whether it is more convenient to propose an initial circuit provided by a construction technique known to perform well or one that is randomly generated. Babin (2005) found that the 2-Opt technique offers significantly better results than a chain exchange technique, when started from randomly generated solutions. However, Perttunen (1994) demonstrated that the performance of the edge exchange techniques, in particular, the 2-Opt technique, gives better results when starting from an initial solution created by a construction technique that is known for its good performance.

The other aspect for consideration is how to select the improvement movement: maybe the *first* improvement *movement* detected in every iteration or else the one that turns out to be the best improvement *movement* after calculating the savings. These issues were partially answered by Hansen and Mladenovic (2004) who conclude that it is slightly better and faster for the *2-Opt* technique to make the first improvement movement than to select the best movement starting from an arbitrarily-generated circuit; the results are inverted when starting from a circuit generated using some construction technique.

In addition to the two aspects mentioned in the above paragraphs, (Johnson et. al. , 2001) describe four rules for saving runtime when implementing the technique in a programming language:

1) avoid the redundancies of the search space: when the *r-Opt* technique is implemented, you can save time by including a calculation that detects those exchanges that do not improve the solution;
2) generate a list of constrained neighbors: only the nearest neighbors to a vertex *p* are considered for making the reconnections (this rule was initially proposed by Zweig (1995);
3) generate a list of "don't look bits" or movements that are known to have been failed in past movements;
4) represent the circuit by means of trees: the use of these types of representations for the circuit speeds up computational times.

## 3.2 Variant of the *2-Opt* technique

The variant proposed by Leeuwen and Schoone (1980) consists of removing and exchanging only those edges that cross each other, demonstrating that a circuit that is free from intersections can be obtained in a polynomial number of steps $O(n^3)$, improving the complexity of the classic *2-Opt* technique.

Leeuwen and Schoone start from the fact that construction techniques generally produce circuits containing edges that cross each other and cannot be optimum circuits as they do not fulfill the *Folk Theorem*. This theorem states that optimum circuits do not contain edges that intersect with each other, thus optimum circuits must be simple polygonal routes apart from the fact that, in an optimum circuit, the points of the convex hull are visited in the order in which they appear in the hull. Using this as a basis, Leeuwen and Schoone propose only exchanging edges that cross each other and not, as in the 2-Opt technique, exchanging edges without considering the existence of possible crossings. They show that a crossings-free circuit can be obtained with this technique in a polynomial number of steps, although they do not mention the quality of the circuit produced.

It is worth mentioning that psychological studies have proven that human beings intuitively know that the absence of crossings is a necessary condition for achieving an optimum circuit. (McGregor, J.N., Ormerod, T., 1996; Graham, S., *et.al.*, 2000; Van Rooij, *et.al.*, 2003) who posed a TSP type problem to a group of 1000 people, classified according to age and experience, who had absolutely nothing to do with the field of operational research, found that only one person included a crossing in the resulting circuit, it being the only way to close the proposed circuit. On the basis of statistical tests, the writers concluded that, regardless of their experience, a person knows that the solution should not contain crossings if one wishes to achieve an optimum circuit.

The identification of the pairs of edges that intersect each other can be done exhaustively (analyzing all the pairs of edges), or by means of a computational geometry algorithm such as, for example, that of Bentley-Ottman (1979); Cormen, (1990), that implements a plane sweep to detect changes in the relative positions of the vertices, only calling on the crossing analysis algorithm when there is a possibility of detecting one.

Three different approaches are proposed for selecting the order in which to uncross the crossings from the list generated (PRIO, FIFO, RANDOM). In the PRIO approach, the pair of edges containing the longest crossed edge is chosen, as the probability of finding more than one crossing is higher than for shorter edges; moreover, the uncrossing of long edges generally achieves a significant reduction in the cost. Whereas the FIFO approach takes the first pair of crossed edges from the list of detected crossings. It is worth mentioning that in this case the order depends on the shape of the initial circuit that in turn depends on the construction technique used. In the RANDOM approach, the pairs of crossed edges are chosen randomly.

It must be noted that there are two ways of proceeding to process the list of detected crossings. It is possible, in a single iteration, to remove all the crossings from the list without importing the new crossings that might be generated or else, after every uncrossing of a single pair of edges, to generate the list again and once again choose the best candidate to be uncrossed according to the selection rule that has already been determined. Depending on the instance being analyzed, the recalculation of the list at each step can double or triple the time required for a complete iteration while, at the same time, obtaining the same final circuit. Therefore this is not to be recommended, in any case.

## 3.3 Example: comparison of 2-Opt, 2-Opt variant and 3-Opt

Consider an example relating to a problem that consists of 12 vertices, whose initial circuit has two crossings, in other words, edge $(x_6, x_{10})$ intersects both with edge $(x_2, x_7)$ and $(x_3, x_7)$, figure 1.

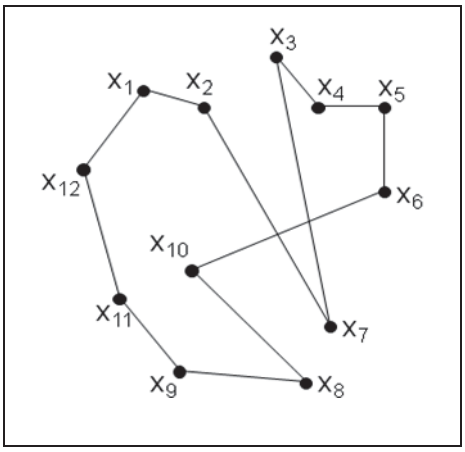
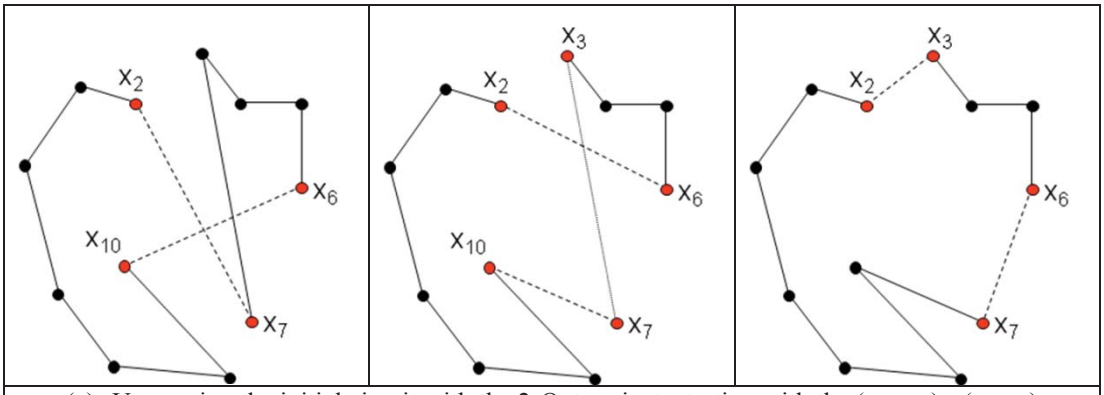
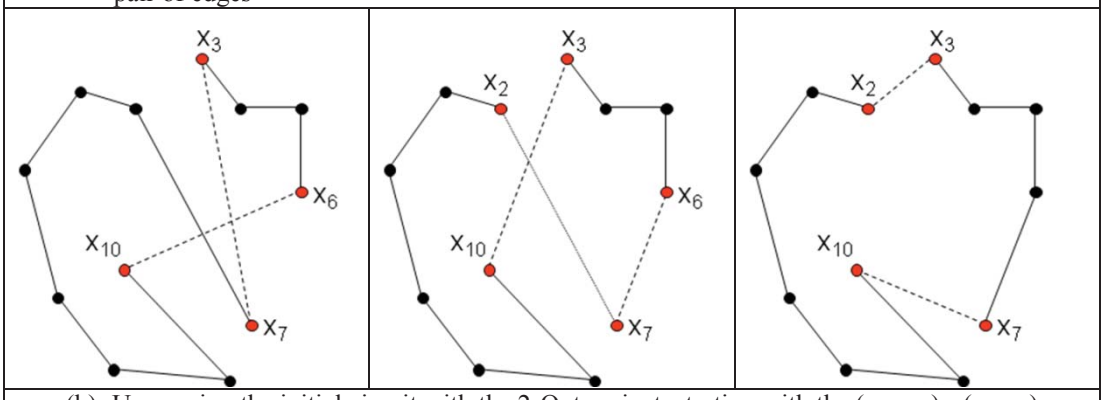



Figure 1. Illustrative problem with 12 vertices.

With the proposed 2-Opt variant, there are two possibilities for uncrossing: first of all, the ($x_6$, $x_{10}$) - ($x_2$, $x_7$) pair or, when applicable, the ($x_6$, $x_{10}$) - ($x_3$, $x_7$) pair can be uncrossed. Two steps are required in this example to generate a crossings-free circuit. Both procedures give the same end result, figure 2(a), although in complex problems this is not necessarily the case. It is worth mentioning that in the final circuit proposed, the $x_2 - x_{3...} - x_6 - x_7$ sequence does not correspond to that of an optimum circuit.



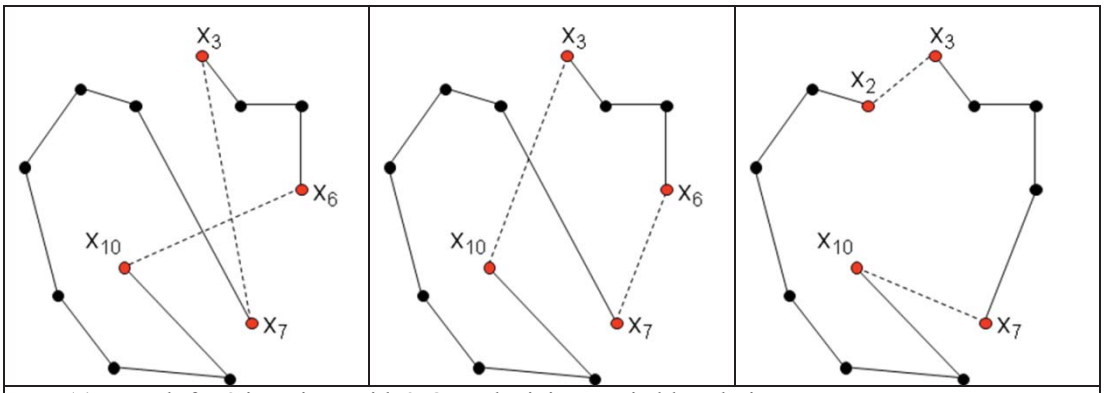

(a) Uncrossing the initial circuit with the 2-Opt variant, starting with the ($x_6$, $x_{10}$) - ($x_2$, $x_7$) pair of edges



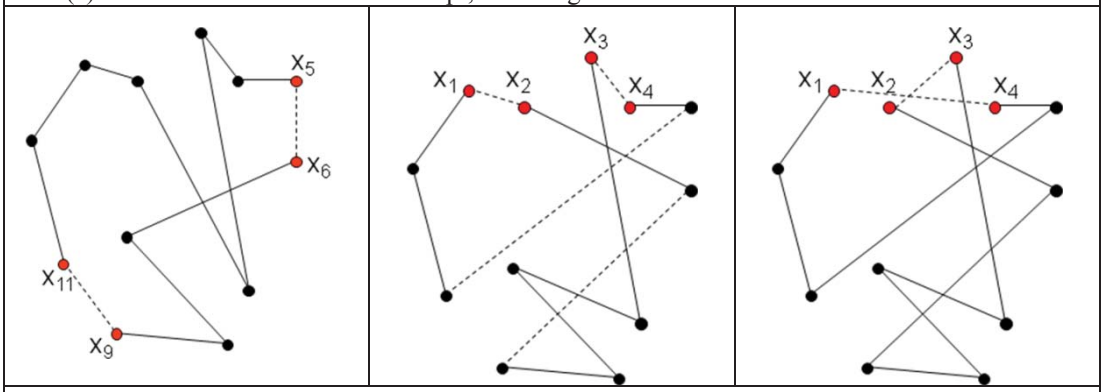

(b) Uncrossing the initial circuit with the 2-Opt variant, starting with the ($x_6$, $x_{10}$) - ($x_3$, $x_7$) pair of edges

Figure 2. Uncrossing the example circuit with the 2-Opt variant in two iterations.

When the same problem is solved using 2-Opt, the final circuit generated in the same two iterations is not necessarily free of crossings, or shorter than that obtained with the 2-Opt variant. Consider figure 3 that presents a good circuit obtained after 2 iterations in (a) and presents a circuit that is much worse than even the initial circuit in (b). Although the fact that 2-Opt permits poor quality intermediate solutions, increases the feasible solution space and, accordingly, the probability of obtaining a better quality final solution than the proposed variant, the number of iterations and the total algorithm runtime required to obtain this solution will, in general, be higher than for the proposed variant.



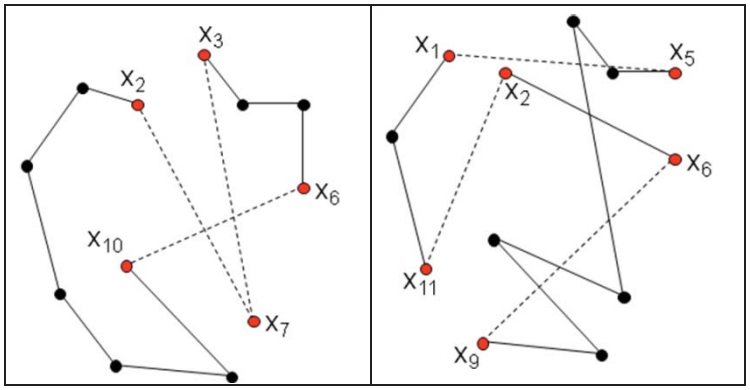

(a) Result for 2 iterations with 2-Opt, obtaining a suitable solution.



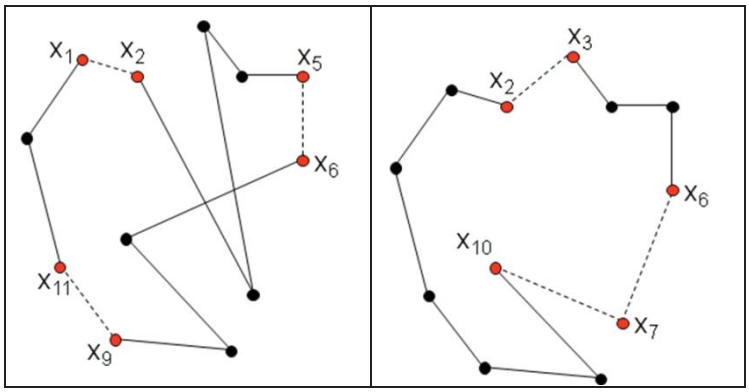

(b) Result for 2 iterations with 2-Opt, obtaining an unsuitable solution.

Figure 3. Result for 2 possible 2-Opt iterations for the example of figure 1. (a) Presents a suitable result, while (b) presents a not very favorable result.

Finally, a single iteration with 3-Opt can achieve the same uncrossed circuit as the 2-Opt variant in two iterations (Figure 4b). However, both 2-Opt and 3-Opt exchange edges in a shortsighted fashion and, once again, the resulting circuit is not necessarily better than the initial one (Figure 4a).
Experimental results show that regardless of the criterion, FIFO, PRIO or RAND, with which one takes the pairs of edges that intersect each other, there is very little difference in the number of iterations and the quality of the tour obtained.




(a) Result for 1 iteration with 3-Opt, obtaining a not very favorable solution.




(b) Result for 1 iteration with 3-Opt, obtaining a suitable solution.

Figure 4. Result for a single 2-Opt iteration for the example of figure 1. (a) Presents a suitable result, while (b) presents a not very favorable result.

## 4. CONDITIONS OF THE EXPERIMENT

For the analysis, we considered 49 Euclidean instances from the TSPLIB library (http://www.iwr.uni-heidelberg.de/groups/comopt/software/TSPLIB95) with less than 1000 nodes and used 6 computers with Intel Pentium IV, at 3GH and 1 Gb RAM. We programmed 6 construction techniques: nearest neighbor, best nearest neighbor (that take as the first city the one that produces a better quality tour), cheapest insertion, farthest, nearest and random insertion, as well as the 2-opt techniques and the proposed modification of this same technique. All these techniques were programmed in JAVA 5.

The algorithms were run in a cluster of the Institute of Nuclear Sciences of the Universidad Nacional Autónoma de Mexico. The program in itself is not parallel, these are serial algorithms and the parameter space we wish to assess was divided manually over the 6 computers of the cluster (basically, each machine has a construction algorithm and varies the rest of the parameters).

There are four parameters: SOLVERS, FINDERS, RECALCS and CROSSES_ITERS, where each parameter, in turn has other parameters that can vary depending on the required calculation. The SOLVERS parameter constitutes the initial solution given by any of the 6 construction algorithms, nearest neighbor (nn), best nearest neighbor (bnn), nearest insertion (in), cheapest insertion (ich), random insertion (ir) and farthest insertion (if). The FINDER parameter indicates the algorithm used for identifying the crossings, brute force which means comparing all the pairs of edges and the Bentley-Otmann algorithm (Benott) that uses the computational geometry strategy of plane sweeping. The RECALC parameter is a flag that indicates whether or not recalculation is involved in the identification of the crossings, in other words, once the list of pairs of edges that cross each other has been identified and a pair of them uncrossed, there is an option that allows the list of pairs of edges that intersect each other to be recalculated, if required. The “false” parameter indicates that it does not recalculate the list of pairs of edges for each iteration and “true” recalculates the pairs of edges as soon as a 2-opt movement is done. The CROSSES_ITER parameter is the criterion whereby the pairs of edges to be uncrossed are taken. Thus, the FIFO (First In First Out) parameter indicates that the first pair of edges is taken from the list to uncross them, RAND indicates that the pairs of edges are taken randomly from the list and PRIORITIZED (PRIO) takes first the pair of edges that contain the longest edge. Figure 5 shows that we have a non-trivial parameter space, as well as the variation of the parameters.

The following information is saved in each iteration: total time in seconds, the initial permutation produced by the construction algorithm, the final permutation produced by the improvement algorithm, and the output from the program that contains additional information, as well as the number of iterations, the crossings found and costs of the initial and final circuits.

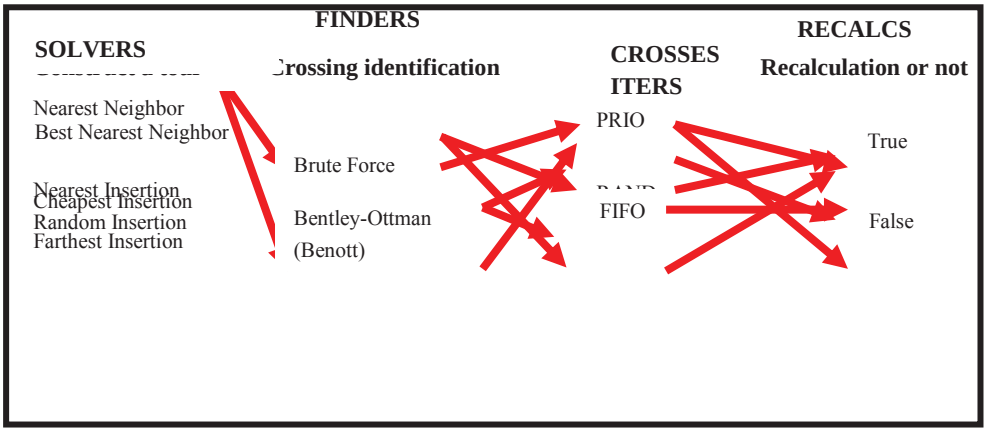



Figure 5. Set of parameters that were calculated in the experimentation.

# 5. PROPOSED METHODOLOGY

The methodology consists of constructing a *circuit* with one of the 6 aforementioned construction techniques. Once the *circuit* is constructed, the edges that cross each other are identified, saved in a list L and then the 2-Opt technique only takes from this list the pairs of edges (that cross each other) to exchange under one of the three criteria, FIFO, RANDOM or PRIO. The algorithm stops when the list L no longer contains edges that cross each other, i.e., the Hamiltonian circuit no longer contains edges that cross each other.

The methodology can be summarized in three steps:

1) Constructing a Hamiltonian circuit using any of the 6 construction techniques
2) Next, this circuit is improved using the variant of the 2-opt technique.
3) A crossings-free Hamiltonian circuit is obtained.

It is worth mentioning that the improvement of the Hamiltonian circuit with the classic 2-opt technique was also calculated in order to compare the final tour, in terms of time and quality, with the proposed variant. The identification of the crossings is the most important part of the proposed methodology, as this is one way of reducing the feasible solution space in the classic 2-opt technique.

## 5.1 Methodology for removing the crossings

To avoid taking all the possible edges for exchange, only a set of highly-promising edges, to wit, the ones that cross each other, are considered and by uncrossing them we can be sure of shortening the length of the Hamiltonian circuits, as triangle inequality is fulfilled. However, when uncrossing a pair of edges, we run the risk of generating further crossings, nevertheless Leeuwen and Schoone demonstrated that the number of steps involved in uncrossing all the edges and achieving a crossing-free *circuit* is $O(n^3)$.

We have considered, in this project, two ways of identifying the edges that cross each other: the brute force algorithm and the algorithm proposed by (Bentley and Ottman, 1979).

In general, for a set of $n$ segments, there are up to $O(n^2)$ intersection points as, if all the segments intersect each other, there would be $(n-1)+(n-2)+...+1=n(n-1)/2=O(n^2)$ intersection points. In the worst case , it would take $O(n^2)$ to calculate them all, in other words the brute force algorithm identifies, in a time $O(n^2)$, the intersection points as well as the edges that contain them.

This algorithm takes all the pairs of edges and asks itself if there is an intersection between them, which requires a lot of calculation time; however, it is efficient when there are just a few intersection points or only one point to be detected.

The Bentley and Ottman algorithm consists of moving an imaginary vertical line along the plane, stopping at the ends of the segments and at the intersection points that are detected. If we observe the situation, said vertical or sweep line cuts or does not cut the given segments, depending on their position, so it is not necessary to compare all the segments with all. This algorithm, in the worst case scenario, identifies the intersection points in a time $O((n+k) \log n)$, where n is the number of edges of the instance and k the intersection points that are found on it. It is worth mentioning that we are not, in this project, interested in finding the intersection point, just the edges that intersect each other, therefore the runtime is shortened to $O(n \log n)$.

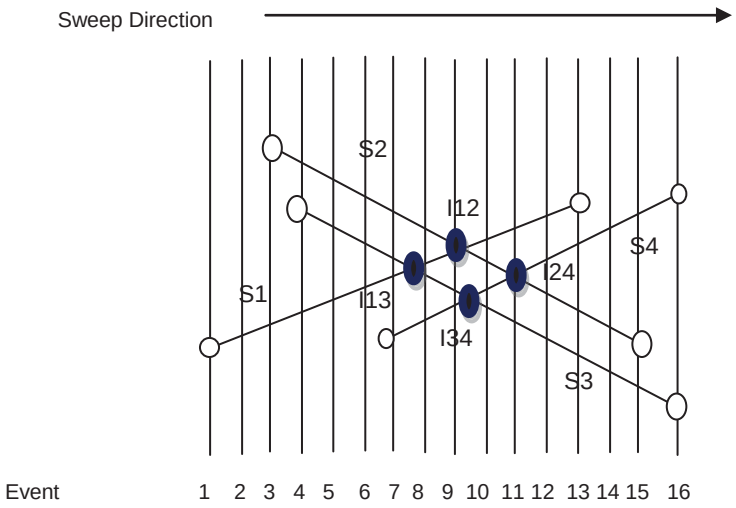



Figure 6. Movement of the sweep line

## 6. EXPERIMENTAL RESULTS

Once we had run all the parameters and their variations, we obtained the following results from the proposed methodology. In table 1 we observed that, in general terms, the brute force algorithm has an advantage in time over the Bentley-Ottman algorithm for instances with less than 500 cities, while the opposite is true for instances with a higher number of cities. We can infer that, in each instance, the geometric arrangement of the cities has a significant influence on the performance of the Bentley-Ottman algorithm.

Table 1. Bentley Ottman vs Brute Force runtime

| Instance | Brute Force Time | Bentley Ottman Time | Instance | Brute Force Time | Bentley Ottman Time | Instance | Brute Force Time | Bentley Ottman Time |
|---|---|---|---|---|---|---|---|---|
| a280 | 0.015 | 0.019 | kroB100 | 0.000 | 0.012 | pr299 | 0.000 | 0.018 |
| berlin52 | 0.000 | 0.012 | kroB150 | 0.000 | 0.015 | pr439 | 0.000 | 0.021 |
| bier127 | 0.000 | 0.012 | kroB200 | 0.012 | 0.012 | pr76 | 0.000 | 0.012 |
| ch130 | 0.000 | 0.020 | kroC100 | 0.000 | 0.012 | rat195 | 0.012 | 0.018 |
| ch150 | 0.000 | 0.020 | kroD100 | 0.000 | 0.012 | rat575 | 0.022 | 0.026 |
| d198 | 0.002 | 0.015 | kroE100 | 0.000 | 0.012 | rat783 | 0.027 | 0.027 |
| d493 | 0.003 | 0.023 | lin105 | 0.000 | 0.012 | rat99 | 0.012 | 0.012 |
| d657 | 0.004 | 0.025 | lin318 | 0.012 | 0.017 | rd100 | 0.012 | 0.015 |
| dsj1000 | 0.004 | 0.026 | p654 | 0.019 | 0.026 | rd400 | 0.018 | 0.024 |
| eil101 | 0.000 | 0.015 | pcb442 | 0.020 | 0.021 | st70 | 0.012 | 0.000 |
| eil51 | 0.000 | 0.012 | pr107 | 0.000 | 0.012 | ts225 | 0.012 | 0.015 |
| eil76 | 0.001 | 0.012 | pr124 | 0.012 | 0.012 | tsp225 | 0.020 | 0.019 |
| fl417 | 0.003 | 0.022 | pr136 | 0.000 | 0.015 | u159 | 0.012 | 0.015 |
| gil262 | 0.002 | 0.018 | pr144 | 0.000 | 0.012 | u574 | 0.022 | 0.024 |
| kroA100 | 0.001 | 0.000 | pr152 | 0.012 | 0.015 | u724 | 0.024 | 0.025 |
| kroA150 | 0.001 | 0.012 | pr226 | 0.015 | 0.015 | | | |
| kroA200 | 0.001 | 0.012 | pr264 | 0.012 | 0.020 | | | |

The following two tables, 2 and 3, give the percent deviations in the length obtained in respect of the optimum value, under the three different strategies for uncrossing the pairs of edges. The results were obtained without recalculating list L and the Bentley-Ottman algorithm was used to identify the edges that intersect each other.

Table 2. Solution time and quality for the techniques that exceed 10% deviation in respect of the optimum value.

| Heuristic | N.N. | | Nearest Insertion | | B.N.N. | |
|---|---|---|---|---|---|---|
| | % Deviation | Execution Time Sec | % Deviation | Execution Time Sec | % Deviation | Execution Time Sec |
| FIFO | 16.65% | 0.63 | 12.49% | 0.24 | 10.93% | 0.38 |
| RAND | 16.06% | 0.71 | 12.47% | 0.27 | 11.07% | 0.5 |
| PRIO | 16.76% | 0.57 | 12.45% | 0.25 | 11.19% | 0.56 |

Table 2 gives the three algorithms that have a percent deviation that exceeds the optimum value by more than 10%, thus the nearest neighbor technique, nearest insertion and the best nearest neighbor are algorithms that produce tours that do not make a good combination with the proposed variant.

Table 3 Solution time and quality for the techniques that exceed a deviation of 10% in respect of the optimum value.

| Heuristic | Random Insertion | | Cheapest Insertion | | Farthest Insertion | |
|---|---|---|---|---|---|---|
| | % Deviation | Runtime Sec | % Deviation | Runtime Sec | % Deviation | Runtime Sec |
| FIFO | 7.27% | 0.89 | 8.85% | 0.16 | 8.05% | 0.79 |
| RAND | 9.00% | 0.1 | 8.85% | 0.13 | 8.05% | 0.72 |
| PRIO | 9.00% | 0.11 | 8.86% | 0.14 | 8.05% | 0.74 |

Table 3 shows the algorithms that produce tours with a percent deviation in respect of the optimum value that is lower than 9%. The best combination is to construct a circuit with a random insertion algorithm using the FIFO (first in first out) strategy for the uncrossing, which has a percent deviation in respect of the optimum value of 7.27%.

Table 4. Solution time and quality of the classic 2-Opt technique with six different construction techniques.

| 2-Opt with | N.N. | Near Insertion | Farthest Insertion +2-Opt | Cheapest Insertion +2-Opt | Random Insertion | B.N.N. |
|---|---|---|---|---|---|---|
| % Deviation | 7.92% | 7.61% | 6.78% | 6.15% | 6.15% | 6.19% |
| Runtime Sec | 3.69 | 1.72 | 3.81 | 1.2 | 1.2 | 2.8 |

Table 4 gives the percent deviations in respect of the optimum value and the runtimes for the classic 2-Opt technique in combination with the six different construction techniques. We can see that the random (RAND) and the cheapest (Cheap) insertion techniques are the construction techniques that give the best performance in combination with the classic 2-Opt technique, producing percent deviations of 6.15% in respect of the optimum value in both techniques.

Table 5 gives the ratio of runtime for the proposed variant in respect of the classic 2-Opt technique. We can see that the runtimes for the classic 2-opt technique are twelve times higher than for the combination (proposed variant) of random insertion, as the construction technique, and RAND strategy. The shorter ratio of time is observed in the combination of the best nearest neighbor and farthest insertion construction techniques with any of the strategies for selecting the pairs of edges to be

uncrossed. This shows that the proposed variant is several orders faster than the classic 2-Opt technique having, as it does in the best case scenario, a percent deviation of 7.61%

Table 5. Ratio of the runtime of the proposed variant in respect of the classic 2-Opt technique.

| Heuristic | N.N. | Nearest Insertion | Farthest Insertion | Cheapest Insertion | Random Insertion | B.N.N. |
|---|---|---|---|---|---|---|
| FIFO | 6 | 7 | 5 | 8 | 1 | 7 |
| RAND | 5 | 6 | 5 | 9 | 12 | 6 |
| PRIO | 6 | 7 | 5 | 9 | 11 | 5 |

## 7. CONCLUSIONS

In general, this experiment shows that, on average, the proposed methodology achieves a runtime that is several orders of magnitude shorter than that obtained using the classic 2-Opt technique, although the cost of the tour is slightly higher. The best construction techniques were found to be the insertion techniques and , in particular, the farthest criterion and random insertion, with the FIFO criterion in the case of the former and any strategy in the case of the latter, giving percent deviations of 7.27% and 8.05% , respectively, in respect of the optimum value. The nearest neighbor and best nearest neighbor techniques give 16% and 12% percent deviations, respectively, in respect of the optimum value, with which we can infer that the construction technique that is used to generate the initial circuits has a major influence on the efficiency of the proposed methodology as well as on the classic 2-Opt technique. This is understandable as both strategies have the same philosophy, i.e., edge exchange. The different methods of prioritization did not have any significant influence on runtime results or on the percent deviation of the final circuit.

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# A $\frac{3}{2}$-Approximation Algorithm for Three Depots Hamiltonian Path Problems

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**Abstract**

In this paper, we propose an approximation algorithm for solving the three depots Hamiltonian path problem (3DHPP). The problem studied can be viewed as a variant of the well-known Hamiltonian path problem with multiple depots (cf., Demange [1] and Malik *et al.* [2]). For a quite broad family of cases of the metric 3DHPP, we show the existence of a $\frac{3}{2}$-approximation algorithm. The proposed algorithm is mainly based on extending the construction scheme already used by Rathinam *et al.* [3].

**Keywords:** Approximation, Hamiltonian path, Traveling salesman.

## 1 Introduction

The multiple traveling salesman problem (MTSP) is a generalization of the well-known traveling salesman problem (TSP), in which there are $m$ salesmen who start and terminate to the depot (cf. Malik *et al.* [2]). The multi depots TSP is an also a generalization of the TSP. Under multiple formulations and variants, TSP has been extensively studied, being a paradigmatic problem in Combinatorial Optimization; however, in this context, the litterature on the generalized multi depots and their variants, though well-motivated from practical applications, is still limited. In this paper, we discuss the three depots Hamiltonian path problem, namely 3DHPP, which can be viewed as a specific case of the well-known Hamiltonian path problem with multiple depots (cf., Demange [1]). For a broad family of instances of the 3DHPP where the values of the distance matrix define a metric, we show the existence of a $\frac{3}{2}$-approximation algorithm. The proposed algorithm is mainly based on extending the construction scheme already used by Rathinam *et al.* [3] and to our knowledge is the first to guarantee a ratio $r < 2$.

### 1.1 Problem formulation

Let $D = \{d_1, d_2, d_3\}$ be the set of vertices representing the three distinct depots, $U = \{1, 2, 3, ...n\}$ be the set of vertices denoting $n$ destinations ($n \geq 3$) and $V = D \bigcup U$. The edge $(i, j)$ joining vertices $i$ and $j$ has a positive rational cost $C_{ij} \in Q_{>0}$. Assume that all costs are symmetric (i.e., $\forall\ (i, j) \in V$, $Cij = Cji$) and they satisfy triangle inequality: $C_{ik} \leq C_{ij} + C_{jk}$, $\forall\ (i, j, k) \in V$. A *path* for salesman $l \in \{1, 2, 3\}$ may be denoted by an ordered sequence of vertices $P_l = \{d_l, v_1^l, v_2^l .....v_{k_l}^l\}$, where $k_l$ represents the number of vertices visited by the $l$-th salesman and $\forall j \in \{1.....k_l\}$, $v_{lj} \in U$ are distinct. The cost of the path $P_l$ traveled by the $l$-th salesman is defined as follows:

$$C(P_l) = \begin{cases} C(d_l, v_1^l) + \sum_{j=1}^{k_l - 1} C(v_j^l, v_{j+1}^l) & \text{if } k_l > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The objective here is to find paths $P_1$, $P_2$ and $P_3$ such that:

1. each destination of $U$ is visited exactly once by any salesman,
2. $\Pi_{l=1}^{3} C(P_l) \neq 0$, ie each salesman visits at least one destination and,
3. the sum of the costs corresponding to the salesman, i.e., $\sum_{l=1}^{3} C(P_l)$, is minimum.

For the rest of the paper, we shall denote the above-defined problem by 3DHPP, for *3-Depots Hamiltonian Path Problem*. The remainder of the paper is organized as follows. First, in Section 2 we discuss the main steps of the approach used for tailoring the approximation algorithm for the 3DHPP. In Subsection 2.1 an example is given in order to illustrate the main steps of the proposed algorithm. Finally, the main result is announced in Section 3 for which the sketch of the proof is given along the example.

## 2 An Approximation Algorithm for the 3DHPP

The proposed algorithm can be viewed as a five-step algorithm in which a series of decompositions, constructions and reconstructions is applied. These steps are described as follows:

1. Find a *minimum cost constrained forest*, namely $F$, such that: there are three trees in the forest, where each depot is of degree one in the forest and each tree contains exactly one depot. This can be done in polynomial time in multiple ways (for example by adding a dummy vertex linked to the depots $d_1, d_2, d_3$) with minimum cost edges and then finding a minimum cost spanning tree by applying a slight modification of Kruskal's algorithm).

   Let's consider the edges in the constrained forest denoted by $E(F)$ and let $O_i$, for $i = 1, 2, 3$, be the odd-degree destination vertices in the $i$-th tree of $F$.

2. Let $G[V_0]$ be the induced subgraph by the set of the odd-degree destination vertices of $F$ in the graph of the instance.

   Notice that, because $|O_1|$, $|O_2|$ and $|O_3|$ are odd, $|V_0| = |O_1| + |O_2| + |O_3|$ is odd too and that $G[V_0]$ is a complete graph.

   Now, find a minimum cost *partial matching* $M$ i.e. in $G[V_0]$ of size $\frac{|V_0|-3}{2}$, ie that matches $|V_0| - 3$ vertices in $V_0$. Of course, such a matching always exists and can be found in polynomial time (trivially, by applying $|V_0|^3$ times any min-cost perfect matching algorithm). Hence, let $m_1$, $m_2$ and $m_3$ denote the three destination vertices that are note matched.

3. Add $M$ provided by step (2) to the *minimum cost constrained forest* reached by step (1), and connect both depots vertices $d_1$ and $d_2$ by using an edge whose cost is equal to zero cost, and also the destinations $m_2$ and $m_3$.

   Consider the following new provided graph:

   $G_n = (V, E(M) \bigcup E(F) \bigcup \{(d_1, d_2, d_3)\})$.

   Notice that $G_n$ might be a multigraph and remark that the degree of each vertex of $G_n$ is even except for the two vertices $m_1$ and $d_3$, meaning that $G_n$ has an Eulerian path. Thus, we proceed to the following step:

4. Find an Eulerian path, namely $E$ in $G_n$,; clearly, the path has extremities vertex $m_1$ and$d_3$ and it passes through each edges in $G_n$ exactly once.

5. Apply a shortcuts on the edges of the Eulerian path previously obtained, to get a Hamiltonian path such that each vertex in $V$ is visited exactly once. The obtained Hamiltonian path is denoted $E_{sc} = (m_1, ...., d_1, d_2, ..., m_2, m_3 ..... d_3)$. Then, return the sequence of destinations which is:

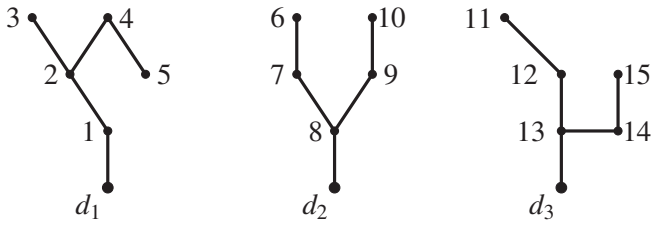
   (a) to assign to the second salesman the subsequence of the provided Hamiltonian path; it starts from $d_2$ and terminates at $m_2$.

   (b) to assign to the first salesman the reverse of the remaining subsequence of the provided Hamiltonian path which starts from $d_1$ and terminates at $m_1$.

   (c) to assign to the third salesman the reverse of the remaining subsequence of the Hamiltonian path reached which has $d_3$ as the starting vertex and $m_3$ its terminal vertex.

## 2.1 Example

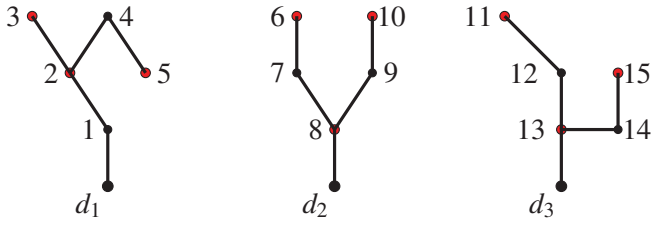
In what follows we show how the different steps of the algorithm can be applied in order to construct the final solution for the algorithm presented in this Section.




**Step 1:**(a) A Minimum cost constrained forest F

Figure 1:

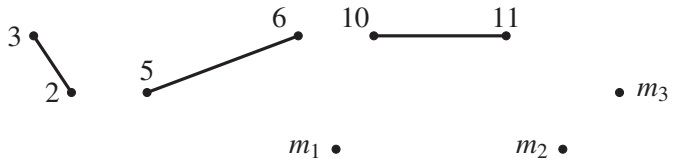
Indeed, according to the first step, Figure 1 can show how the three threes can be obtained and how each of these trees can be relied to the depot vertex, i.e., $d_1$, $d_2$ and $d_3$, respectively.




**Step 1:** (b) The odd-degree destination vertices
$|V_0| = |O_1 \bigcup O_2 \bigcup O_3| = 9$

Figure 2:

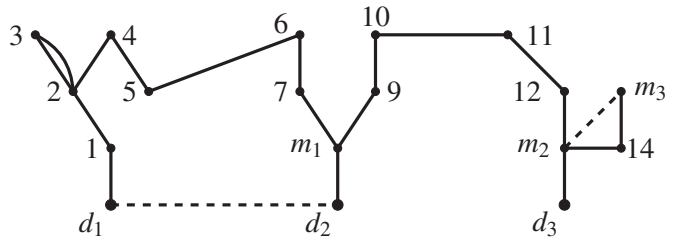
Of course, we can observe that such a forest (noted $F$) realizes a minimum cost constrained forest. We also can remark that the construction respects the conditions of step 1.




**Step 2:** The minimum cost partial matching
of cardinality equal to $\frac{|V_0|-3}{2} = 3$

Figure 3:

We recall that step 2 serves to find a partial min-cost matching of a suitably defined cardinality. Indeed, Figure 3 illustrates such a matching satisfying the cardinality of 3 fixed at step 2.

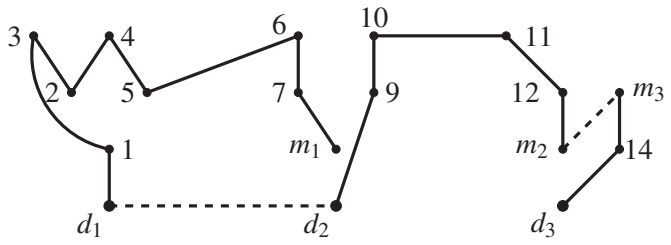



**Step 3 and 4:** Add the edges of M to F. Also add a zero cost edges $(d_1, d_2)$ and $(m_2, m_3)$

Eulerian path=$\{m_1, 7, 6, 5, 4, 2, 3, 2, 1, d_1, d_2, m_1, 9, 10, 11, 12, m_2, m_3, 14, m_2, d_3\}$

Figure 4:

Following both steps 3 and 4, we can observe (cf. Figure 4) how we use the adding-phase in order to build the Eulerian path, namely $E$.

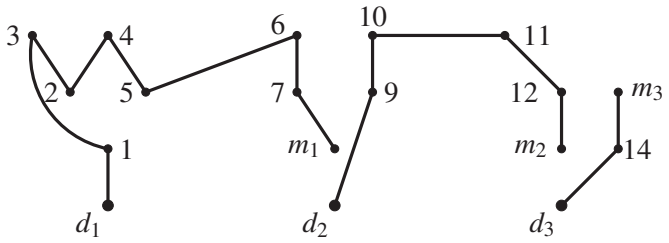



**Step 5:** Shortcut the edges of the Eulerian path to find a

Hamiltonian path $=\{m_1, 7, 6, 5, 4, 2, 3, 1, d_1, d_2, 9, 10, 11, 12, m_2, m_3, 14, d_3\}$

Figure 5:

By applying the shortcutting phase of step 5, we can observe the result (cf. Figure 5) the new Hamiltonian path, namely $H$, obtained from the previously Eulerian path $E$.




**Step 5:**Remove the zero cost edges to find the paths for the 3 salesmen:

$\{d_1, 1, 3, 2, 4, 5, 6, 7, m_1\}$, $\{d_2, 9, 10, 11, 12, m_2\}$, $\{d_3, 14, m_3\}$

Figure 6:

Finally, according to the last three points of step 5, which corresponds to the removing phase, Figure 6 illustrates the final solution provided by the algorithm.

# 3 Main result

To our knowledge, there is no approximation algorithm better than 2 for metric 3DHPP. Herein, we propose an algorithm which admits an approximation ratio of $\frac{3}{2}$ for a broad family of metric cases and guarantees a ratio $r < 2$ in the general metric case. An outline of the proof for the aforementioned result is sketched by the detailed example presented in subection 2.1.

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# AN EXPERIMENTAL INVESTIGATION OF A PRIMAL-DUAL EXTERIOR POINT SIMPLEX ALGORITHM

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**Abstract**

The aim of this paper is to present an experimental investigation of a Primal-Dual Exterior Point Simplex Algorithm (PDEPSA) for Linear Programming problems (LPs). There was a huge gap between the theoretical worst case complexity and practical performance of simplex type algorithms. The algorithm bases on interior points to move from one basic solution to another. The main advantage of PDEPSA is its computational performance. This performance stems from the fact that PDEPSA can deal much better with the problem of stalling and cycling. Moreover, the use of interior points is responsible for the reduction of iterations and the CPU time and especially in linear degenerate problems. A computational study is also presented with experiments on randomly generated sparse optimal linear problems, which increases the reliability of the conclusions. The algorithm is very encouraging and promising due to the fact that it proved its superiority to the exterior point algorithm and the primal revised simplex algorithm on the computational study.

**KEYWORDS**

Linear programming, Primal Simplex algorithms, Exterior Point Algorithms, Computational results.

## 1. INTRODUCTION

One of the most significant and well-studied optimization problems is the Linear Programming problem (LP). Linear programming consists of optimizing, (minimizing or maximizing) a linear function over a certain domain. The domain is given by a set of linear constraints. The simplex algorithm is the oldest and most well-known solution method for the LP. George B. Dantzig is regarded as the founder of linear programming and the person who set the fundamental principles of optimization. The simplex starts from a feasible solution and moves from one vertex to an adjacent one until an optimum solution is computed (Dantzig, 1963). The two main drawbacks of the simplex algorithm are the stalling and/or the cycling problem. Moreover, the phenomenon of cycling is very often in many practical problems for the simplex algorithm. In order to avoid the problem of cycling, researchers have introduced many anti-cycling pivoting rules (Terlaky & Zhang, 1993). The simplex algorithm performs effectively in practice only under specific circumstances and especially on small or medium size LPs. It has been proved that the average number of iterations in simplex algorithm can be estimated with a polynomial bound (Borgwardt, 1982). Despite that the worst case complexity of the simplex method has exponential behavior (Klee & Minty, 1992).

Until the decade of 1980 simplex algorithm and barrier methods were known as the only solution approach for the LP. This situation changed in 1984 when the first try of interior point methods for linear programming appeared (Karmarkar, 1984). Next years researchers focused their attention on how the simplex algorithm and the Interior Point Methods (IPMs) can be combined in order to improve the computational behavior of software packages (Erling et al., 1996). In contrast to the simplex method, IPMs compute an optimal solution by moving inside the feasible region. Moreover, the research adopted primal-dual algorithms based on IPMs and simplex method which were regarded as the most significant and useful algorithms. The primal-dual methods for linear programming have interesting theoretical properties. On the

computational side, Mehrotra's predictor corrector algorithm was the main idea for the most interior-point software (Wright, 1997). Primal-dual methods have good computational performance and they can be extended to wider classes of problems in mathematical programming (Gondzio, 1996).

Apart from the IPMs, another approach to solve LPs is to modify the main idea of simplex algorithm to move from one basic vertex to another in the exterior of the feasible region. The algorithm can construct basic infeasible solutions instead of feasible. The algorithm which includes these kinds of features are called exterior point simplex algorithm (EPSA). The first attempt for an exterior point algorithm was introduced in 1991 for the assignment problem (Paparrizos, 1991). Since then, many papers and articles have been presented in order to enhance the capabilities of the exterior point algorithms. A significant improvement is the primal-dual versions of the algorithm which reveal its superiority to simplex algorithm.

The main idea of exterior point algorithm is that it relies on two paths for estimating the optimal solution. The first past refers to a number of feasible solutions until the optimal is found and the other path includes basic points which lead to the exterior path. The exterior point algorithm has two main disadvantages (Paparrizos et al., 2003B). The first refers to the construction of moving directions which should lead the algorithm close to the optimal solution. The creation of a direction with these features is a difficult process. The second disadvantage is the fact that there is no known method which can reveal the path that leads into the interior of the feasible region, something which would make easier the search of a computational good direction. These advantages can be avoided if the exterior path is replaced with a dual feasible simplex path.

This method has been introduced by Paparrizos et al. (Paparrizos et al., 2003A). The main idea of the Revised Primal Dual Simplex Algorithm (RPDSA) is based on the process of moving from any interior point to an optimal basic solution. The advantage of this algorithm is that the optimal solution is not computed by an interior point method but it can be used only in the first stage. After the first few iterations IPMs do not result in great enhancements of the objective function's value. RPDSA can be applied in the second stage and computes the optimal solution in a few iterations, much faster from an interior point method. Although this algorithm is better from the exterior point algorithm and it deals very well with the two disadvantages which were described above.

The algorithm of our paper introduces a new technique which deals with the problems of stalling and cycling and improves the performance of the RPDSA. This algorithm has been developed by Samaras (Samaras, 2001). This algorithm is primal-dual, meaning that it simultaneously solves both the primal and dual problem. RPDSA begins with a boundary point of the feasible region. This boundary point is used in order to compute the leaving variable. It has been observed that in this stage the problem of stalling can arise very often. This weakness can be overcome if the boundary point is replaced by an interior point. The transfer into the interior of feasible region disappear the problem of cycling. In order to gain an insight into the practical behavior of the proposed algorithm, we have performed some computational experiments. Preliminary results on randomly generated LPs show that the reduction in the computational effort is promising.

The paper is organized as follows. Following introduction, in section 2 we briefly describe the general framework of the proposed algorithm. In section 3 the computational study is presented. Finally, in section 4 there are the conclusions and possible enhancements of the proposed algorithm.

## 2. ALGORITHM DESCRIPTION

In this section we briefly describe the Primal-Dual Exterior Point Simplex Algorithm (PDEPSA). For solving general LPs see (Paparrizos et al., 2003B) and for a full description of PDEPSA see (Samaras, 2001). Consider now the following linear program in the standard form.

$$\begin{aligned} \min \quad & c^T x \\ \text{s. t.} \quad & Ax = b \qquad (LP) \\ & x \geq 0 \end{aligned}$$

where $A \in \Re^{mxn}$, $c, x \in \Re^n$, $b \in \Re^m$ and T denotes transpose. Assume that A has full rank, rank(A)=m (m<n). The dual problem associated with (LP) is

$$\begin{aligned} \max \quad & b^T x \\ \text{s. t.} \quad & A^T w + s = c \qquad (DP) \\ & s \geq 0 \end{aligned}$$

where $w \in \Re^n$ and $s \in \Re^n$.

**Step 0 (Initialization):**
A) Start with a dual feasible basic partition (B, N) and an interior point y of (LP).
Set:

$$P = N, Q = \varnothing$$

and compute

$$x_B = (A_B)^{-1} b, w^T = (c_B)^T (A_B)^{-1}, (s_N) = (c_N)^T - w^T A_N$$

B) Compute the direction $d_B$ from the relation: $d_B = y_B - x_B$

**Step 1 (Test of optimality and choice of the leaving variable):**
If $x \geq 0$, STOP. The (LP) is optimal. Else, choose the leaving variable $x_k = x_{B[r]}$ from the relation:

$$a_l = \frac{x_{B[r]}}{-d_{B[r]}} = \max\{\frac{x_{B[r]}}{-d_{B[r]}} : d_{B[i]} > 0 \wedge x_{B[i]} < 0\}$$

**Step 2 (Computation of the next interior point):**
Set:

$$a = \frac{a_1 + 1}{2}$$

Compute the interior point: $y_B = x_B + a d_B$

**Step 3 (Choice of the entering variable):**
Set: $H_{rN} = (B^{-1})_r . A_{.N}$
Choose the entering variable $x_l$ from the relation:

$$\frac{-s_l}{H_{rN}} = \min\{\frac{-s_j}{H_{rj}} : H_{rj} \wedge j \in N\}$$

Compute the pivoting column: $h_l = B^{-1} A_{.l}$
If $l \in$ P,
P $\leftarrow$ P\{l}
Else
$Q \leftarrow Q \setminus \{l\}$

**Step 4 (Pivoting):**
Set:

$$B[r] = l \text{ and } Q \leftarrow Q \cup \{k\}$$

Using the new partition (B, N) where N = (P, Q), compute the new basis inverse $B^{-1}$ and the variables:

$$x_B = (A_B)^{-1} b, w^T = (c_B)^T (A_B)^{-1}, (s_N) = (c_N)^T - w^T A_N$$

Go to step 0B.

## 3. COMPUTATIONAL STUDY

In order to check and test the performance and practical effectiveness of our algorithm, a computational study is presented below. The problems which were included in the computational study were small and medium scale sparse instances. The computational study includes the revised simplex algorithm, the Exterior Point Simplex Algorithm (EPSA) and the Primal Dual Exterior Point Simplex Algorithm (PDEPSA).

All implemented algorithms are running in MATLAB Professional R2010b. MATLAB (MATrix LABoratory) is a powerful programming environment and as its name suggests, is especially designed for

matrix computations like, solving systems of linear equations or factorizing matrices. Also, it is an interactive environment and programming language for numeric scientific computing. The computing environment includes an Intel(R) Core™ i7 3.00 GHz (2 processors) and 16.384 MB RAM. The operating system was Microsoft Windows 7 Professional SP1 edition. All times in the following tables are measured in seconds with the "`cputime`" built-in function of MATLAB, including the time spent on scaling. All runs were made as a batch job. Two techniques are used in order to improve the performance of memory-bound code in MATLAB. These techniques are: (i) Pre-allocate arrays, and (ii) Store and access data in columns.

Three different density classes of LPs are used in the computational study: 5%, 10% and 20%. In each dimension 10 different instances are tested. These LPs include only randomly generated matrices. The initial basis consists of only the slack variables.

Tables 1-3 present the average execution time in seconds (CPU time) and the average number of iterations (niter) that each of the competitive spent to solve the LPs in request.

Table 1. Results for randomly generated sparse LPs with dimension nxn and density 5%

| Density 5% | PDEPSA | | EPSA | | Simplex | |
|---|---|---|---|---|---|---|
| | CPU_time (sec) | Niter | CPU_time (sec) | Niter | CPU_time (sec) | Niter |
| **500 x 500** | 0,8502 | 406 | 3,7924 | 1.536 | 3,9617 | 1.660 |
| **600 x 600** | 1,4492 | 520 | 6,8079 | 2.042 | 8,5560 | 2.487 |
| **700 x 700** | 2,3447 | 684 | 11,4099 | 2.627 | 17,9035 | 3.706 |
| **800 x 800** | 3,5225 | 831 | 19,4112 | 3.238 | 33,4693 | 5.187 |
| **900 x 900** | 5,2479 | 1.010 | 27,5498 | 3.591 | 48,6587 | 6.034 |
| **1000 x 1000** | 7,7650 | 1.218 | 39,6211 | 4.148 | 82,6693 | 7.962 |
| **1100 x 1100** | 11,3865 | 1.319 | 59,9605 | 4.991 | 138,3463 | 10.331 |
| **1200 x 1200** | 15,6400 | 1.552 | 75,3610 | 5.406 | 197,5266 | 12.372 |
| **1300 x 1300** | 21,9790 | 1.749 | 103,0355 | 6.151 | 300,8046 | 15.533 |
| **1400 x 1400** | 27,4468 | 1.916 | 125,4295 | 6.676 | 416,6580 | 18.237 |
| **1500 x 1500** | 34,0847 | 2.071 | 151,6782 | 7.120 | 574,1572 | 21.802 |
| **1600 x 1600** | 46,9520 | 2.428 | 186,7300 | 7.712 | 748,6919 | 24.400 |
| **1700 x 1700** | 54,4100 | 2.576 | 225,6000 | 8.388 | 1012,0000 | 28.642 |
| **1800 x 1800** | 64,1663 | 2.696 | 270,6056 | 9.015 | 1366,8000 | 34.026 |
| **1900 x 1900** | 80,0582 | 2.988 | 328,8189 | 9.713 | 1643,0000 | 36.365 |
| **2000 x 2000** | 91,4950 | 3.175 | 375,5810 | 10.164 | 2057,0000 | 41.011 |
| **Total** | 468,7981 | 27.139 | 2011,3925 | 92.518 | 8650,2031 | 269.755 |

As it is obvious from the first table, PDEPSA is clearly superior to the other two algorithms. The execution time of PDEPSA is much lower than EPSA and simplex algorithm. For example, PDEPSA needs almost 8 seconds to solve a 1000 x 1000 problem when EPSA demands almost 40 seconds and even worse Simplex Algorithm needs 83 seconds to complete the computations. Likewise, the number of iterations is much greater for Simplex Algorithm and much less from PDEPSA.

In addition, in a 2000 x 2000 problem the simplex algorithm requires in average 41.000 iterations when PDEPSA claims for 3.200 iterations and EPSA 10.200 algorithms. It is clear that PDEPSA can lead to significant reductions of the execution time and the number of iterations. In the same problem, PDEPSA can solve the linear problem almost in 90 seconds when Simplex algorithm needs 2.100 seconds. One may infer a growth in the relative speed of PDEPSA with respect to simplex and EPSA as problem sizes increase.

Table 2. Results for randomly generated sparse LPs with dimension nxn and density 10%

| Density 10% | PDEPSA | | EPSA | | Simplex | |
|---|---|---|---|---|---|---|
| | CPU_time (sec) | Niter | CPU_time (sec) | Niter | CPU_time (sec) | Niter |
| **500 x 500** | 0,9641 | 420 | 3,7627 | 1.529 | 5,9177 | 2.281 |

| | | | | | | |
|---|---|---|---|---|---|---|
| **600 x 600** | 1,7690 | 569 | 7,0434 | 1.996 | 13,9170 | 3.557 |
| **700 x 700** | 2,5460 | 653 | 11,8650 | 2.378 | 24,2287 | 4.658 |
| **800 x 800** | 4,2027 | 828 | 19,6858 | 2.817 | 41,2261 | 6.001 |
| **900 x 900** | 6,7361 | 1.049 | 26,6060 | 3.186 | 71,3937 | 7.869 |
| **1000 x 1000** | 9,0184 | 1.109 | 38,6524 | 3.646 | 112,3835 | 9.647 |
| **1100 x 1100** | 13,8825 | 1.290 | 55,6768 | 4.195 | 174,3109 | 11.878 |
| **1200 x 1200** | 20,6296 | 1.539 | 67,4408 | 4.485 | 245,1684 | 13.907 |
| **1300 x 1300** | 25,7230 | 1.644 | 97,9452 | 5.265 | 387,0143 | 17.688 |
| **1400 x 1400** | 31,8000 | 1.762 | 111,0493 | 5.484 | 514,3469 | 19.897 |
| **1500 x 1500** | 42,4307 | 2.048 | 139,9000 | 5.889 | 646,8230 | 22.105 |
| **1600 x 1600** | 49,3322 | 2.077 | 166,4187 | 6.305 | 878,5583 | 25.665 |
| **1700 x 1700** | 63,1305 | 2.351 | 208,5203 | 6.852 | 1137,4827 | 28.744 |
| **1800 x 1800** | 77,4171 | 2.567 | 251,1772 | 7.338 | 1441,0763 | 32.301 |
| **1900 x 1900** | 90,2918 | 2.680 | 296,2397 | 3.505 | 1773,6488 | 35.054 |
| **2000 x 2000** | 104,0885 | 2.803 | 332,9217 | 8.147 | 2300,1632 | 40.319 |
| **Total** | 543,9622 | 25.389 | 1834,9050 | 76.706 | 9767,6594 | 281.571 |

In less sparse problems the PDEPSA continues to present its surprisingly good performance. Its superiority is clear and with no doubt it is an effective and promising algorithm. In all dimensions PDEPSA has the shortest execution time and the smallest number of iterations.

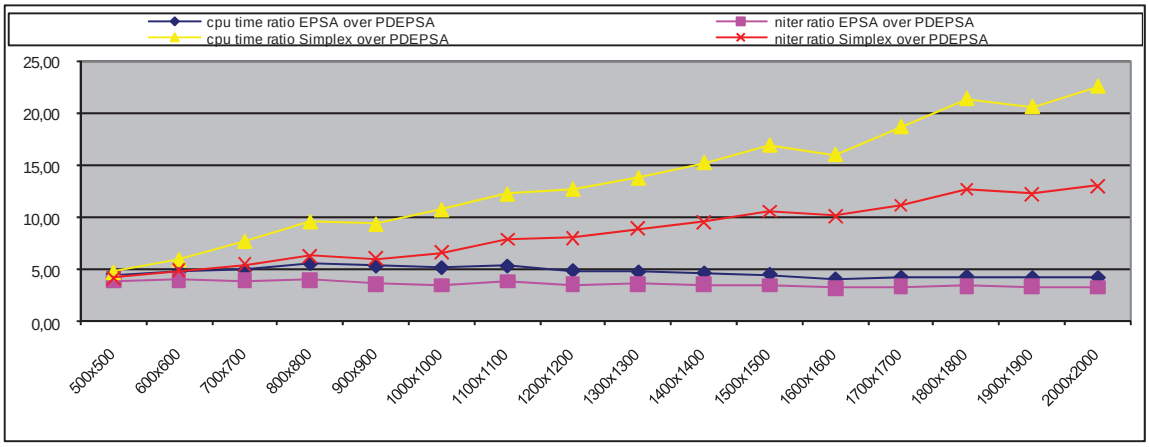
Table 3. Results for randomly generated sparse LPs with dimension nxn and density 20%

| **Density 20%** | **PDEPSA** | | **EPSA** | | **Simplex** | |
|---|---|---|---|---|---|---|
| | **CPU_time (sec)** | **Niter** | **CPU_time (sec)** | **Niter** | **CPU_time (sec)** | **Niter** |
| **500 x 500** | 1,1747 | 403 | 3,4476 | 1.313 | 6,5975 | 2.270 |
| **600 x 600** | 2,2386 | 527 | 5,4132 | 1.548 | 13,6264 | 3.255 |
| **700 x 700** | 3,4632 | 634 | 11,9684 | 1.964 | 26,4489 | 4.469 |
| **800 x 800** | 5,2323 | 762 | 17,7794 | 2.302 | 43,8315 | 5.597 |
| **900 x 900** | 8,6003 | 905 | 25,5483 | 2.598 | 74,2187 | 7.137 |
| **1000 x 1000** | 12,6954 | 1.043 | 35,2547 | 2.937 | 125,6610 | 8.928 |
| **1100 x 1100** | 16,7529 | 1.088 | 50,4632 | 3.304 | 184,9891 | 10.597 |
| **1200 x 1200** | 22,9415 | 1.246 | 63,3988 | 3.555 | 231,0437 | 11.292 |
| **1300 x 1300** | 31,5106 | 1.415 | 87,3496 | 4.099 | 349,3742 | 13.595 |
| **1400 x 1400** | 42,2669 | 1.604 | 104,9980 | 4.250 | 441,9010 | 15.130 |
| **1500 x 1500** | 51,6083 | 1.733 | 129,9145 | 4.673 | 600,8061 | 17.246 |
| **1600 x 1600** | 63,8855 | 1.850 | 171,6245 | 5.232 | 840,5429 | 20.385 |
| **1700 x 1700** | 76,7353 | 1.963 | 210,9352 | 5.653 | 1097,0000 | 23.049 |
| **1800 x 1800** | 90,8534 | 2.068 | 251,1585 | 6.049 | 1364,2000 | 24.985 |
| **1900 x 1900** | 112,5890 | 2.291 | 297,1148 | 6.407 | 1720,4000 | 27.967 |
| **2000 x 2000** | 135,8129 | 2.462 | 362,7834 | 6.951 | 2183,1451 | 30.733 |
| **Total** | 678,3608 | 21,994 | 1829,1521 | 62,835 | 9303,7861 | 226,635 |

In addition, in the last group of problems with bigger density PDEPSA keeps to be the algorithm with the best performance by far. A characteristic example, with a 1500 x 1500 problem PDEPSA requires in average 52 seconds when EPSA needs 130 seconds and the simplex algorithm 600 seconds. Likewise, the number of iterations is much less in PDEPSA which is able to estimate the optimal solution only within 1.730 iterations when EPSA needs to make 4.670 and the simplex algorithm almost 17.200 iterations.

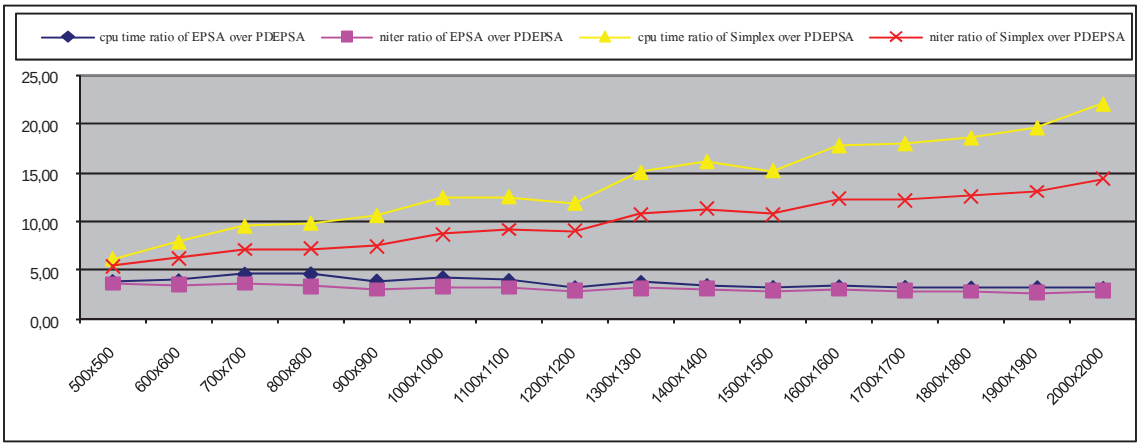
Furthermore, below figures can show more clearly the superiority of PDEPSA over EPSA and the Simplex Algorithm. Figures 1-3 present the ratios (execution time of the EPSA)/(execution time of the PDEPSA), (iterations of the EPSA)/(iterations of the PDEPSA), (execution time of the Simplex Algorithm)/(execution time of the PDEPSA) and (iterations of the Simplex Algorithm)/(iterations of the PDEPSA) for the corresponding densities and dimensions. The above ratios indicate how many times PDEPSA is better than EPSA and the Simplex Algorithm.

Figure 1. Speed-up ratios for nxn LPs and density 5%



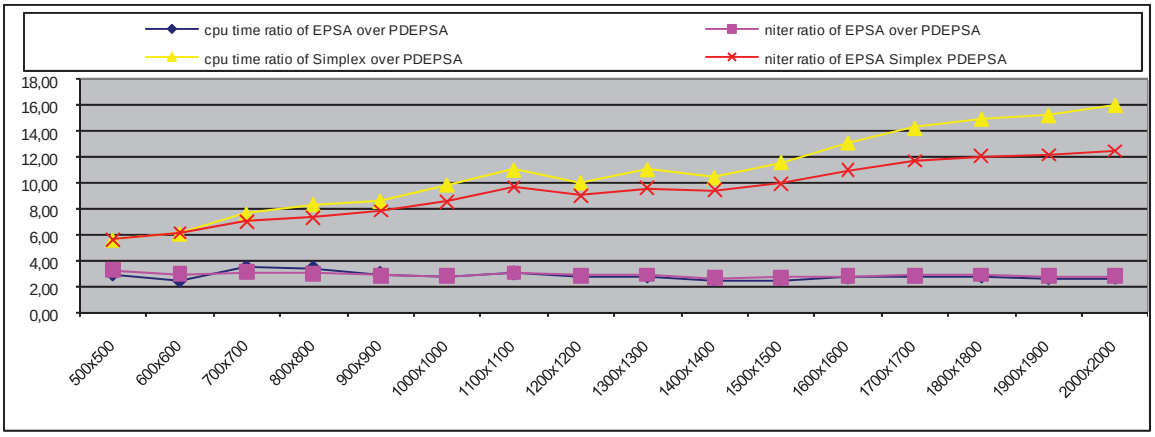

From the above figure, it is clear that as the problem dimensions increases the superiority of PDEPSA over EPSA does not vary a lot and we can claim that PDEPSA is five times faster than EPSA and it requires five times less iterations to complete its computations. On the other hand, the superiority of PDEPSA over simplex algorithm increases proportionally with the dimension of the problems. With small size LPs PDEPSA is almost five times faster and requires five times less iterations than simplex method. In contrast, in bigger LPs, like 2000 x 2000 LPs PDEPSA is almost 23 times faster and it needs 13 times less iterations.

Figure 2. Speed-up ratios for nxn LPs and density 10%




As the density of problems increases, PDEPSA continues its superiority and the results are as satisfactory as in more sparse problems. Comparatively to EPSA, PDEPSA is 4 times faster and it demands 4 times less iterations in small size LPs. However, in bigger scale LPs the difference decreases to 3 times for the execution time and the number of iterations. In regard to the simplex method, the results are very similar with the previous group of problems (density 5%), the superiority of PDEPSA over simplex algorithm increases analogically with the dimension of problems.

Figure 3. Speed-up ratios for nxn LPs and density 20%




In the last group of problems with the density at the level of 20%, the results do not differ a lot from the previous. PDEPSA is almost 3 times better than EPSA both in the execution time and the number of iterations. On the other hand, PDEPSA is almost 6 times faster and it requires 6 times less iterations in small size LPs like 500 x 500 and 600 x 600 dimension problems. In bigger scale LPs, the difference increases and it reaches 16 times faster referring to execution time and the 10 times less iterations for PDEPSA.

The results of the computational study clearly proved that the PDEPSA can perform faster and with a minor number of iterations in perspective to the performance of the simplex algorithm. Nevertheless, it is significant to mention the fact that as the density increases the superiority of PDEPSA decreases. More specific, in 2000 x 2000 dimension the PDEPSA is 23 times faster than the simplex algorithm in the 5 % density when in 20% density PDEPSA is almost 16 times faster. Despite the fact that this looks like a drawback because the performance of PDEPSA decreases comparing to simplex algorithm, in fact is not. In real world, the vast majority of problems are extremely sparse problems, and their density does not reach the level of 5%. Consequently, this computational performance of PDEPSA is a remarkable advantage and clearly shows its worth.

## 4. CONCLUSION

The current paper investigates the practical behavior of the PDEPSA. The PDEPSA which is presented in this paper refers to the attempt to avoid the problem of stalling and/or cycling. The elimination of these disadvantages can lead to a more effective algorithm with better computational performance.

Apart from the description of the algorithm, we presented an extended comparative computational study between the Revised Primal Simplex Algorithm, the Exterior Point Simplex Algorithm and the Primal-Dual Exterior Point Simplex Algorithm. In this computational study randomly generated sparse optimal linear problems are used and all the implementations were accomplished with the help of MATLAB programming environment.

Regarding to the results, PDEPSA has proved its superiority to EPSA and the simplex method in all densities and sizes. PDEPSA indicates the same behavior comparing to EPSA in all sizes of the LPs. Their difference is not affected by the dimensions of the linear problems. In contrast to this, in all densities the performance of PDEPSA is getting better comparing to the simplex algorithm while the size of the LPs increases. Moreover, the computational performance of PDEPSA is much better in very sparse problems. As the results clarify, PDEPSA can perform little better in 5% density than in 20% density. The difference between simplex algorithm and PDEPSA is greater in sparser problems. This is a strong and significant advantage of PDEPSA because in real problems the level of density is very low.

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# An Experimental Study of Bicriteria Models for Robust Timetabling

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**Abstract**

We conduct an experimental study for a fundamental case of the timetabling problem in a public railway network under disruptions. Three bicriteria optimization problems, modeling the robustness of the timetable towards delays, are experimentally evaluated against various waiting time rules at stations. Our results constitute the first proofs-of-concept for these models.

## 1 Introduction

The construction of a timetable is one of the most important phases in a public transportation network and has been extensively studied since quite some time; see e.g., [4, 5, 6, 8]. The timetabling problem asks for determining the departure and arrival times of public transportation media (e.g., trains) in order to serve, in a timely fashion, the customer demand. Typically the goal is to find an optimal solution, that is, a timetable that minimizes the overall traveling time of passengers. On the other hand, quite often there are disruptions to the normal operation of a public transportation network (e.g., due to signaling problems, maintenance work, weather conditions, accidents, etc) and this affects the timetable. As a consequence, there is a recent shift of focus towards robustness issues [2, 3], that is, providing a timetable that is robust to disruptions rather than an optimal timetable for an ideal case.

Many notions of robustness have been introduced and used for the timetabling problem (see e.g., [2]). Most of them use a given level of robustness that has to be determined beforehand. In this work, we concentrate on a new robustness approach presented recently in [7]. In that paper, the robust timetabling problem is studied as a bicriteria optimization problem, where its first objective is the objective of the nominal (undisrupted) problem, while its second objective is a suitably defined function measuring the robustness of the provided solution. The new approach has the advantage that one can use any robustness function as a second objective. Based on common practical waiting time rules at train stations, three such robustness measures are proposed in [7], leading to three bicriteria optimization problems. The theoretical study in [7] uses as building block a fundamental case with two different trains, one arriving and the other departing at/from a given station, and a transfer activity between them.

In this work, we provide the first implementation and experimental evaluation of this fundamental case that forms the building block for other more involved cases. The purpose of our study is to provide a proof-of-concept for the three bicriteria optimization problems for the fundamental case, modeling the robustness of the timetable towards delays against various waiting time rules at stations. Our experimental study is based on real data taken from the German railway network and constitutes the first proof-of-concept for these models.

The rest of the paper is organized as follows. In Section 2, we review the definition of the timetabling problem, as well as the robustness measures and the corresponding three optimization problems as introduced in [7]. In section 3, we review the fundamental case studied in [7] along with the specific formulation of the optimization problems. In section 4, we present the results of our experimental study.

## 2 Preliminaries

We follow the exposition in [7]. A timetable is defined with the use of the so-called event-activity network, i.e., a directed (connected) graph $G = (E, A)$ whose node set $E$ represents events (departures, arrivals), while its edge set $A \subseteq E \times E$ represents activities (transfer, stop, drive). A *timetable* $\Pi \in \mathbb{R}^{|E|}$ assigns a time $\Pi_i$ to each event $i \in E$. Each activity $a = (i, j)$, linking two events $i, j \in E$, is associated with a lower $l_a$ and an upper time bound $u_a$ that have to be respected. A timetable is called *feasible* if $\Pi_j - \Pi_i \in [l_a, u_a]$, for all $a = (i, j) \in A$. A path $P$ in $G$ is a sequence of events $(i_1, \ldots, i_n)$ such that either $(i_k, i_{k+1}) \in A$ (forward activity) or $(i_{k+1}, i_k) \in A$ (backward activity), for $1 \leq k < n$. The forward (resp. backward) activities of $P$ are denoted as $P^+$ (resp. $P^-$). A cycle $C$ is a path with $i_1 = i_n$. The *slack times* of a timetable are defined as $s_a = \Pi_j - \Pi_i - l_a$, $\forall a \in A$. The slack time $s_a$ is the available additional time for activity $a$ and it is used to reduce delays.

Let $m_a = u_a - l_a$, and let $w_a$ be the number of passengers traveling along an activity $a$. There are two equivalent definitions of the *timetabling* problem that aims at minimizing the overall traveling time of passengers (for details, see [7]).

– Timetabling using variables $\Pi_i, i \in E$

$$\min F(\Pi) = \sum_{a=(i,j)\in A} w_a(\Pi_j - \Pi_i)$$
$$s.t.\ l_a \leq \Pi_j - \Pi_i \leq u_a \quad \forall a = (i, j) \in A,\ \Pi_i \in \mathbb{N}.$$

– Timetabling using variables $s_a, a \in A$

$$\min F(s) = \sum_{a\in A} w_a s_a$$
$$s.t.\ 0 \leq s_a \leq m_a$$
$$\sum_{a\in C^+} s_a - \sum_{a\in C^-} s_a = -\sum_{a\in C^+} l_a + \sum_{a\in C^-} l_a$$

The robustness of a timetable is its sensitivity to unforseen delays. Before defining robustness formally, we need to see what happens when some delay occurs. Delays typically affect transfer activities. When a transfer activity $a = (i, j)$ takes place, there are two possibilities: either the outgoing train waits for the delayed incoming train so that the transfer is maintained, or the train departs on time and no delay is transferred. The problem of determining which transfers must be maintained and which must not is known as the delay management problem. As it is customarily assumed, we consider this problem solved, i.e., that we are given beforehand some waiting time rules (WTRs). We consider three typical such rules.

Let $i$ represent an arrival event of train A at a station, let $j$ represent a departure event of another train $B$ at the same station, let $a = (i, j)$ be a transfer activity from train A to train B, and let $a_{next} = (j, k)$ be the next driving activity of train B. Assume that train A arrives at $i$ with a delay $\delta_i$. The next three WTRs determine whether train B waits for train A, or departs on time.

**WTR1:** Train B is not allowed to have a delay at its next station. The transfer is maintained if and only if $\delta_i \leq s_a + s_{a_{next}}$.

**WTR2:** Train B can wait $n$ minutes, where $n$ is fixed. The transfer is maintained if and only if $\delta_i \leq s_a + n$.

**WTR3:** Train B is not allowed to have a delay of more than $m$ minutes at its next station. The transfer is maintained if and only if $\delta_i \leq s_a + s_{a_{next}} + m$.

We are now ready to define robustness. We assume that only one WTR is used within a public transportation system and that the timetabling problem is described using variables $s_a$. In particular, let a fixed WTR be given, let $s \in \mathbb{R}^{|A|}$ be a timetable, and consider a set of source-delayed events $E_{delayed}$ with delays $\delta_i \leq \delta$, for all $i \in E_{delayed}$, for some given $\delta$. Three robustness functions are defined.

- $R(s)$ measures the robustness of a timetable if all of its transfers are maintained whenever all source delays are smaller than or equal to some value $R$.
- $R_{no}(s, \delta)$ is the maximum number of passengers who miss a transfer if all source delays are smaller than $\delta$.
- $R_{del}(s, \delta)$ is the maximum sum of all passengers' delays[1], if all source delays are smaller than $\delta$.

The first definition of robustness computes the maximum source delay for which no transfer is missed. Clearly, this robustness measure needs to be maximized. The other two definitions of robustness evaluate how badly the passengers are affected by source delays. In these cases, the robustness measures need to be minimized.

The above definitions lead naturally to three bicriteria optimization problems that incorporate robustness to the timetabling problem. The first objective function is common for all models and represents the total traveling time of the passengers ($F(s)$). The second objective function is one of the robustness functions just defined ($R(s), R_{no}(s, \delta), R_{del}(s, \delta)$). To summarize, the three bicriteria optimization problems are as follows:

$$\begin{array}{rll} (P) & [\min F(s), \max\ R(s)] & \text{s.t. } s \text{ is a feasible timetable} \\ (P_{no}) & [\min F(s), \min R_{no}(s, \delta)] & \text{s.t. } s \text{ is a feasible timetable} \\ (P_{del}) & [\min F(s), \min R_{del}(s, \delta)] & \text{s.t. } s \text{ is a feasible timetable} \end{array}$$

Note that each problem can be considered for all three WTRs. In all problems we seek for *Pareto optimal* (or non-dominated) solutions, i.e., timetables $s$ that there does not exist another timetable $s'$ which is not worse in one of the two objectives and strictly better in the other one. A timetable is called *weak Pareto* (or weakly non-dominated) if there does not exist another timetable which is strictly better in both objectives. A Pareto solution is always weak Pareto, but the reverse is not true in general.

## 3 A Fundamental Case

The simplest case – which forms the building block for more involved scenarios - is to model a single transfer between two trains [7]; see Fig. 1. We assume that train 1 travels from station $F$ to station $M$, train 2 travels from station $M$ to station $K$, there is one possible transfer activity at station $M$, and a delay of size $\delta$ occurs at station $F$.

Train 1 reaches station $M$ with a delay of $[\delta - s_1]^+$. Passengers transferring to train 2 will arrive with a delay of $[\delta - s_1 - s_2]^+$. Let us see how the transfer is maintained w.r.t. each WTR.

- Using WTR1, train 2 can wait at most $s_3$ minutes. Hence, the transfer is maintained if and only if $\delta - s_1 - s_2 \leq s_3 \Leftrightarrow \delta \leq s_1 + s_2 + s_3$.

[1] If a passenger misses a transfer, then the delay is assumed to be $T$ minutes, provided that the timetable is repeated after $T$ minutes and hence the passenger can use the transfer of the next period.

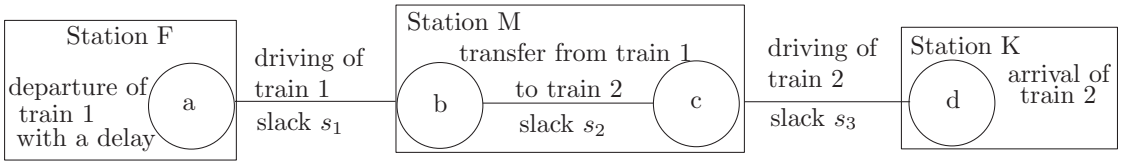


Figure 1: Fundamental case; nodes $a, c$ (resp. $b, d$) represent departure (resp. arrival) events.

- Using WTR2, train 2 can wait at most $n$ minutes. Hence, the transfer is maintained if and only if $\delta - s_1 - s_2 \leq n \Leftrightarrow \delta \leq s_1 + s_2 + n$.
- Using WTR3, train 2 can wait at most $m$ minutes and is not allowed to have a delay at its next station. Hence, the transfer is maintained if and only if $\delta - s_1 - s_2 \leq s_3 + m \Leftrightarrow \delta \leq s_1 + s_2 + s_3 + m$.

The above suggest that the first robustness function is defined as:

$$R(s_1, s_2, s_3) = \begin{cases} s_1 + s_2 + s_3, & \text{for WTR1} \\ s_1 + s_2 + n, & \text{for WTR2} \\ s_1 + s_2 + s_3 + m, & \text{for WTR3} \end{cases}$$

To determine the other two robustness functions, $R_{no}$ and $R_{del}$, we have to take into account the number of passengers $w_{FM}, w_{MK}, w_{FK}$ traveling among stations $F$, $M$, and $K$ (which is assumed given). For $R_{no}$ only passengers traveling from $F$ to $K$ can miss the transfer. Hence, for any WTR, $R_{no}$ becomes:

$$R_{no}(s, \delta) = \begin{cases} w_{FK} & \text{, if } \delta > R(s_1, s_2, s_3) \\ 0 & \text{, if } \delta \leq R(s_1, s_2, s_3). \end{cases}$$

For $R_{del}$, observe that passengers from $F$ to $M$ gain a delay of $[\delta - s_1]^+$, passengers from $F$ to $K$ gain a delay of $T$ if they miss the transfer, or they get the same delay as the passengers from $M$ to $K$, which is $[\delta - s_1 - s_2 - s_3]^+$.

$$R_{del}(s, \delta) = \begin{cases} w_{FM}[\delta - s_1]^+ + Tw_{FK} & \text{, if } \delta > R(s_1, s_2, s_3) \\ w_{FM}[\delta - s_1]^+ + (w_{FK} + w_{MK})[\delta - s_1 - s_2 - s_3]^+ & \text{, if } \delta \leq R(s_1, s_2, s_3). \end{cases}$$

The first objective function $F(s)$ concerns the minimization of the passengers' traveling time and is therefore common to all three problems $(P), (P_{no}), (P_{del})$.

$$F(s_1, s_2, s_3) = (w_{FM} + w_{FK})s_1 + w_{FK}s_2 + (w_{FK} + w_{MK})s_3.$$

After determining the objective functions, we now turn to the specific problem formulations. The first problem $(P)$ becomes:

$$\begin{aligned} &\min F(s_1, s_2, s_3) \\ &\max R(s_1, s_2, s_3) \\ &\text{s.t. } 0 \leq s_i \leq m_i, \text{ for } i = 1, 2, 3 \end{aligned}$$

As it is proved in [7], for the solutions $(s_1, s_2, s_3)$ of $(P)$ it holds that $s_1 \in \{0, m_1\}$, $s_2 \in \{0, m_2\}$, and $s_3 \in \{0, m_3\}$ – the specific values depend in the used WTR. The interpretation of these solutions is that the slack should be put on the transfer and not on the driving activities.

For the second problem ($P_{no}$), a binary variable $z$ is used with $z = 0$ if train 2 waits for train 1, and $z = 1$ otherwise. Then, ($P_{no}$) becomes:

$$\begin{aligned}
&\min F(s_1, s_2, s_3)\\
&\min zw_{FK}\\
&s.t.\ \delta z + R(s_1, s_2, s_3) \geq \delta\\
&\quad 0 \leq s_i \leq m_i,\ \text{for } i = 1, 2, 3\\
&\quad z \in \{0, 1\}.
\end{aligned}$$

Depending on the particular WTR used, we have the following. If the transfer is maintained, $z = 0$ and $R_{no} = 0$. If the transfer is missed, $z = 1$ and $R_{no} = w_{FK}$. As it is proved in [7], the Pareto solutions are of the form $(F(s), 0)$, if the transfer is maintained, and of the form $(0, w_{FK})$, otherwise (in the latter case the traveling time of passengers is zero, because they missed the transfer and hence do not travel). The interpretation of these solutions is that either distribute no slack at all, or distribute the minimum amount of slack to maintain the transfer.

For the third problem ($P_{del}$), we also use the binary variable $z$ and recall that the sum of all delays of the passengers need to be taken into account. Then, ($P_{del}$) becomes:

$$\begin{aligned}
&\min F(s_1, s_2, s_3)\\
&\min (\delta - s_1)w_{FM} + zTw_{FK} + (1 - z)(w_{FK} + w_{MK})(\delta - s_1 - s_2 - s_3)\\
&s.t.\ \delta z + R(s_1, s_2, s_3) \geq \delta\\
&\quad s_1 + s_2 + s_3 \leq \delta\\
&\quad 0 \leq s_i \leq m_i,\quad i = 1, 2, 3\\
&\quad z \in \{0, 1\}.
\end{aligned}$$

This model is quadratic and the linear formulation w.r.t. the three WTRs is as follows. For WTR1, if $z = 0$, then the first constraint implies that $R(s_1, s_2, s_3) \geq \delta \Leftrightarrow s_1 + s_2 + s_3 \geq \delta$; this in combination with the second constraint ($s_1 + s_2 + s_3 \leq \delta$) gives that $\delta - s_1 - s_2 - s_3 = 0$. If $z = 1$, then the term $(1 - z)(w_{FK} + w_{MK})(\delta - s_1 - s_2 - s_3)$ is zero. For WTR2 and WTR3, an auxiliary variable $q$ is used. In summary, ($P_{del}$) simplifies to the following (left for WTR1, right for WTR2 and WTR3) models:

$$\begin{aligned}
&\min\quad F(s_1, s_2, s_3)\\
&\min\quad (\delta - s_1)w_{FM} + zTw_{FK}\\
&s.t.\quad \delta z + R(s_1, s_2, s_3) \geq \delta\\
&\quad s_1 + s_2 + s_3 \leq \delta\\
&\quad 0 \leq s_i \leq m_i,\ i = 1, 2, 3\\
&\quad z \in \{0, 1\}
\end{aligned}
\qquad
\begin{aligned}
&\min\quad F(s_1, s_2, s_3)\\
&\min\quad (\delta - s_1)w_{FM} + zTw_{FK} + (w_{FK} + w_{MK})q\\
&s.t.\quad q + \delta z + s_1 + s_2 + s_3 \geq \delta\\
&\quad \delta z + R(s_1, s_2, s_3) \leq \delta\\
&\quad s_1 + s_2 + s_3 \leq \delta\\
&\quad 0 \leq s_i \leq m_i,\ i = 1, 2, 3\\
&\quad z \in \{0, 1\}\\
&\quad q \geq 0
\end{aligned}$$

It can be proved that the two linear formulations are equivalent to the original model [7].

## 4 Implementation and Experiments

Our implementations were conducted using `C++` (compiler gcc version 4.1.2) and CPLEX 10.1. For our experiments we used real-world data from the German Railways [1]. In particular, we considered the case of passengers traveling from Frankfurt (F) to Kaiserslautern (K) with an intermediate transfer at Mannheim (M), using ICE (intercity express) and IC (intercity) trains.

We used the timetable of August 2010 (for one particular day) and considered the capacity of ICE trains to 415 passengers and that of IC trains to 1000 passengers. The overall 23 routes and the train connections are shown in Table 1. The number of passengers $w_{FM}$, $w_{FK}$, $w_{MK}$ traveling in each activity (F→M, F→K, and M→K, resp.) was a random value in $[1, 415]$ if at least one ICE train was involved, and a a random value in $[1, 1000]$ otherwise. We also assumed that when a transfer is made from train 1 to train 2, the latter train had the same or bigger capacity than the former. Each bicriterion optimization problem had to be solved for every

| Routes (id's) | Departure Frankfurt(Main)Hbf | Arrival Mannheim Hbf | Departure Mannheim Hbf | Arrival Kaiserslautern Hbf |
|---|---|---|---|---|
| 1 | 06:50 | 07:28 | 07:56 | 08:59 |
| 2 | 07:50 | 08:28 | 08:56 | 09:59 |
| 3 | 09:13 | 10:20 | 10:26 | 11:28 |
| 4 | 10:05 | 10.42 | 10:55 | 11:59 |
| 5 | 10:50 | 11:28 | 11:56 | 12:59 |
| 6 | 11:13 | 12:20 | 12:26 | 13:28 |
| 7 | 12:05 | 12:42 | 12:56 | 13:59 |
| 8 | 13:13 | 14:20 | 14:26 | 15:28 |
| 9 | 14:05 | 14:42 | 14:56 | 15:59 |
| 10 | 14:50 | 15:28 | 15:56 | 16:59 |
| 11 | 15:13 | 16:20 | 16:26 | 17:28 |
| 12 | 16:05 | 16:42 | 16:56 | 17:59 |
| 13 | 17:13 | 18:20 | 18:26 | 19:28 |
| 14 | 18:05 | 18:42 | 18:56 | 19:59 |
| 15 | 18:50 | 19:28 | 19:56 | 20:59 |
| 16 | 19:13 | 20:20 | 20:26 | 21:26 |
| 17 | 20:05 | 20:42 | 20:50 | 21:33 |
| 18 | 21:05 | 21:53 | 22:08 | 23:14 |
| 19 | 21:50 | 22:28 | 22:40 | 23:45 |
| 20 | 22:05 | 22:42 | 23:12 | 00:18 |
| 21 | 23:00 | 23:41 | 00:01 | 01:43 |
| 22 | 23:13 | 00:20 | 00:31 | 02:07 |
| 23 | 00:18 | 01:10 | 01:16 | 02:19 |

Table 1: Timetable for a single day (August 2010) for ICE and IC trains.

WTR and there are 23 routes to be considered in each case, leading to a vast amount of results. Due to space constraints, we report here on selected results representing the most interesting cases (concentrating on the strictest rule WRT1 and on WRT2).

We start with $(P)$ using WTR1. The data used and the computed slack times are shown in Table 2 and concern routes 5-10 of Table 1. The results are reported in Fig. 2, with the horizontal axis representing function $R$ (that needs to be maximized) and the vertical axis representing function $F$ (that needs to be minimized). The legend in Fig. 2 shows the values of $F$ and $R$. The results confirm the theoretical solutions discussed in Section 3. Observe that (see Table 2) in every triple $(s_1, s_2, s_3)$ the slack variables get either their lower bound or their upper bound. In all triples except the first one, variables $s_2$ get their maximum slack, which confirms the theoretical observation that slack should be put on transfer activities.

Pareto solutions for $(P_{no})$ using WTR2 are shown in Table 3. The data shown are: $n$ (number of minutes regarding WTR2), $\delta$ (source delay in minutes), $w_{FM}, w_{FK}, w_{MK}$ (number of passengers for the corresponding activities), the values of $F$ and $R$ when the transfer is missed $(z = 1)$, and when is maintained $(z = 0)$. In the latter case, the slack times are also shown with their upper bounds. The data correspond to routes 20-22. There are only two non-dominated solutions and hence the results confirm the theoretical solutions discussed in Section 3. In

| Input Data | | | | | | | Solutions |
|---|---|---|---|---|---|---|---|
| Routes | $w_{FM}$ | $w_{FK}$ | $w_{MK}$ | $s_1$ | $s_2$ | $s_3$ | $(s_1, s_2, s_3)$ |
| 5 | 93 | 77 | 105 | [0,7] | [0,28] | [0,6] | {(0,0,0),(0,28,0),(7,28,0),(7,28,6)} |
| 6 | 55 | 13 | 55 | [0,3] | [0,6] | [0,1] | {(0,0,0),(0,6,0),(3,6,1)} |
| 7 | 158 | 242 | 307 | [0,3] | [0,14] | [0,5] | {(0,0,0),(0,14,0),(3,14,0),(3,14,5)} |
| 8 | 80 | 19 | 235 | [0,2] | [0,6] | [0,7] | {(0,0,0),(0,6,0),(2,6,0),(2,6,7)} |
| 9 | 183 | 150 | 457 | [0,8] | [0,14] | [0,3] | {(0,0,0),(0,14,0),(8,14,0),(8,14,3)} |
| 10 | 539 | 107 | 350 | [0,7] | [0,28] | [0,8] | {(0,0,0),(0,28,0),(0,28,8),(7,28,8)} |

Table 2: Input data and computed slack times for $(P)$ for routes 5-10.

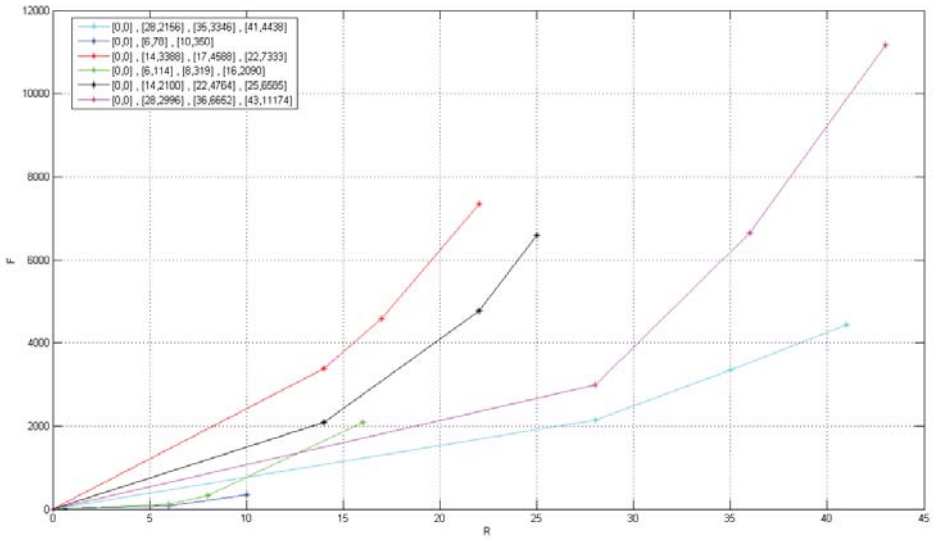


Figure 2: Pareto solutions for $(P)$ concerning routes 5-10 using WTR1.

routes 20-22 the time provided by WTR2 reduces the delay $\delta$ but does not absorb it, so either the minimum amount of slack is given in order to maintain the transfer or there is no slack at all and the transfer is missed.

| | | | | | | $z=0$ | | | | | | | | $z=1$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Routes | $n$ | $\delta$ | $w_{FM}$ | $w_{FK}$ | $w_{MK}$ | $F$ | $R$ | $s_1$ | $m_1$ | $s_2$ | $m_2$ | $s_3$ | $m_3$ | $F$ | $R$ |
| 20 | 8 | 42 | 100 | 50 | 210 | 1800 | 0 | 2 | 3 | 30 | 30 | 2 | 10 | 0 | 50 |
| 21 | 1 | 28 | 88 | 77 | 96 | 2695 | 0 | 7 | 7 | 20 | 20 | 0 | 9 | 0 | 77 |
| 22 | 11 | 13 | 400 | 219 | 100 | 438 | 0 | 0 | 4 | 2 | 11 | 0 | 4 | 0 | 219 |

Table 3: Input data and Pareto solutions for $P_{no}$ regarding routes 20-22.

Finally, we report results for $(P_{del})$ using WTR1. The data used and the computed slack times are shown in Table 4, and concern routes 11-16. The results are reported in Fig. 3, with the horizontal axis representing function $R$ and the vertical axis representing function $F$ (both need to be minimized). Note that there are solutions with $F = 0$. This means that the transfer was missed and the sum of all passengers' delays is $R$. When the transfer is maintained, the values of $R$ are lower, but $F$ increases as we are dealing with two conflicting functions. The results suggest that we should distribute slack equal to the delay when the transfer is maintained in order to absorb it. When the transfer is missed, either distribute no slack at all or distribute an amount of slack to minimize the maximum sum of all passengers' delays.

| Input Data | | | | | | | | Solutions |
|---|---|---|---|---|---|---|---|---|
| Routes | $\delta$ | $w_{FM}$ | $w_{FK}$ | $w_{MK}$ | $s_1$ | $s_2$ | $s_3$ | $(s_1, s_2, s_3, z)$ |
| 11 | 5 | 237 | 73 | 40 | [0,2] | [0,6] | [0,3] | {(0,5,0,0),(2,3,0,0),(0,0,0,1),(2,0,0,1)} |
| 12 | 19 | 607 | 302 | 41 | [0,3] | [0,14] | [0,3] | {(2,14,3,0),(3,14,2,0),(0,0,0,1),(3,0,0,1)} |
| 13 | 7 | 78 | 108 | 61 | [0,2] | [0,6] | [0,2] | {(0,6,1,0),(1,6,0,0),(2,5,0,0),(0,0,0,1),(2,0,0,1)} |
| 14 | 18 | 109 | 350 | 200 | [0,10] | [0,14] | [0,10] | {(4,14,0,0),(10,8,0,0),(0,0,0,1),(2,0,0,1)} |
| 15 | 25 | 300 | 309 | 102 | [0,10] | [0,28] | [0,10] | {(0,25,0,0),(10,15,0,0),(0,0,0,1),(10,0,0,1)} |
| 16 | 10 | 50 | 80 | 152 | [0,3] | [0,6] | [0,7] | {(3,6,1,0),(0,0,0,1),(3,0,0,1)} |

Table 4: Input data and computed slack times for $(P_{del})$ regarding routes 11-16.

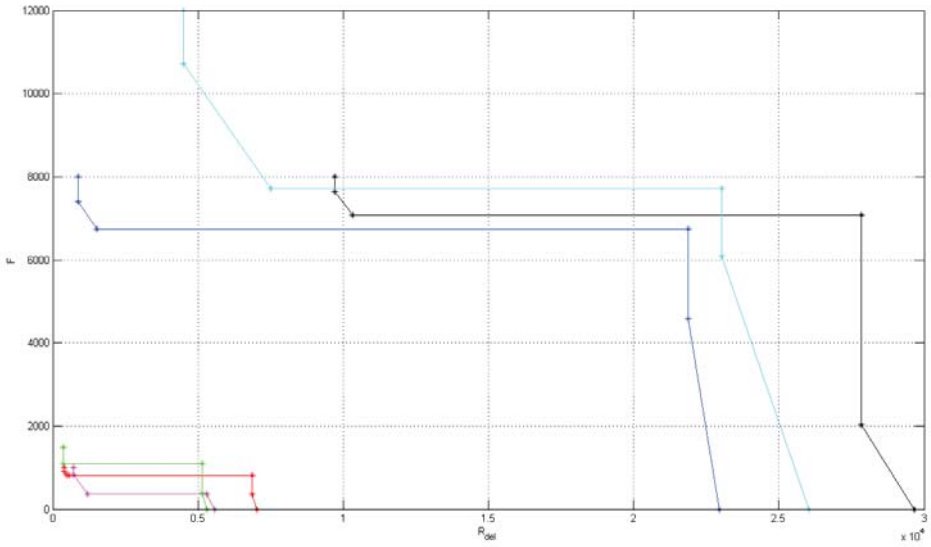


Figure 3: Weak Pareto solutions for $(P_{del})$ for the routes 11-16 using WTR1.

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# On variable priorities for solving systems of nonlinear equations

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**Abstract**

In real world applications, the variables involved in a nonlinear system of equations have different importance. Motivated by the concept of lexicographic methodology which gives priorities in functions, we give priorities in the variables of the system, before an iterative solver be applied. The way of giving priorities corresponds some variables of low priority to be substituted by other ones of high priority. Specifically, after a priority is given in the variables of each function component, a new point for each of them is produced. This process may be considered as a preprocessing step for the corresponding iterative scheme which has determined to be applied. Evidently, the preprocessing step points may be exploited by several ways during the chosen iterative process. In the present work we experiment with Newton's method. The new points are only used for updating the Jacobian matrix at each iteration. Through the approximated Jacobian matrix, Newton's method results to a modified one, that is a quasi Newton type method, named Variable Lexicographic Nonlinear System method. The convergence theorem of the produced iterative form is given. From a geometrical point of view, Variable Lexicographic Nonlinear System method corresponds to Newton's method in which a proper rotation of the tangent plane has be done. As the numerical results show, this rotation, generally, induces an acceleration of the standard method as well as better results from points far away from the solution or in singular cases of Jacobian matrix. The first promising results, encourage us to consider that more search about rules priorities on variables of the system and about the optimal of them may be done.
*Keywords:* lexicographic optimization; variable priorities; Newton's method; quasi Newton method; systems of nonlinear equations;

## 1 Introduction

Consider a system of nonlinear equations:

$$\text{given} \quad F : R^n \to R^n, \quad \text{find} \quad x^* \in R^n \quad \text{such that} \quad F(x^*) = 0 \tag{1}$$

where $F = (f_1, \ldots, f_n)^T$ is assumed to be continuously differentiable. We denote $J(x) = F'(x)$ for all $x \in R^n$.

There exist a lot of iterative methods in order to manage the problem (1), such as the classical Newton's method and quasi Newton methods in which an approximation of Jacobian matrix is done in order to avoid to compute it at each iteration. The general iterative scheme for a quasi Newton method is given by:

$$x^{p+1} = x^p - B_p^{-1} F(x^p), \quad p = 1, 2, ... \tag{2}$$

The matrix $B_p$ is an approximation to the Jacobian matrix $J(x_p)$. For several approaches of choosing $B_p$ see [5, 7]. When $B_p = J(x^p)$ the relation (2) becomes

$$x^{p+1} = x^p - J^{-1}(x^p) F(x^p), \quad p = 1, 2, ... \tag{3}$$

and describes the Newton's iterative scheme.

Recently, the transformation of problem (1) to an equivalent optimization problem has been proposed in [2, 8] or to a multiobjective optimization one in [1].

Motivated by the real world applications, where in systems of nonlinear equations the involved variables have different relative importance, the concept of lexicographic methodology is formulated in an iterative process for solving nonlinear systems. Thus, before an iterative algorithm for solving the system (1) is applied, proper priorities are given to the variables $x_1, x_2, \ldots, x_n$, under some rules. In the proposed idea - giving priority to variables - each variable is involved in all iterations, without ignoring any of them in the sense of discarding it, as it is done for functions in lexicographic methodologies. Moreover, the number of priorities may be less than the number of variables.

An important role in this paper plays the way of defining rules of priorities on variables. For this reason, in some sense, we imitate the priority idea of the lexicographic multiobjective optimization. Instead of a strict application of lexicographic methodology to be applied in systems of nonlinear equations, defining a set of subproblems that must be solved following the given priorities, we experiment with priorities given to the variables of equations. Also, the concept of priority is not applied in the sense of choosing some high priority variables and their appearance to be only in the last subproblems. In this work, the term priority corresponds to indicate the low priority variables that must be substituted by high priority ones. The purpose of this process is the generation of new points from the current one, to be used for an appropriate approximation of the Jacobian matrix. As it seems, this inherent information plays an important role in the iterative process.

The problem of locating a good initial point for Newton's method, especially in real world applications, destroys its good feature of quadratic convergence. It is known that its quadratic convergence is succeeded only in its convergence area which is a close region around the solution point. Thus, the seeking for better points is an important and difficult issue for such problems. In the present paper new points are produced at each iteration from the current point, via a technique according to which the values of variable components are modified and substituted. It is obvious that changing the value of some variables components a new variable is defined and consequently a new point is determined. Notice that, in this paper the new points do not substitute in whole the current point but only in the calculation of Jacobian matrix. This means that an approximation of Jacobian is done, which as it seems from the first numerical results, it contributes in acceleration of Newton's method by reducing the number of iterations, which in some cases is more less than half of original Newton's one.

In Section 2 is presented the proposed preprocessing step and how it can convert Newton's method to a quasi Newton one. Furthermore, the convergence and the rate of convergence are discussed. Some numerical applications to well-known systems of nonlinear equations are given in Section 3. Conclusions and further research aspects are presented in Section 4.

## 2 The proposed methodology

Let $x^p = (x_1^p, x_2^p, \ldots, x_n^p)$ denotes the current approximation. Without loss of generality, we fix the first $n-1$ variables of $x^p$ and the $n^{th}$ component of $x^p$ is substituted by the variable $x_i^p$. Then the corresponding vectors $u^{i,p}$ are defined as:

$$u^{i,p} = (u_1^{i,p}, u_2^{i,p}, \ldots, u_n^{i,p}) = (x_1^p, x_2^p, \ldots, x_{n-1}^p, x_i^p),$$

and hence,

$$x^p - u^{i,p} = (0, 0, \ldots, 0, \ldots, x_n^p - x_i^p)\,.$$

Expanding the functions $f_i(x^p)$ at the corresponding points $u^{i,p}$ we have:

$$\begin{aligned} f_i(x^p) &\approx f_i(u^{i,p}) + \sum_{j=1}^{n} \partial_j f_i(u^{i,p})(x_j^p - u_j^{i,p}) \\ &= f_i(u^{i,p}) + \sum_{j=1}^{n-1} \partial_j f_i(u^{i,p})(x_j^p - x_j^p) + \partial_n f_i(u^{i,p})(x_n^p - x_i^p) \\ &= f_i(u^{i,p}) + \partial_n f_i(u^{i,p})(x_n^p - x_i^p) \end{aligned} \tag{4}$$

Now expanding the functions $f_i$ at the corresponding points $u^{i,p}$ we have:

$$\begin{aligned} f_i(x) &= f_i(u^{i,p}) + \sum_{j=1}^{n} \partial_j f_i(u^{i,p}) \left(x_j - u_j^{i,p}\right) \\ &= f_i(u^{i,p}) + \sum_{j=1}^{n-1} \partial_j f_i(u^{i,p}) \left(x_j - x_j^p\right) + \partial_n f_i(u^{i,p}) \left(x_n - x_i^p\right) \\ &= f_i(u^{i,p}) + \sum_{j=1}^{n-1} \partial_j f_i(u^{i,p}) \left(x_j - x_j^p\right) + \partial_n f_i(u^{i,p}) \left(x_n - x_i^p + x_n^p - x_n^p\right) \\ &= f_i(u^{i,p}) + \sum_{j=1}^{n-1} \partial_j f_i(u^{i,p}) \left(x_j - x_j^p\right) + \partial_n f_i(u^{i,p}) \left(x_n - x_n^p\right) + \partial_n f_i(u^{i,p}) \left(x_n^p - x_i^p\right) \\ &= f_i(u^{i,p}) + \partial_n f_i(u^{i,p}) \left(x_n^p - x_i^p\right) + \sum_{j=1}^{n} \partial_j f_i(u^{i,p}) \left(x_j - x_j^p\right) \\ &\overset{(4)}{=} f_i(x^p) + \sum_{j=1}^{n} \partial_j f_i(u^{i,p}) \left(x_j - x_j^p\right) \quad i = 1, 2, \ldots, n \end{aligned}$$

Hence, we have the equation

$$f_i(x^p) + \sum_{j=1}^{n} \partial_j f_i(u^{i,p}) \left(x_j - x_j^p\right) = 0$$

from which we have the linear system

$$\sum_{j=1}^{n} \partial_j f_i(u^{i,p}) \left(x_j - x_j^p\right) = -f_i(x^p)$$

or in matrix form

$$J(u^{i,p}) \left(x - x^p\right) = -F(x^p)$$

and thus, we have the iterative form:

$$x^{p+1} = x^p - J^{-1}(u^{i,p}) F(x^p) \tag{5}$$

The relation (5) corresponds to a quasi Newton iterative form with

$$B_p = J(u^{i,p})$$

Notice that, although the iterative scheme (5) is a quasi Newton method, it demands the Jacobian matrix computation at the points $u^{i,p}$ and its computational cost is the same with Newton's method given by relation (3)

The new proposed Variable Lexicographic Nonlinear System (VLNS) method is described by VLNS Algorithm 1.

**Algorithm 1** Variable Lexicographic Nonlinear System (VLNS) method

**Require:** $\{x^0, F(x), F'(x), \epsilon, MIT\}$ $\{x^0$ is the initial guess, $F(x)$ the set of functions, $\epsilon$ is the predefined accuracy and $MIT$ is the maximum number of iterations.$\}$
  $p = 1$
  **repeat**
    **for** $i = 1$ to $n$ **do**
      Create the points $u^{i,p}$ according to the priority rules of $x_j^p$.
    **end for**
    Calculate $x^{p+1} = x^p - F'(u^{i,p})^{-1} \cdot F(x^p)$
  **until** Stopping criteria met
  Set $x^* = x^{p-1}$
  **return** $x^*$

**Remark 2.1.** *Since, the proposed method given by (5), is a quasi Newton method, the convergence Theorem is similar to Dennis Newton-Kantorovich Theorem 2, p.327, in [3], under the corresponding same assumptions, thus it is missed. So, under this Theorem the sequence* $\{x_n\}$ *of VLNS method converges. Due to the Corollary 2.2 , p.328, in [3], the rate of convergence is* $q \in (1, 2]$ *and its proof is similar. Furthermore, more about the rate of convergence of such class of methods may be found in [4].*

## 3 Numerical Results

We apply the proposed approach with the preprocessing step to a set of nonlinear systems given in [6]. The priority to each variable is given heuristically. The numerical results are quite satisfactory.

In Tables 1,2,3 we present the results obtained, for various initial points, by Newton's and VLNS methods with accuracy $\varepsilon = 10^{-14}$.

- *'IT'* indicates the number of the iterations,
- *'FE'* the number of the function evaluations (including derivatives) and
- *'$r_i$'* the root to which each method converges.

The convergence criteria are:

1. $\|f_i(x^p)\|_\infty \leq \varepsilon$ and $\left\|x^{p+1} - x^p\right\|_\infty \leq \varepsilon$,
2. $MIT = 1000$, where $MIT$ is the maximum number of iterations.

**Example 3.1.** *The first system is given by:*

$$\begin{aligned} f_1(x_1,x_2,x_3) &= x_1^3 - x_1x_2x_3 = 0 \\ f_2(x_1,x_2,x_3) &= x_2^2 - x_1x_3 = 0 \\ f_3(x_1,x_2,x_3) &= 10x_1x_3 + x_2 - x_1 - 0.1 = 0 \end{aligned}$$

*has two roots:*

$$r_1 = (0.1, 0.1, 0.1), \qquad r_2 = (-0.1, -0.1, -0.1)$$

*and its Jacobian matrix is nonsingular and and cannot be achieved accurately, at some points. This feature constitutes the difficulty of the system.*

*The Jacobian matrix has been calculated at the points:* $u^{1,p} = u^{2,p} = u^{3,p} = (x_1^p, x_2^p, x_2^p)$. *The numerical results are presented in table (1).*

| | | | *Newton* | | | *VLNS* | | |
|---|---|---|---|---|---|---|---|---|
| $x_1^0$ | $x_2^0$ | $x_3^0$ | $IT$ | $FE$ | $r_i$ | $IT$ | $FE$ | $r_i$ |
| -4 | -2 | 1 | 53 | 636 | $r_2$ | 14 | 168 | $r_2$ |
| -1 | -2 | 0.6 | 51 | 612 | $r_1$ | 9 | 108 | $r_2$ |
| 0.5 | 2 | 1 | 54 | 648 | $r_1$ | 20 | 240 | $r_1$ |
| 10 | -2 | -2 | 39 | 468 | $r_1$ | 17 | 204 | $r_2$ |
| -2 | -2 | 1 | 33 | 396 | $r_2$ | 9 | 108 | $r_2$ |

Table 1: Comparison between Newton's and VLNS method for Example 1

**Example 3.2.** *The second system is given by:*

$$\begin{aligned} f_1(x_1,x_2,x_3) &= x_1x_3 - x_3e^{x_1^2} + 10^{-4} = 0 \\ f_2(x_1,x_2,x_3) &= x_1(x_1^2 + x_2^2) + x_2^2(x_3 - x_2) = 0 \\ f_3(x_1,x_2,x_3) &= x_1^3 + x_3^3 = 0, \end{aligned}$$

*has the root:*

$$r = (-0.99990001 \cdot 10^{-4}, -0.99990001 \cdot 10^{-4},\ 0.99990001 \cdot 10^{-4})$$

*and its Jacobian matrix is singular.*

*The Jacobian matrix has been calculated at the points:* $u^{1,p} = x^p = (x_1^p, x_2^p, x_3^p)$, $u^{2,p} = (x_2^p, x_2^p, x_3^p)$ *and* $u^{3,p} = (x_3^p, x_2^p, x_3^p)$. *The numerical results are presented in table (2).*

| | | | *Newton* | | *VLNS* | |
|---|---|---|---|---|---|---|
| $x_1^0$ | $x_2^0$ | $x_3^0$ | $IT$ | $FE$ | $IT$ | $FE$ |
| 2 | 2 | 2 | 42 | 504 | 24 | 288 |
| 3 | 3 | 3 | 122 | 1464 | 29 | 348 |
| 3 | 3 | 15 | 612 | 7344 | 22 | 264 |
| 3 | -2 | 15 | 612 | 7344 | 22 | 264 |
| 4 | 4 | 4 | 73 | 876 | 36 | 432 |

Table 2: Comparison between Newton's and VLNS method for Example 2

**Example 3.3.** *The third system is the well-known Brown's almost linear system and given by:*

$$\begin{aligned} f_1(x_1,x_2,x_3,x_4,x_5) &= 2x_1+x_2+x_3+x_4+x_5-6=0\\ f_2(x_1,x_2,x_3,x_4,x_5) &= x_1+2x_2+x_3+x_4+x_5-6=0\\ f_3(x_1,x_2,x_3,x_4,x_5) &= x_1+x_2+2x_3+x_4+x_5-6=0\\ f_4(x_1,x_2,x_3,x_4,x_5) &= x_1+x_2+x_3+2x_4+x_5-6=0\\ f_5(x_1,x_2,x_3,x_4,x_5) &= x_1x_2x_3x_4x_5-1=0. \end{aligned}$$

*Its Jacobian matrix is nonsingular and ill-conditioned, which constitute the difficulty of the system.*
*The presented numerical results in table (3), are for $n=5$ whereas the system has the following three roots:*

$$\begin{aligned} r_1 &= (1,1,1,1,1)\\ r_2 &= (0.91635458253385,\ldots,0.91635458253385,1.41822708733080)\\ r_3 &= (-0.57904308849412,\ \ldots,-0.57904308849412,\ \ 8.89521544247060) \end{aligned}$$

*The Jacobian matrix is calculated at the point $u^{i,p}=(x_4^p,x_2^p,x_3^p,x_4^p,x_5^p)$, for all the functions $f_i$, $i=1,...,5$.*

| | | | | | *Newton* | | | *VLNS* | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $x_1^0$ | $x_2^0$ | $x_3^0$ | $x_4^0$ | $x_5^0$ | $IT$ | $FE$ | $r_i$ | $IT$ | $FE$ | $r_i$ |
| -8 | -3 | 4 | 2 | 1.5 | 85 | 2550 | $r_3$ | 66 | 1980 | $r_1$ |
| 10 | 3 | 4 | 2 | 1.5 | 83 | 2490 | $r_3$ | 59 | 1770 | $r_3$ |
| -0.5 | -0.6 | 4 | 2 | 1.5 | 36 | 1080 | $r_3$ | 50 | 1500 | $r_1$ |
| -0.2 | -0.2 | -0.2 | -0.2 | -0.2 | 36 | 1080 | $r_3$ | 36 | 1080 | $r_3$ |
| 0.1 | -0.1 | 0.1 | -0.1 | 4 | 41 | 1230 | $r_3$ | 31 | 930 | $r_1$ |

Table 3: Comparison between Newton's and VLNS method for Example 3

**Remark 3.1.** *In examples 3.1 and 3.2 the proposed VLNS method gives remarkable less number of iterations while in example 3.3, where the most partial derivatives are independent from the variables, the VLNS method does not affect a surprising improvement in number of iterations.*

## 4 Conclusions and further research

As it has already been mentioned the proposed VLNS iterative scheme, given by (5), although it is a quasi Newton method, it indicates at each iteration the computation of Jacobian matrix. Thus, at each iteration it is of the same computational cost compared with Newton's method. Nevertheless, the above examples show that in many cases the VLNS algorithm induces a remarkable reduction in number of iterations with promising results, even in cases of initial points are far away from the solution or when the Jacobian is singular. Nowadays, especially in solving systems of nonlinear equations, in most practical applications, the computation of derivatives using automatic differentiation is not expensive [7]. It is notable that in some cases, from the same initial point the proposed VLNS method and Newton's one, result to different roots of the system.

Under the proposed methodology, taking some variables to have the same priority or generally to be a linear combination of some others, a projection of functions in lower dimension is easily produced. From this projection an important information is extracted, which contributes in the

acceleration of the algorithm. So, a further investigation about the variable priorities in solving systems of nonlinear equations it is worth to be done. Furthermore, the proposed variable priorities may be tested under some existing Newton types methods.

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# Logarithmic-approximations for the relocation problem

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**Abstract**

This paper studies the minimization of makespan in the relocation problem. The relocation problem, formulated from an area redevelopment project. It also can be treated as a resource constrained scheduling problem and provides a generalization of conventional resource constraints [2], [3], [5] by allowing that the amount of resource returned by a job can be less than, equal to or greater than the amount the job has required. In this paper, we present the first polynomial-time $O(\log n)$-approximation algorithm for the relocation problem with arbitrary processing times.

## 1 Introduction

The relocation problem is a resource constraint scheduling problem typically rising in house redevelopment projects. For example, in XX Century a large-scale public housing project was initiated in East Boston Area [7, 8] to rehabilitate a set of old buildings. Before demolishment of an old house, all its residents must be relocated to some temporary houses. Then, the old house is destroyed, a new house is built, and the residents are relocated again to their new accommodations. Given the capacities of the houses (old, new and temporary), the authority has to schedule house renovation activities such that at any moment in time all residents are (at least temporarily) housed. Here, the typical objective is to minimize the makespan, i.e., the project duration.

The significance of the relocation problem could be attributed to its potential applications in database management [1] and financial planning [13]. The relocation problem provides a generalization of conventional resource constraints [2], [3], [5] by allowing that the amount of resource returned by a job can be less than, equal to or greater than the amount the job has required.

Moreover, the relocation problem is a natural generalization of the non-preemptive parallel job scheduling problem. An instance of this problem is given by a list of jobs and for each job an execution time and the number of required machines is given [4]. It is easy to see that the relocation problem can be used to model the non-preemptive parallel job scheduling problem by using the available processors as the single resource.

### 1.1 Problem formulation

Formally, the problem can be described as follows. We are given a set of activities $J = \{1, 2, \ldots, n\}$ with processing requirement $p_j$ for each activity $j \in J$. There is a single resource needed for processing of

the activities. Let $\Omega_0$ be the initial amount of the resource. For each activity $j \in J$, we are given the start time $S_j$. At time $S_j$ activity $j$ consumes $\alpha_j$ units of the resource. Clearly, this amount of the resource should be available at $S_j$, and it should be devoted exclusively for activity $j$. At its completion time $C_j = S_j + p_j$, activity $j \in J$ releases $\beta_j$ units of the resource, and this amount becomes freely available for any other activities. No activity preemption is allowed. Given the resource constraint, the task is to find a schedule $\sigma = \{S_j : j \in J\}$ minimizing the makespan $C_{\max} = \max_{j \in J} C_j$. Notice, we have no limits on the amount of activities being processed at the same time. We also assume that all numbers in the input are non-negative integers. We denote this problem as $RP||C_{max}$ using standard scheduling notation [11].

### 1.2 Related and our results

In a basic model of the relocation problem, the temporal parameter $p_i$ is not included, i.e., given a fixed amount of the resource, the problem is to plan a feasible redevelopment sequence of the buildings. The minimization counterpart of the feasibility problem is to determine the minimum initial budget required for the existence of a feasible sequence. Kaplan and Amir in [1] showed that this minimization problem is equivalent to the well-known makespan minimization in a two-machine flowshop [6]. Kaplan in [7] and Amir and Kaplan in [1] addressed the deployment of multiple working crews. If there are available crews and sufficient resource, the development of several buildings can overlap. Kononov and Lin in [9] showed that minimizing the makespan is strongly NP-hard even when there are only two working crews, all buildings have the same processing time and the new capacity of each building is no less than its original capacity. They also proposed approximation algorithms and analyzed the associated performance ratios. In [10] Kononov and Lin considered the relocation problems with the total weighted completion time criteria. They showed the NP-hardness of the considered problems and presented an 2-approximation algorithm for two restricted special cases. Sevastyanov et al. in [12] investigated the relocation problem of makespan minimization subject to release dates. They analyzed the complexities of several cases and also developed a pseudo-polynomial time algorithm, based on a multi-parametric dynamic programming technique, for the case where the number of different release dates is constant.

Regarding the computational complexity, the relocation problem is hard to approximate within a factor better than $3/2$, unless $P = NP$, even if all activities have unit processing times; see [9]. Kononov and Lin in [9] discuss approximability of the more restricted versions of the problem. There, the crucial additional assumption is that the processing times are the same for all jobs. In this setting, polynomial-time constant-approximation algorithms are developed. In contrast, in this paper we develop the first polynomial-time $O(\log n)$-approximation algorithm for the relocation problem with arbitrary processing times.

## 2 Approximation algorithms with performance guarantees $O(\log p_{\max})$ and $O(\log n)$

Notice, without loss of generality we may assume $\alpha_j \leq \beta_j$ for all $j \in J$, for otherwise we make use of the schedule mirroring argument as in [9]. Moreover, let the activities be arranged in non-decreasing order of the release-to-consumption ratios, i.e., $\beta_1/\alpha_1 \leq \beta_2/\alpha_2 \leq \ldots \leq \beta_n/\alpha_n$. Here, if $\alpha_j = 0$ for some $j \in J$, we naturally assume that the ratio is $+\infty$.

Now, we are ready to present the ideas for the approximation algorithms. We start with a polynomial-time $O(\log p_{\max})$-approximation algorithm, where $p_{\max} = \max_{j \in J} p_j$.

**Algorithm A**

1. We slightly modify the instance rounding up the processing times of the activities to the nearest power-of-two numbers. Then, let the set of activities having processing time $2^i$, $i \in \{0,1,\ldots,\lceil \log_2 p_{\max} \rceil\}$, be denoted by $J_i$.

2. At each step of the algorithm we select several activities from a single set $J_i$, $i \in \{0,1,\ldots,\lceil \log_2 p_{\max} \rceil\}$, such that the aggregated resource requirement of selected activities does not exceed the available resource. Then, selected activities are started and completed simultaneously. Let the completion time of the activities scheduled in step $k \geq 1$ of the algorithm be denoted by $t_k$. Finally, we compute the amount $\Omega_k$ of the resource available at time $t_k$.

   If activities of set $J_i$ are scheduled at step $k$, for simplicity, we say: *set $J_i$ is scheduled at step $k$*. Further we specify the choice of a set to be scheduled at step $k \geq 1$ of the algorithm, and the choice of activities to be processed at step $k$.

   At step $k = 1$ we consider activities from set $J_0$. We pick an activity $j \in J_0$ that has the lowest index (equivalently, the highest value of the release-to-consumption ratio). If the amount of available resource is at least $\alpha_j$, we start processing $j$ at time 0 and we decrease the amount of available resource by $\alpha_j$. Otherwise, we disregard $j$ and recurse on the (not-disregarded) rest of set $J_0$. Since all activities from set $J_0$ have unit processing times, the completion time $t_1$ of the scheduled activities is 1. Let $\Omega_1$ be the amount of resource becoming available at time $t_1 = 1$.

   Now, consider step $k \geq 2$ of the algorithm. First, we specify which set should be scheduled at step $k$. This is done according to the following rule: for all $0 \leq i \leq \lceil \log_2 p_{\max} \rceil - 1$, set $J_i$ must be scheduled exactly four times between any two consecutive scheduling of set $J_{i+1}$ (for completeness we assume that prior to time 0 all sets are scheduled). Given the initial set $J_0$, this rule uniquely defines the sequence of scheduled sets. For instance, if there are only two sets, $J_0$ and $J_1$, the sequence is formed by repetition of subsequence $S_2 = [J_0, J_0, J_0, J_0, J_1]$. If the number of sets is 3, the sequence is generated by repetition of $S_3 = [S_2, S_2, S_2, S_2, J_2]$. Recursively, given subsequence $S_\ell$ determined for $\ell$ sets, the sequence for $\ell + 1$ sets is formed by repetition of $S_{\ell+1} = [S_\ell, S_\ell, S_\ell, S_\ell, J_\ell]$.

   Given a set to be scheduled at step $k \geq 2$, similarly to step 1, we select the activities from the set according to non-decreasing order of the release-to-consumption ratios such that the total amount of consumed resources is up to $\Omega_{k-1}$. We start processing the selected activities at time $t_{k-1}$.

3. We stop the algorithm when all activities are scheduled.

Clearly, the algorithm runs in time $O(n \log n)$. To provide an intuition for the correctness of the claimed performance guarantee, we have the following propositions.

**Proposition 1.** *When rounding up the processing times of the activities to the nearest power-of-two numbers, we increase the makespan by at most factor of* 2.

**Proof** Let $\sigma$ be a feasible schedule of the $RP||C_{max}$ problem. We use an oriented graph $G = (V, A)$ to represent $\sigma$. Each activity $j$ is presented as a node $j$. Each node $j$ has a weight $p_j$. An arc $(i, j)$ specifies that activity $i$ completes before activity $j$ starts , i.e. $C_i \leq S_j$ in $\sigma$. Let us consider a schedule $\bar{\sigma}$ in which each activity starts at the earliest possible time equal to the length of the longest path in $G$ from a source to the node corresponding to that activity. Feasibility of $\sigma$ implies a feasibility of $\bar{\sigma}$ and the makespan of $\bar{\sigma}$ don't exceed the makespan of $\sigma$.

After rounding up the processing times of the activities to the nearest power-of-two numbers the length of the longest path in $G$ increases by at most factor of 2. □

Further we assume that the processing times of all activities are power-of-two numbers.

**Proposition 2.** *There exists a nested (laminar) schedule of the activities such that its makespan is at most four times the optimal makespan.*

**Proof.** We show that there exists a schedule $\sigma$ in which each job $j$ starts at time $lp_j$ for some integer $l$ and the makespan of $\sigma$ is at most four times the optimal makespan.

Indeed, let us stretch out the optimal schedule to twice its length. In a new schedule each activity occupies an interval twice bigger than its processing time. Such stretching does not change the order of activities. It follows that all resource constraints are satisfied in the new schedule. Suppose that activity $j$ starts at time $\tau$ and completes at time $\tau + 2p_j$. Let $l$ be a minimal integer such that $lp_j \geq \tau$. We have $lp_j \leq \tau + p_j$. We schedule job $j$ in interval $[lp_j, (l+1)p_j]$. Repeating this procedure for all activities we get the feasible schedule $\sigma$.

If the processing times of all activities are power-of-two numbers, $\sigma$ is a nested schedule. Taking into account the result of Proposition 1 we obtain the required statement. □

**Proposition 3.** *The makespan of the schedule returned by the algorithm is at most $O(\log p_{\max})$ times the makespan of an optimal nested schedule.*

**Proof.** Intuitively, the last proposition follows from the following facts. First, any nested schedule can be *stretched*, i.e., rescheduled in such a way that at any moment in time only activities with the same processing requirements are executed, and the precedence order of the activities is preserved, i.e., if one activity starts after completion of another in the nested schedule, the same order takes place in the stretched schedule. One can show that *stretching* affects the makespan by at most factor of $K$, where $K$ is the number of distinct processing times. Second, the algorithm above represents one such stretching. Third, the sufficiency of the resources is guaranteed by non-decreasing order of the release-to-consumption ratios. Therefore, the algorithm outputs a feasible schedule with makespan at most $O(\log p_{\max})$ times the optimum. □

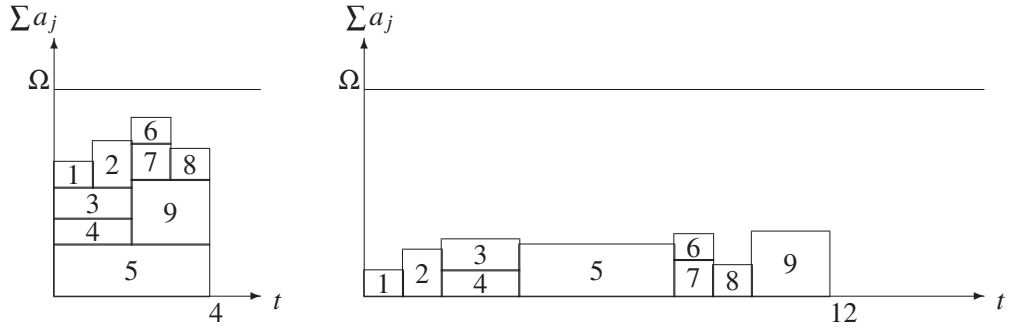


Figure 1: Stretching of the nested schedule

We illustrate the proof of Proposition 3 on figure 1. We present each activity $j$ as a rectangle. The width of the rectangle is equal to $p_j$ and the height of the rectangle is equal to $\alpha_j$. For simplicity, we suppose that $\alpha_j = \beta_j$. Figure 1 shows the transformation of the nested schedule with nine activities to the feasible stretched schedule.

Finally, using standard scaling techniques we turn the $O(\log p_{\max})$- into $O(\log n)$-approximation: at small additional cost to the makespan, we transform the processing times of the activities to numbers polynomial in $n$. Indeed, we set $p'_j = \frac{np_j}{p_{max}}$. Now we have $p_{max} = n$ and $OPT \geq n$, where $OPT$ stands for the optimum of the new instance. Next we round up processing times of all activities to the nearest integer. Similar to reasoning in the proof of Proposition 1 we can conclude that the length of any feasible

schedule increases at most $n$ time units. It follows that the makespan of an optimal schedule for the modified instance at most twice of the makespan of an optimal schedule for the original instance. Now, applying Algorithm A to the modified instance we obtain $O(\log n)$-approximation solution.

This brings us to the main theorem:

**Theorem 4.** *The relocation problem admits $O(\log p_{\max})$- and $O(\log n)$-approximation algorithms, both running in $O(n \log n)$-time.*

# 3 Conclusion

In this paper we have considered the relocation problem. We have presented the first polynomial-time $O(\log p_{max})$- and $O(\log n)$-approximation algorithms for the relocation problem with arbitrary processing times. As we mention in Introduction the relocation problem is hard to approximate within a factor better than $3/2$, unless $P = NP$, even if all activities have unit processing times. It would be interesting to design a polynomial time approximation algorithm for $RP||C_{max}$ with constant worst-case performance or provide a stronger inapproximability result.

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# Optimization of agriculture aviation in Republic of Serbia

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**Abstract**

In Republic of Serbia agriculture is a significant strategic factor, which can hire large number of people, but it is not adequately used. Radical changes in food production have to be made if Serbia is going to be concurrent in European Union. Implementation of best or at least appropriate technologies in agricultural production could create possibility to get higher returns of good quality products. This way of production would create a lower product price and Serbia could reduce food import and, at the same time increase food export. The fact that food safety is the key of success of every country cannot be disregarded. The subject of this paper is application of nonlinear programming in optimization of agriculture aviation. The problem was to determinate optimal number of airplanes by category and by carrying capacity, depending on size and type of technological operations, and all of this separately for central Serbia and for Vojvodina. Mathematical model for agriculture aviation optimization was created and problem was solved using solver integrated in MS Excel. Database was also created in MS Excel and was used for model solving. The results of this research could serve as an initial base for future optimization of technically - technological systems in agriculture aviation, where solutions could increase productivity and decrease energy supplying.

**KEYWORDS**

Agriculture aviation, model formulation, optimal number of airplanes, technically - technological systems.

## 1. INTRODUCTION

Agriculture has long been the mainstay of Serbia's economy. Although less than one-fourth of economically active Serbs are now employed in farming (compared with nearly three-fourths in 1948), cropland occupies nearly two-thirds of Serbia's territory. The principal area of commercial agriculture is the Vojvodina region and lowlands south of the Sava and Danube rivers, including the valley of the north-flowing Morava River. Three-fourths of sown crops in Serbia are grains. Corn predominates, occupying around one-third of the cropland, and wheat is next in importance. Other noteworthy crops are sugar beets, sunflowers, potatoes, oilseeds, hemp (Encyclopedia Britannica Online, 2011).

Despite the obvious needs, agriculture aviation in Serbia isn't appropriately developed. Necessity for research and analysis of technically - technological systems in agriculture aviation stems from the economic importance of its further development. At this moment (Serbia is in transition), aviation isn't appropriately used (average flight of airplane is 36 hours in the last five years) and has no clear direction of development (Jakovljevic, 2006). It is therefore necessary to perform a systematic analysis of parameters that will further influence on the choice including the number, types of aircraft and the extent of their application.

Agriculture aviation is used for nutrition and protection of agricultural crops. Future trend in modern agriculture is application of protection products with lower volumes of fluid use. If the development is followed with improvement of application methods of protection products, significant results will be obtained during the exploitation of agriculture aviation. These results will be primary obtained in production and labor.

Climatic conditions and soil management are also important and they demand usage of aircraft in agriculture. Exploitation of aircraft for fertilizers distribution has been reduced mainly to the nutrition of winter crops, and for a long time there are no major improvements. The protection is lowest in years that are dry, like the year 2000th, when the protection was conducted on 36,807 ha.

Application of mathematical models and methods for improvement of the use of aircraft in agriculture of Serbia will be presented in this paper. We will briefly describe the formulated problem, and then we'll show formulated mathematical model for optimization of agricultural aviation. The model is nonlinear and for solving we used software module - solver, which is integrated into MS Excel. Later in this paper, we will present the optimal solution and in the end, we will provide guidelines for further research.

## 2. PROBLEM DESCRIPTION

Organized use of aircraft in agriculture of Serbia began in the eighties (Bajovic, 1982). The total number of aircraft was 100 and these aircrafts were located in ten different units fully independent and insufficiently connected in terms of procurement and maintenance. Agriculture aviation was using more than ten types of aircrafts and utilization rate was approximately 200 hours per year (Stanojevic, 1987).

During the nineties the countries in the region remained without agricultural aviation, whilst Serbia kept aircrafts. These aircrafts now have little involvement in agriculture, and the number of flying hours is reduced to a minimum. This certainly reflects to the technology of growing culture and impossibility of compliance with the terms of terrestrial soil management techniques, and all of these factors led to yield losses. Serbia now has nine organizational units of the aviation and five types of aircraft.

The current production and economic conditions in agriculture demand analysis of use of agriculture aviation, possible reconstructions, starting from the organization of agricultural aviation, new legislation of ownership of agricultural land, i.e. size of property that becomes property of private persons and provides opportunities for the application of aviation.

According to WHO analysis, level of underground waters in Vojvodina increases and will increase over the next 17 years. Use of aircraft will be necessary in the production process of cultures, particularly in nutrition and protection. Underground water represents great problem and limiting factor for successful fulfillment of agro technical deadlines. These deadlines are especially important in corn production, since corn is cultivated on 62.72% sawn area. Use of airplanes in the technology of growing these crops would give better results in terms of yield and quality.

During the nutrition of corps weather conditions are variable, so the deadlines could not be fulfilled without the active participation of agricultural aviation, especially in the Vojvodina (because of high underground water), and in the Morava River basin. In the production of grain a lack of agricultural machinery in volume and structure is observed, and this lack leads to decrease in yield. Therefore, agricultural aviation has to take its place in the production of grains, because proper application can lead to a significant increase in yields in these crops. Yields decreased in Serbia; in the nineties they were 7 t/ha for grain, and now 3.7 t/ha in central Serbia and in Vojvodina is 4.1 t/ha. The reasons for this decrease in yield were mainly lies mainly in non-compliance with agricultural deadlines.

For purposes of this paper, research has included assessment of the state of agricultural aviation in Serbia, data collection and analysis of agricultural aviation organizations, the number of available aircraft, aircraft carrying capacity, lifecycle and schedule of use of agricultural aircraft.

We had to perform analysis and optimization of agricultural aviation separately for central Serbia and for Vojvodina. The reasons are more than obvious: in Vojvodina the consolidation was carried out (merging of parcels), and the therefore parcels are larger than in central Serbia. Another reason is the terrain - in central Serbia, the terrain is hilly and mountainous and the Vojvodina is lowland.

After studying the situation in agriculture and agriculture aviation in Serbia, we gathered all the available data, which included number and types of aircraft available and used in Serbia. We also collected data on technical and technological operations that are performed for the cultures and, of course, the data on the cultures itself. We considered two types of operations that are realized with agricultural aviation: nutrition and protection. Protection has to be done for 12 different diseases and pests, and it is necessary to protect all considered culture. Nutrition is done for two cultures: barley and wheat. The analysis was conducted for the crops that are grown mostly in Serbia: barley, potatoes, corn, wheat, soybean, sunflower and sugar beet, and the fruit and grape growing.

We observed a one year period - from February to late October. Fertilization operations and plant protection should be repeated several times depending on the type of the culture.

## 3. MODEL FORMULATION

The research subject in this paper is to determine the number of aircraft by category and carrying capacity, depending on the amount and type of agro-technical operations, with minimal total power of the aircrafts.

Based on the current planting structure of Serbia, i.e. the volume and type of work, we need to choose an optimal number of aircraft and these aircrafts have to carry out technological operations of nutrition and protection. Solving this problem is possible through the optimization of agricultural aviation fleet, where the objective function is taken as the total power of the aircrafts.

As an agricultural aircraft optimization criterion we choose the total power of the aircrafts by category and by region in kwh. Agricultural aircrafts are divided into categories where we have three categories: light (capacity up to 500kg), medium (capacity up to 900kg) and heavy (capacity up to 1500kg) aircrafts.

To optimize the technically - technological systems in agriculture aviation, we used the database formed by collecting the data directly during the research, which lasted for 5 years.

During consultations with experts in agricultural aviation in Serbia, we concluded that the most important limiting factors are the total area in hectares in certain regions. In these areas it is necessary to perform certain technological operations (nutrition or protection) in a given period.

Formulated model for agriculture aviation optimization:

$$(\min) f(x) = \sum_{i=1}^{n} a_i \sum_{j=1}^{m} (\max_{p} \{x_{ijp}\}) \tag{1}$$

*po*

$$\sum_{i=1}^{n} \sum_{k=1}^{s} a_{ikl} b_{il} r_{kl} x_{ijp} \geq a_{jlp} \ , \quad j = 1,...,m, l = 1,...,v, p = 1,...,t \tag{2}$$

$$x_{ijp} \geq 0, \ \text{integer} \ , \ i = 1,...,n, j = 1,...,m, p = 1,...,t \tag{3}$$

where:

$x_{ijp}$ - represents total number of *i*-th category airplanes in *j*-th region in *p*-th period,

$a_i$ - power of *i*-th category airplane in kwh,

$r_{kl}$ - time limit for realization of *l*-th type of operation for *k*-th culture (in days),

$b_{il}$ - effective working hours per day of *i*-th category airplane for *l*-th type of operation (without refueling),

$a_{ikl}$ - *i*-th category airplane performance for *k*-th culture for *l*-th type of operation (ha/h),

$a_{jlp}$ - total area in ha in *j*-th region where we need to execute *l*-th type of operation in *p*-th period,

Objective function (1) represents total power of aircrafts which must be minimized. Constraints (2) represent the minimum total area in hectares of arable land on which it is necessary to make certain technical operation. All variables are integers (3).

Using nonlinear programming we will determine optimal number of aircrafts for each category, so the planned work (type and deadlines) can be successfully realized. It is clear that the agricultural work may employ more than one category of aircraft, but we need to select the most rational structure of the aircraft, to ensure successful execution of agricultural activities on schedule.

# 4. DATABASE AND PROBLEM SOLUTION

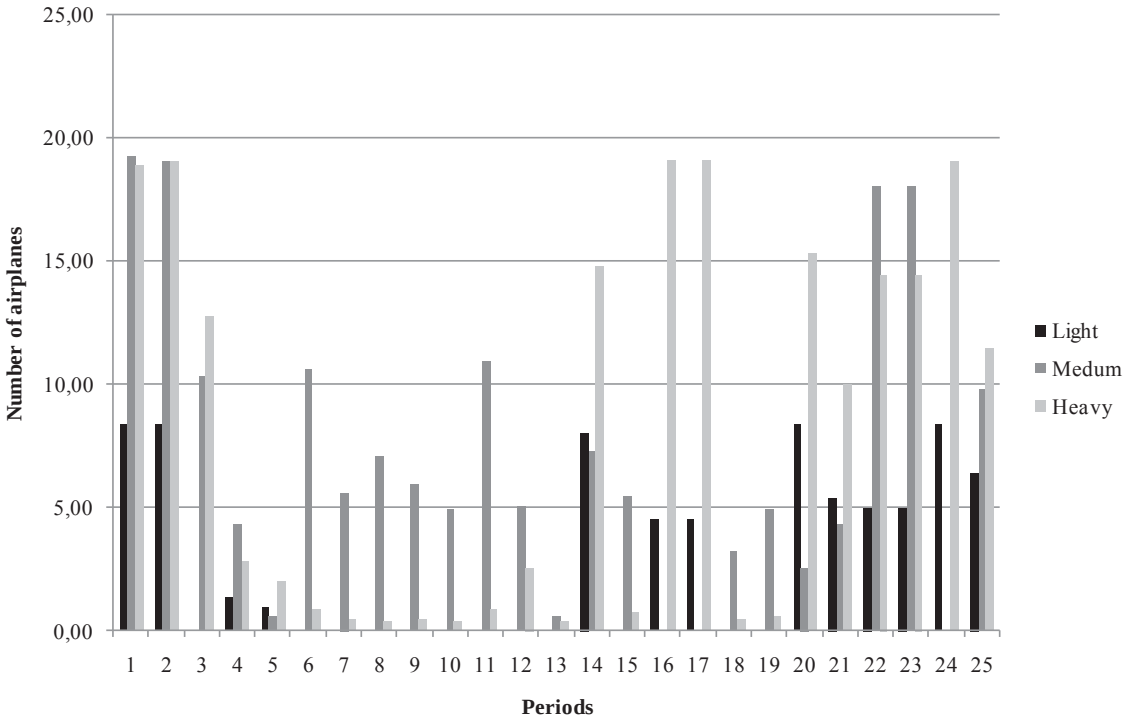
All data were collected directly from experts and arranged in MS Excel. Table 1 represents a part of a database for central Serbia. The data were sorted by period, by type of culture and technology operations.

Table 1: Part of a database for central Serbia in MS Excel

| Periods | Culture | Operation | Aircraft performance (ha/h) | | | Effective number of hours (h/day) | | | Period days) | Area (ha) |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | light | medium | heavy | light | medium | heavy | | |
| 21.2-28.2 | Grain | first nutrition | 36 | 81 | 99 | 4 | 3.4 | 3.4 | 7 | 110.000 |
| 21.2-28.2 | Grain | protection | 50 | 90 | 106 | 3.8 | 3.2 | 3.2 | 5 | 3.600 |
| 21.2-28.2 | Barley | protection | 50 | 90 | 106 | 3.8 | 3.2 | 3.2 | 7 | 3.600 |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |

Model for optimization of agricultural aircraft has a total of 75 variables and 50 constraints. The problem is solved on a computer Pentium - 4 CPU 1500 MHZ, 1.5 GHZ with 256 MB of RAM. Solutions were obtained for central Serbia and Vojvodina. In figure 1 we can see the required number of aircraft by category for each period for central Serbia.

Figure 1: Optimal number of aircraft by category and by periods for central Serbia



We can see that in central Serbia we should mostly use middle aircrafts. That is because terrain is hilly and mountainous, so the aircraft need to make many turns during the flight to do the planned operation of protection or nutrition, and that is easier with smaller aircrafts.

Optimal number of aircrafts in central Serbia is 46. Annual utilization indicators are shown in Table 2.

Table 2: Annual utilization indicators for central Serbia

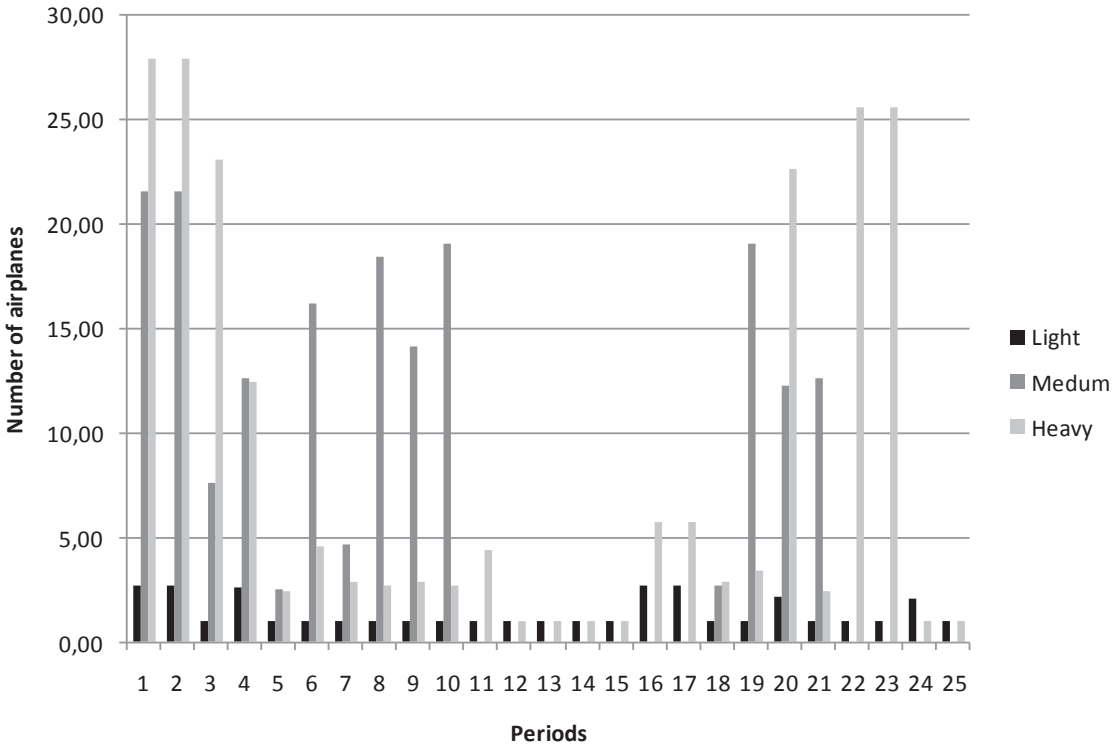
| Aircraft category | Number of aircrafts | Total working hours [h] | Working hours per plane [h] | Total working hours [%] |
|---|---|---|---|---|
| Light | 8 | 1.082,85 | 135,35 | 13,20 |
| Medium | 19 | 4.275,28 | 225,01 | 52,11 |
| Heavy | 19 | 2.846,58 | 149,82 | 34,69 |
| Total | 46 | 8.204,71 | | 100,00 |

Agricultural aircraft in Serbia should be processing 613,430 ha. The total number of hours needed for processing this area is 8,204.74. 52.10% of the total number of aircraft hours is engaged in medium category aircraft.

All three categories of aircraft are engaged in cultivating (light 6,239 ha, 12,863 ha of medium and heavy 12,718 ha).

In figure 2 we can see the required number of aircraft by category for each period for Vojvodina.

Figure 2: Optimal number of aircraft by category and by periods for Vojvodina




We can see that in Vojvodina we should mostly use large aircrafts. That is logical, because terrain is flat and aircraft can cultivate very large piece of land with only one take off. The largest number of airplanes is needed at the beginning and at the end of observed period.

Optimal number of aircrafts in Vojvodina is 51. Annual utilization indicators are shown in Table 3.

Table 3: Annual utilization indicators for Vojvodina

| Aircraft category | Number of aircrafts | Total working hours [h] | Working hours per plane [h] | Total working hours [%] |
|---|---|---|---|---|
| Light | 3 | 1.006,05 | 335,35 | 9,29 |
| Medium | 21 | 4.796,71 | 228,41 | 44,28 |
| Heavy | 27 | 5.031,00 | 186,33 | 46,43 |
| Total | 51 | 10.833,76 | | 100,00 |

Agricultural aircraft in Vojvodina should be processing 950,020 ha. The total number of hours needed for processing this area is 10833.76. 44.27% of the total number of aircraft hours is engaged in medium category aircraft.

All three categories of aircraft are engaged in cultivating (light 10.614 ha, 18.836 ha of medium and heavy 17.612 ha).

## 5. CONCLUSION

The results of this research should serve as initial base for optimization of technically - technological systems in agriculture aviation, which will give higher productivity and lower energy supplying for all the foreseen works.

The obtained results (Table 4) show that we increased the present employment of airplane from 53 hour to 178 hours in Central Serbia, and from 33 hours to 212 hours in Vojvodina. At the same time, the energy supplying would decrease from 0,1737 kW/ha to 0,035 kW/ha in Central Serbia, and from 0,2948 kW/ha to 0,0280 kW/ha in Vojvodina.

Table 4: Number of aircrafts and their usage before and after optimization

| Region | Current | | | | Optimization results | | | |
|---|---|---|---|---|---|---|---|---|
| | Number of aircrafts | Number of flight hours per year per aircraft | Number treated hectares per aircraft | Energy supplying [kW/ha] | Number of aircrafts | Number of flight hours per year per aircraft | Number treated hectares per aircraft | Energy supplying [kW/ha] |
| Central Serbia | 4 | 53 | 4.073 | 0,1737 | 46 | 178 | 13.335 | 0,035 |
| Vojvodina | 40 | 33 | 2.184 | 0,2948 | 51 | 212 | 18.627 | 0,028 |

After presenting our results to the experts in this field, we got very positive reviews. It is obvious that with low cost we can more efficiently use our agricultural aviation fleet.

We are currently developing an extended model for optimizing of agricultural aviation – by adding constraints for the number of acres per year which can one particular category of aircraft process. We will introduce the restrictions on the available number of aircraft – possibly we will take into account not only the existing fleet, but the possibility of renting aircraft from neighboring countries.

In the near future we will continue to work on the problems of use of aircrafts in agriculture in Serbia, and most likely we will use the already collected data to determine the location of agricultural airports in Serbia.

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# Design and shape optimization of a Turkish Tirkeş bow

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**Abstract**

The Turkish composite bow is a recurved bow. This shape provides more power by its short length and gives advantage for horseman at war. Traditional bows made by using organic materials such as horn, sinew, wood and fish glue by sandwiching techniques takes very long time to produce. They are also suffering from tip twist when the bow set into draw position. The present work aims optimizing a replica of Turkish Tirkeş bow. Thus using state of the art composite materials bow can be produced with short production times. Using composite allows all the control of the fiber directions, thus offering the chance of reducing tip twisting.

There exist no technical draw for traditional Turkish bow and also description of Turkish bows in the literature is indefinite. The initial design is therefore necessarily based on the examination of existing bows in museums collections. The characteristic recurved profile, the length and the cross section of a typical Turkish bow are adapted as design criteria. Based on the developed data of typical bow is done in a CAD environment. The effects of these variables on output parameter are determined Force vs. draw distance behavior is identified with the expected consequence of the ability to store more energy with identical draw forces and lengths involved in bows of different nature and designs. The inherent recurved shape of Turkish Tirkeş profile allows the ability to draw the bow less effort. The difference between the stored energy in the bow and the kinetic energy of the propelling arrow is clear indication of higher efficiency of the Turkish bow.

**KEYWORDS**

Turkish bow, archery, design, shape optimization

## 1. Introduction

Archery plays an important role in the history of mankind. The earliest people known to have used the bow and arrow are the ancient Egyptians, who adopted it at least 5000 years ago for hunting and warfare. In the course of time, a steady improvement of shape of the bow and material took place. For the last few centuries, archery has had no military meaning and, in modern times, has turned out to be a recreational Olympic sport (http://www.archery.org).

The bow was simple in concept, yet it represented an extremely sophisticated technology. It was almost certainly the earliest mechanical device to achieve greater speed in a projectile than could be attained by throwing it. In its most basic form, the bow consisted of a stave of wood slightly bent by the tension of a bowstring connecting its two ends. The bow stored the force of the archer's draw as potential energy, and then transferred it to the bowstring as kinetic energy, imparting velocity and killing power to the arrow. The bow could store no more energy than the archer was capable of producing in a single movement of the muscles of his back and arms, but it released the stored energy at a higher velocity, thus overcoming the arm's inherent limitations.

Traditional bows made by using organic materials such as horn, sinew, wood and fish glue by sandwiching techniques takes very long time to produce. They are also suffering from tip twist when the bow set into draw position. The present work aims optimizing a replica of Turkish Tirkeş. Thus using state of the art composite materials bow can be produced with short production times. Using composite allows all the control of the fiber directions, thus offering the chance of reducing tip twisting.

There exists no technical draw for traditional Turkish bows. The current design is therefore necessarily based on the examination of existing bows in museums and adapts reverse engineering skills. However the final design is optimized with parameters of Turkish bow.

The characteristic recurved profile, the length and the cross section of a typical Turkish bow are adapted as design criteria and optimization carried out. Based on the developed data of typical bow is done in a CAD environment. Force & draw distance behavior is identified with the expected consequence of the ability to store more energy with identical draw forces and lengths involved in bows of different nature and designs. The inherent recurved shape of Turkish Tirkeş profile allows the ability to draw the bow less effort. The difference between the stored energy in the bow and the kinetic energy of the propelling arrow is clear indication of higher efficiency of the Turkish bow.

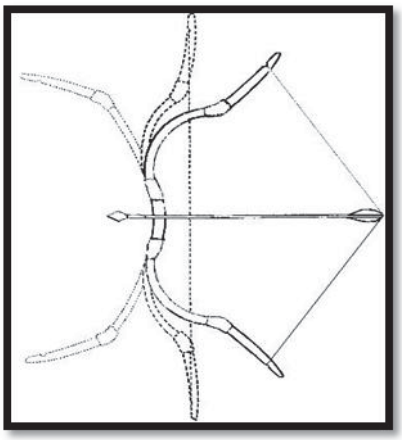

**Fig. 1** Three profile of Turkish composite bow. (Unstrung, braced, strung)

## 2. Design

Each section of a Turkish bow can be named separately. We can begin with the symmetric point, middle of the bow where is named as kabza (qabza). The straight bending section after the qabza is sal (limb) section. Rigid, bent section towards to the tip is kasan (kassan) and tip sections are named as baş (ear). The transition point between limb and grip is kabza boğazı (grip throat). This point is used to side the arrow to the bow when shooting. The three profile of the Turkish composite bow is showed in Fig. 1 (Unstrung, braced, drawn).

## 2.1 Profile

There are numbers of beautiful examples of bow in military museum and Topkapi museum in Istanbul. These bows have very wide variety from 80 lb to 240 lb draw weight. Since above 120 lb bow is very high to drawn, these bow weight is out of interest. However the draw weight of a bow is not affect profile of the bow. A bow can be strengthening with only increasing their thickness, width and material. For these reasons, all bow profile is valid to consider a new bow profile.

The qualities to reach a best bow are;

- Arrow speed
- Accuracy
- Comfort of draw and release
- Durability
- Suitability for its use environment
- Beauty
- Ease of construction
- Ease of maintenance
- Cost, or availability of materials
- Unobtrusiveness

In the beginning this all qualities can be seen overwhelming. Since higher arrow speed has benefits i.e. 1) flatter path, 2) more penetrating power etc. the most important quality for a bow is initial arrow speed. All bows are made to launch fastest arrow possible. Everything being equal, higher draw weight stores more energy and arrow can be shooting with higher speed. However, in this study our aim is not make a too-heavy bow that can propel an arrow faster. We want to make a bow that can shoots faster per kg of draw weight.

What is then the best design for a composite bow? Here we can summarize the design characteristics of a fast bow, made to shoot low mass arrow;

1) Make it short as possible and the limbs, including the bending section as thick and as narrow possible.
2) Can be drawn as far as possible.
3) Reflex the limbs as much as possible (effect on draw weight).

Profile is the most influent feature that affects the bow draw weight after material properties. Turkish bow has a different profile than other types of bows. This effect can be shown on an example. The shown figure below illustrates two types of bow which are English long bow and Turkish composite bow. At the equal draw length (50 lb) long bow shoots an arrow with a speed 150 fps and Turkish bow shoots 175 fps. Despite both take equal weight to keep at full draw, the only differences between its energy storage at braced position. English long bow at braced position bow cannot shoot an arrow but Turkish bow can. This condition causes to shift upward Turkish bows' draw weight-draw length curve. The area between the curve and the X-axis represents the energy stored in the bow.

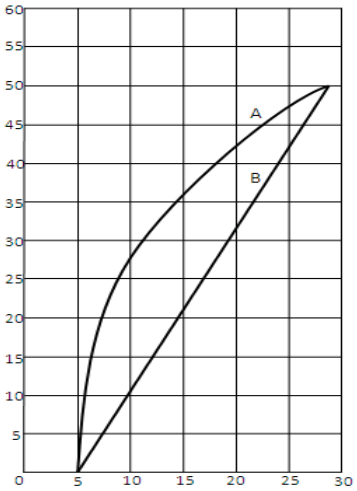


**Fig. 2** Turkish composite bow and English long bow.

The necessity of the reflex section is proved above. However, some improvements are also needed in the reflex profile. There are many types of reflex profile. Some of them make a bow very strong and also some of them are not efficient. The schematic illustrations of reflex bow types are shown below.

In the design A, the reflex section and the limbs are very long and this causes to use more material. Since the working part of the bow is limb section, more material usage in limb means the bow efficiency and limb speed is low. Now to abbreviate this section, some part can be straighten, as shown B but the kink at the transition may cause failure and if this section made straight until ears of the bow will be lighter and shorter. The resulting design C, however it is not enough bent and less reflex if compared with design A and B. To achieve this bow as strong as with others, bow is more reflex as in design D.

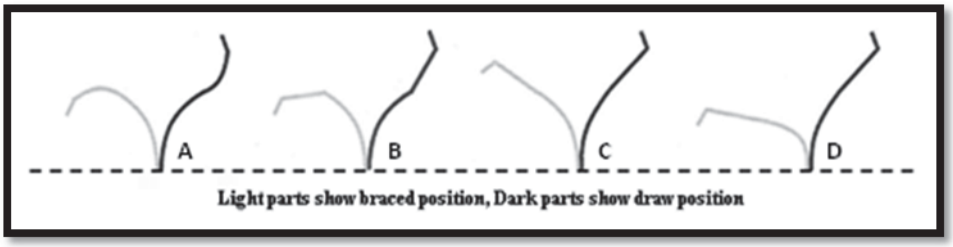


**Fig. 3** The schematic drawing of reflex bow (Adam karpowicz).

When the reflex section is shifted towards to the ears, the draw weight of the bow will decreased. Design C seems to be very close of this design. However, since the length of the bow is same for all design the horizontal length of bow will be very long compared to the other. With this consideration it is concluded that the bow is reflex only transition section between kassan and ear. In this type of bow the bending section of limbs have no reflex, the transitional section is well reflex, and additionally the tips have a kink.

The length of the bow is desired to be less as possible, but there is a limitation. There is a relationship between the length of the bow and the length of the draw weight or otherwise the arrow length. Traditional examples show that a 70 cm arrow is used for 106 – 112 cm bow. This length is measured from string nock to other string nock. In our design bow has no reflex and then their draw weight is low. To increase draw weight at this design bow length is extended 4 cm for both side. Thus bow store more energy when drawing. Also this change allows using 76 cm arrow in this bow. 126 cm length is concluded with the tip extension which has negligible effect. Up to this point bow profile is a replica of traditional examples.

In traditional bows the cross sections of the bow through bending section represent half circle or half ellipse shape with the sandwiched horn, sinew and wood. However, if we look from engineering side, this cross section of limb is not optimal form. Since organic materials has less strength respect to modern material, to reach higher draw weight, traditional examples has to be very thick. To decrease bow mass the edge side of bow is trimmed. However, we have high strength material. The most efficient shape of cross section in this working part is rectangular. It is known from strength of material that if limb section supposed to be a cantilever beam the force need to bend beam is increase with the cube of thickness but in proportion to the width. Because of this if we want to increase draw weight we only increase thickness.

The rigid section of composite is usually made with a ridge along symmetric center of bow. This ridge allows to be stiffening this part. In this section the triangular cross section maximize the width of stiff part for better stability, and also allow minimizing mass while still maintains same resistance against to twist and bending force when the bow is drawn.

## 3. Optimization

Since the shape of the bow has symmetry both x and y direction, optimization is carried out on ¼ part of bow. The fixed point of the bow is qabza section. The force acts on ear section. The desired draw weight at the end of the study is about 33 kg or higher (75 lb). The optimization variables are grouped into three categories for profile, length and width. The parameters that given to the program are kassan outer radius, kassan horizontal length, grip throat width, thickness of bow, qabza width, limb length and ear length. The lower and upper bound at parameters are determined by program. The output parameters for the goal driven optimization are deformation-X and equivalent stress.

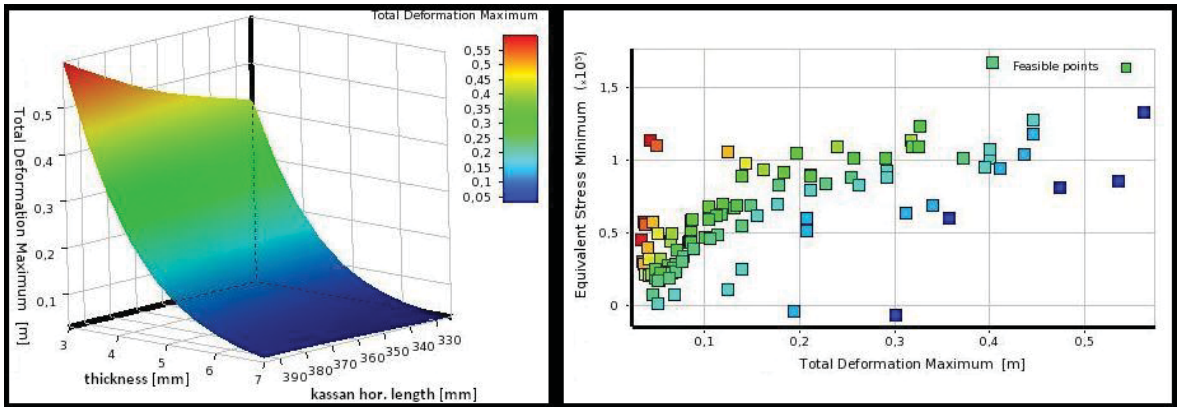


**Fig. 4** The response surface and sensitivity analysis of bow.

When the program solves the entire experiments, the local sensitivity graph shows that the most important parameter that affects the bow draw length is thickness as predicted. The limb length and kassan horizontal length are other most influential parameters. However, there is approximately 5 times difference between them. This means that we can change draw weight only controlling thickness. At this point other parameters are optimized one more time to achieve minimum mass.

The next point after mass optimization is simulation study. At each practice bow is drawn about 60 – 70 times and the weakest point (grip throat) generally will fail. This point has to be strengthening but since arrow is held at this point when shooting, this part cannot be expanded. The elliptic shape of qabza can be extended up to this point. The table below shows optimization's best candidate which is very close to the produced bow in the previous study.

| | Kassan R. | Kassan L. | Grip W. | Thickness | Qabza W. | Limb L. | Ear L. |
|---|---|---|---|---|---|---|---|
| Best Candidate (mm) | 101.09 | 115.35 | 10.03 | 3.16 | 12.58 | 283.27 | 109.14 |

## 3.1 Stability

It can be faced to limb twist in all composite bows. The reflex shape of bow and the great force acting on it also misalignment in fiber direction in production causes sideway flexing of limbs. This twisting usually occurs middle and end of the drawing. Twisting can be decreased but not eliminated. As can be seen on simulation study below twisting begins to occur on limb section (Actually grip throat). The wider limbs can solve this problem but as mentioned before wider limb decrease bow efficiency. Because of this twisting can be tolerated by fiber addition with different angle to the acceptable level.

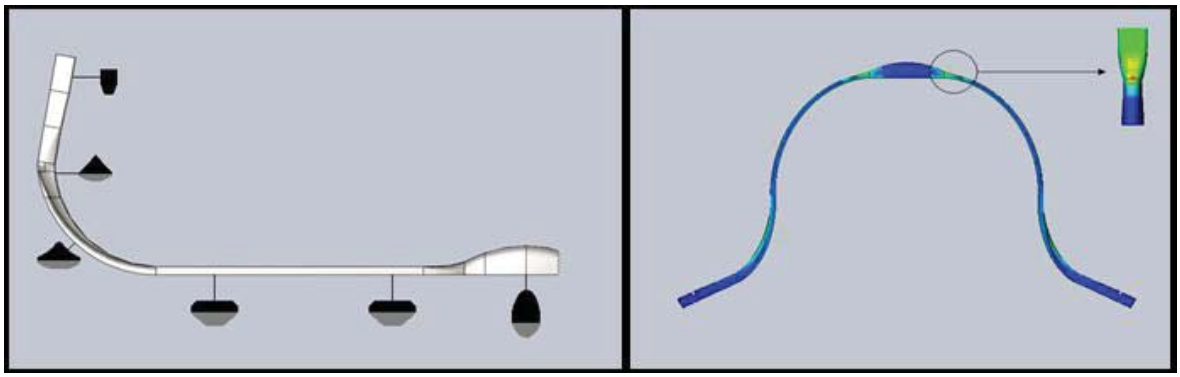

**Fig. 5** The cross section of designed bow and simulation result. (Lighter part added later)

## 4. The recurvature effect on the limb section

In this study, as can be seen on above figure, our design has not recurved profile. However, the best feature of the Turkish composite bow is their recurved profile. In the braced position since the ears are bended reverse direction to the unstrung position, bow has an initial string force in vertical position. At this situation, there is no horizontal force to propel the arrow. Here it seems to be there is no advantage, but when the archer starts to draw the string the vertical force which has no effect on shooting at first begins to make an angle lower than 90 degree between horizontal direction and string. From the literature it is known that early in the draw, such bows are very hard to pull. During mid-draw, when draw weight would normally become uncomfortably high, strings lift off their contact points, letting their retroverted tips works as levers, keeping final draw weight at tolerable levels. In effect, such tips work as cams.

The effect of the recurved profile on the limb section is observed by a simulation study. Between the limb section start and end point, bow is flexed at various degrees to $90^{\circ}$. 333 N (75 lb) of total forces are acted on both nocks. When the program solves the simulation study the total strain energy is calculated. Thus the affect of the recurvature is observed.

The total strain energy calculation of the bow shows us the flex angle and the strain energy has a linear relationship. The total strain energy difference between no flex and recurved bow is %67,6. Since we do not want high draw weight, recurved profile is not added. This high difference appears due to the great strain involved in bracing such severely reflexed limbs.

## 5. Concluding remarks and recommendation

The design of the bow firstly depends on museum collections. The other types of bow can be designed by changing its main characteristic and dimensions. However the thickness of the bow and some part like qabza and ears and maybe kassan section are very same for all types of Turkish bow. Bow made with modern material has very low thickness related to traditional examples. In this work the designed bow has a thickness only 3 mm and 75 lb draw weight at 70 cm draw distance. To achieve higher thickness to increase comfort grabbing low strength material can be added back side of the bow. Otherwise the more thickness means the more draw weight bow has.

The bow with reflex shape store more energy than other type of bow which has same draw weight but straight shape. If the bow had an extreme recurvature profile, draw weight will be doubled and this makes it useless. In recurved profile, bow follows more distance when setting to braced position than non-recurved profile. This more bending increase bow twisting force. The low thickness cannot resist this force. The simulation studies show that the maximum stress is occurred on grip throat. This section has minimum width and bow begin to twist from there. Fibers are laid on whole bow in only lateral direction but at this section the angled fiber prevent failure and twisting.

The optimization result shows us thickness increasing is enough for higher draw weight. Other parameter affect on draw weight is very low with respect to thickness. Since one output parameters is deformation maximum, our designed bow is long (131 cm). For short bow limb length can be decreased. However these operation increase bows draw weight and also stress at grip throat.

Finally bow design is whole study that covers many disciplines. In the previous study we see that bow efficiency is changing dramatically by arrow weight. From literature we know that arrow speed is affected by many number of parameter; archer, string, humidity, wind, arrow type etc. This study is a first step to future works about bow design and other types of Turkish bow.

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# Iterative local search methods for the talent scheduling problem

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**Abstract**

We study the talent scheduling problem which can be reformulated as the column permutation problem for a binary matrix. The problem is NP-hard in the strong sense. Modified greedy algorithm and iterative local search metaheuristics are developed for this problem. Computational experiments show that we can find optimal or near optimal solutions easily by these methods if number of columns and number of rows of the matrix are at most 100.

**Keywords** Local search, greedy algorithms, metaheuristics.

## 1 Introduction

In the talent scheduling problem we assume that a film production project is divided into $n$ independent shooting days (scenes). There are $m$ talents (actors) involved at different scenes of the project. An $(m \times n)$ binary matrix $A$ indicates the scenes for each talent. The element $a_{ij}$ of the matrix is set to 1 if talent $i$ is needed for scene $j$ and it is 0 otherwise. The per-diem payment for talent $i$ is a positive value $w_i$. Each talent starts to job in his first shooting day and finishes in his last shooting day. We wish to find a permutation of scenes to minimize the total payment for the talents.

If there is a permutation without delays of the talents (free days) then the total payment is $\sum_{i=n}^{n} w_i \sum_{j=1}^{m} a_{ij}$ and the problem is polynomially solvable [1, 2]. If it is not the case, the problem is NP-hard. The branch and bound method for the problem is presented in [3]. It allows us to solve the problem for small dimensions only, $n = m \leq 15$. Three integer linear programming formulations of the problem are studied in [4]. They can be used for solving the problem by commercial software. Unfortunately, the integrality gap is large for these formulations and solver CPLEX 11.0 can not find global optimum for $n = m = 30$ during 24 hours on Pentium 2.8 GHz. The dynamic programming method for the problem is presented in [5]. We can solve the problem for small dimensions by the method again, $n = m \leq 30$. The first metaheuristic approach, a genetic algorithm, in presented in [6]. It is not strong and tested for $m \leq 40, n \leq 70$ only.

In this paper we develop three metaheuristic algorithms: a hybrid simulated annealing, stochastic tabu search and genetic local search. We show that these algorithms can discover the optimal solutions for random generated instances with $n = m \leq 100$ quickly. They use a new constructive heuristic to create good starting solutions and three neighborhoods *Insert, Rotate*, and *k-Opt* for improving a current solution or population of solutions. These metaheuristics incorporate idea of Variable Neighborhood Search method [7] to change neighborhoods during the iterative process.

The paper is organized as follows. Section 2 introduces the notations and the mathematical model. Section 3 describes the constructive greedy heuristics. Three neighborhoods for local search are presented in Section 4. The stochastic tabu search, hybrid simulated annealing, and genetic local search are described in Sections 5, 6 and 7 correspondingly. Computational results are discussed in Section 8. In final Section 9, we give some conclusions and directions for future research.

## 2 Mathematical model

Let us introduce the following variables:
$e_i$ is the earliest shooting day for talent $i$;
$l_i$ is the latest shooting day for talent $i$;
$x_{jt}$ is the scheduling for the project, i.e.

$$x_{jt} = \begin{cases} 1 & \text{if scene } j \text{ is scheduled in day } t \text{ of shooting} \\ 0, & \text{otherwise.} \end{cases}$$

We wish to minimize the total payment $F(e,l,x)$ defined by

$$F(e,l,x) = \sum_{i=1}^{m} w_i(l_i - e_i + 1)$$

subject to

$$\sum_{j=1}^{n} x_{jt} = \sum_{t=1}^{n} x_{jt} = 1, \qquad 1 \le j,t \le n;$$

$$a_{ij}e_i \le \sum_{t=1}^{n} a_{ij}t x_{jt} \le l_i, \qquad 1 \le i \le m, \;\; 1 \le j \le n;$$

$$x_{jt} \in \{0,1\}, \qquad 1 \le j,t \le n;$$

$$l_i, e_i \ge 1, \text{ integer}, \qquad 1 \le i \le m.$$

The objective function is the total payment for the talents. The first restriction requires one day for each scene and one scene for each day. The second restriction defines the earliest and latest days for each talent. Other restrictions are integrality constraints. It is easy to see that the last restriction can be omitted. The optimal solution is not changed in this case. Thus we have got the mixed integer linear program with $n^2$ binary variables $x_{jt}$ and nonnegative $2m$ continuous variables $e_i$ and $l_i$. Unfortunately, this formulation has large integrality gap as the previous pure integer formulations [4] and we can solve the problem by CPLEX software for small dimensions only. For large scale instances we develop metaheuristics which can discover optimal or near optimal solutions.

## 3 Fast constructive algorithms

As we have mentioned above, the problem is polynomially solvable if there is a scheduling without free days of the talents. For testing this property, a PQ–tree data structure is applied [2]. It is polynomial and sophisticated algorithm. It carries out at most $m$ iterations. In each iteration we consider a subset of talents, and the current permutation of scenes is modified in order to remove free days for an additional talent. If it is not possible, the algorithm terminates with answer ”No”. If we do that for all talents, we obtain the desired permutation and the algorithm terminates with answer ”Yes”. Below we present two greedy algorithms for the general case. The first naive algorithm carries out $n$ iterations. It starts by selecting a scene for the first day. In each iteration a subset of scenes is considered and we select an additional scene for this subset to minimized an additional payment for the talents.

There are at least two weak points in this framework. It is not clear how to select the first scene. But we can test all scenes and return the best found solution. In each iteration, we consider a subset of scene and calculate the objective function for this subset only. It may be not correct. For some talents we must pay for free days because they will be needed later. Now we present a modified greedy algorithm where

in each iteration we check whether a talent will be needed for unscheduled scenes or not. An auxiliary set $J$ is used for the unscheduled scenes and permutation $\sigma$ is a result of the algorithm.

**Modified Greedy Algorithm**

1. Put $J \leftarrow \{1,\dots,n\}$, $I \leftarrow \emptyset$.
2. Select a scene $j$ for the first day, put $\sigma(1) \leftarrow j$, $J \leftarrow J \setminus \{j\}$.
3. For $i$:=1 to $m$ do if $a_{ij} = 1$ then $I \leftarrow I \cup \{i\}$. { Identify the involved talents }
4. For each day $t := 2$ to $n$ do
   4.1. For each $i \in I$, if $\sum_{j \in J} a_{ij} = 0$ then $I \leftarrow I \setminus \{i\}$. { Remove unused talents }
   4.2. Find a scene $j_0$ with minimal additional payment

$$\Delta_{j_0} = \min_{j \in J} \Big\{ \sum_{i \in I} w_i (1 - a_{ij}) + \sum_{i \notin I} w_i a_{ij} \Big\}$$

   4.3. Put $\sigma(t) \leftarrow j_0$; $J \leftarrow J \setminus \{j_0\}$.
   4.4. For each $i \in I$, if $a_{ij_0} = 1$ then $I \leftarrow I \cup \{i\}$. { Involve new talents}
5. Return the permutation $\sigma$.

Figure 1 shows computational results for the greedy algorithms. Each point at the curves corresponds to the average deviation from the optimum for 30 randomly generated instances, $n = m = 100$. For each instance we apply the algorithms 100 times with each scene for the first day and return the best found solution. As we can see, the algorithms produce near optimal solutions for instances with high density, $d = \sum_{ij} a_{ij}/mn$. For low density $d = 0.05$ the average deviation is large, it is about 50% for the naive greedy algorithm and about 10% for the modified greedy algorithm. The instances with low density are the most difficult. Thus, the remain part of the paper is devoted to metaheuristics for this difficult case.

Figure 1: Average deviations of the greedy algorithms

# 4 Neighborhoods

For a permutation $\sigma$ we define the following neighborhoods.

The *Insert* neighborhood is the set of all permutations which can be obtained from $\sigma$ by insertion an element (scene) to a new position (day).

The *Rotate* neighborhood is the set of all permutations which can be obtained from $\sigma$ by choosing two elements and using inverts order for these elements and all elements between them.

The *k-Opt* neighborhood is the set of all permutations which differ from $\sigma$ in at most $k$ positions.

It is easy to verify that the neighborhoods have the following important property. For two arbitrary permutations $\sigma'$ and $\sigma''$ there is a finite number $q$ and permutations $\sigma^l, l = 0,\dots,q$ such that $\sigma^0 = \sigma'$, $\sigma^q = \sigma''$ and each permutation $\sigma^l$ is neighboring for $\sigma^{l-1}$. In other words, using each neighborhood we can reach arbitrary solution (for example, optimal one) from arbitrary starting solution. This property is important for asymptotic properties of the iterative processes based on the Markov Chains [8, 9].

## 5 Stochastic tabu search

In this section we present a variant of the well-known tabu search algorithm [10] which uses three randomized neighborhoods. For a neighborhood $N$, the randomized neighborhood $N_p$ with parameter $p$ is a subset of $N$. Each element of $N$ is included in $N_p$ with probability $p$ independently from other elements. The neighborhood $N_p$ may be found empty or coincide with $N$. By definition of the *Insert*, *Rotate* and *k-Opt* neighborhoods, each neighboring solution is defined by at most $k$ positions in current permutation. We store these positions and use three tabu lists, one for each neighborhood. The length of each list is a constant. Following [9], we use small lists for small $p$. Below we show the pseudocode for the stochastic tabu search heuristic. Parameters $\Delta_{in}$ and $\Delta_{div}$ define number of iterations for intensification and diversification stages. The earliest and latest days for each talent are easily calculated by the scheduling $x_{jt}$. Thus we use notation $F(x)$ for the objective function instead of $F(e,l,x)$.

**Stochastic Tabu Search**

**Initialization**: Create a starting solution $(x_{jt})$ and empty tabu lists;
put $(x^*_{jt}) \leftarrow (x_{jt})$, $N \leftarrow Rotate$, $t \leftarrow 1, t_{in} \leftarrow -\Delta_{in}, t_{div} \leftarrow -\Delta_{div}, p \leftarrow p^0$;

**Iterative process**: **While** not termination condition **do**

1. If $t > t_{in} + \Delta_{in}$ then $N \leftarrow Rotate$; If $t = t_{div} + \Delta_{div}$ then $p \leftarrow 2p$;
2. Find the best solution $(x'_{jt})$ in $N_p(x_{jt})$ without forbidden elements;
3. Put $(x_{jt}) \leftarrow (x'_{jt})$; $t \leftarrow t+1$; update the corresponding tabu list;
4. If $F(x^*_{jt}) > F(x_{jt})$ then $(x^*_{jt}) \leftarrow (x_{jt})$;
5. If intensification condition met then $t_{in} \leftarrow t$; $N \leftarrow Insert \cup 2\text{-}Opt$; $(x_{jt}) \leftarrow (x^*_{jt})$;
6. If diversification condition met then $t_{div} \leftarrow t$; $p \leftarrow p/2$;

**end while**

Out computational results show that the *Rotate* neighborhood produces better local optima than the *Insert* and 2-*Opt* neighborhoods. We use the *Rotate* neighborhood for local search and other neighborhoods for intensification. In order to diversify the search process we decrease the size of the randomized neighborhood at Step 6. As a result, the local search becomes more chaotic, and after $\Delta_{div}$ iterations we continue the search in a new area of the feasible domain. The starting value $p^0$ is selected as 0.10.

## 6 Hybrid simulated annealing

This algorithm explicitly combines ideas of two well-known metaheuristics: simulated annealing [8] and variable neighborhood search [7]. We use framework of the annealing and systematically change neighborhoods in this iterative process. The local improvement algorithm is applied to intensify the search after decreasing the temperature. For diversification we apply a random walk by the 3-*Opt* neighborhood. Below we show the pseudocode for our hybrid algorithm.

**Hybrid Simulated Annealing**

**Initialization**: Create a starting solution $(x_{jt})$; put $t \leftarrow 1, (x^*_{jt}) \leftarrow (x_{jt})$,
define starting temperature $c$ and cooling factor $\gamma$;
**Iterative process**: **While** not termination condition **do**

1. For $t := 1$ to $T_{max}$ do
   1.1. Select a neighborhood;
   1.2. Pick out a neighboring solution $(x'_{jt})$ and calculate $\Delta = F(x'_{jt}) - F(x_{jt})$;
   1.3. If $\Delta \leq 0$ then $(x_{jt}) \leftarrow (x'_{jt})$ else $(x_{jt}) \leftarrow (x'_{jt})$ with probability $e^{-\Delta/c}$;
   1.4. If $F(x^*_{jt}) > F(x_{jt})$ then $(x^*_{jt}) \leftarrow (x_{jt})$;
   1.5. If diversification condition met then apply a random walk under 3-$Opt$ neighborhood.
2. Decrease the temperature $c \leftarrow c\gamma$;
3. Apply the local improvement algorithm under *Insert* and 2-$Opt$ neighborhoods;

**end while**

The good starting solution we obtain by modified greedy algorithm. It allows us to start with low temperature. The maximal number of iterations $T_{max}$ for the given temperature is defines as the sire of the $Rotate$ neighborhood. The cooling factor $\gamma$ we put 0.99, the starting temperature is 18.

# 7 Genetic local search

As we have mentioned in Introduction, the first genetic algorithm for this problem is presented in [6]. In this algorithm the reproduction operator selects a number of solutions from the current population for crossing depending on their fitness. This is done by constructing a Monte Carlo wheel. A linear scaling of the fitness function is used for the wheel. Crossover is the well-known randomized two points operator. *Mutation* is a random step under the 2-$Opt$ neighborhood. The starting population is generated at random. Below we show another genetic algorithm which dominates the previous one. The main operators are modified in order to use the scheduling structure of the problem. Moreover, we apply the local improvement algorithm for each element of population to intensify the search process.

**Genetic Local Search**

**Initialization**: Create a starting population, put $(x^*_{jt})$ as the best solution in the population;
**Iterative process**: **While** not termination condition **do**

1. For $l := 1$ to $L$ do
   1.1. Select two parents from the population;
   1.2. Produce an offspring solution;
   1.3. Apply mutation operator with a small probability;
   1.4. Include the obtained solution into the population;
2. Remove $L$ the worst solutions from the population;
3. Update the best found solution $(x^*_{jt})$;
4. If intensification condition met then apply local improvement algorithm under the randomized *Insert* and 2-$Opt$ neighborhoods for each element of the population;

**end while**

The starting population is generated by modified greedy algorithm. We create $n$ solutions using each scene for the first day and select an appropriate number of the best solutions. The parents are taking at random but one of them is the best in the current population. The offspring solution is produced as follows. We consider two permutations and move $\lceil n/2 \rceil$ random positions from one of them into the new permutation. The last positions are fulfilled according to the second permutation. The mutation operator is a moving to neighboring solution under one of *Insert*, *Rotate* or 2-*Opt* neighborhoods. The local improvement is applied if the best found solution is not changed during some iterations. For computational experiments we use populations from 10 solutions and $L = 10$. For randomized neighborhoods we put $p = 0.05$.

# 8 Computational results

The developed algorithms are coded in C++ and tested on random generated test instances with known optimal solutions and instances from the previous publications [3, 5, 6]. For random instances we put $n = m = 100, w_i \in [1,100], d = 0.05$. For each talent we generate two scenes $j_1, j_2$ at random and put $a_{ij} = 1$ for $j = j_1, j_1 + 1, \ldots, j_2$ and $a_{ij} = 0$ otherwise. Thus we have optimal solution. Then we apply a random permutation for the columns of the matrix and use the result as the benchmark. Table 1 show the average deviation from the optimum for such 10 test instances. In brackets it is shown the number of trials from 10 when the optimal value is discovered. As we see these heuristics show small average deviation and can find global optima. The termination condition is the same for all methods: $15 \cdot 10^5$ objective function calculations. The running time is a few minutes on Pentium 2.8 GHz under the Windows XP Professional operating system.

Table 1. Average deviations from the optimum (%)

| N | Naive Greedy Algorithm | Hybrid Simulated Annealing | Genetic Local Search | Stochastic Tabu Search |
|---|---|---|---|---|
| 1 | 1 | 0 (10) | 0.001 (9) | 0 (10) |
| 2 | 22.1 | 0 (10) | 0.006 (7) | 1.82 (6) |
| 3 | 54.35 | 0 (10) | 0 (10) | 4.73 (6) |
| 4 | 24.73 | 0 (10) | 0 (10) | 1.10 (4) |
| 5 | 53.8 | 0 (10) | 0 (10) | 2.92 (7) |
| 6 | 63.46 | 0.23 (8) | 0.006 (9) | 3.87 (4) |
| 7 | 4.81 | 0 (10) | 0 (10) | 0.01 (9) |
| 8 | 16.2 | 0.15 (9) | 0.006 (8) | 1.20 (5) |
| 9 | 5.74 | 0 (10) | 0 (10) | 0 (10) |
| 10 | 88 | 0 (10) | 0 (10) | 0 (10) |

It is interesting to note that when we change the test instances and include one or two zeros between $j_1$ and $j_2$ for some talents and apply our heuristics, we discover solutions with better value than $\sum_{i=1}^{m} w_i \sum(j_2 - j_1 + 1)$. We do not know optimal values for this case but we have good near optimal values. Nevertheless, our iterative methods can improve these values.

To compare our methods with previous ones we test them on the instances available from internet [5, 6, 11]. For all cases we have $n \leq 70, m \leq 100$. Our methods have found the best known solutions quickly. Thus, we conclude that our local search methods can find the optimal or best known solutions with high frequency.

## 9 Conclusion and future research

In this paper we consider the talent scheduling problem and present iterative local search methods for solving this problem. We show that these methods can find the optimal solutions or the best known solutions if the number of talents and the number of scenes are at most 100.

For future research it is interesting to study a more general case of the problem when each scene has a time duration and we can shoot some scenes every day. Again, each talent starts to job in his first shooting day and finishes in his last shooting day. This new problem includes some aspects of the classical bin packing problem and may be more interesting from practical point of view. For theoretical research some questions are still open. For example, it is not clear how difficult the local search problem for each *Insert*, *Rotate*, and *k-Opt* neighborhoods. If daily payments are the same for all talents, in particular, $w_i = 1$ for all $i = 1, \ldots, m$, we can find local minimum in polynomial time by the simple local improvement algorithm. But for general case this local search problem may be PLS-complete [12].

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# Lower bounds for the two stage multimedia problem with an active prefetch

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**Abstract**

We consider a buffer-constrained flow shop problem. It can be applied to schedule media objects presentation on a portable device (a player). The player takes objects from the remote database and prefetchs them into the buffer. The size of buffer is limited. The presentation of an object cannot be started earlier than the finish of its loading. An object leaves the buffer after the completion of its presentation. The goal is to minimize the completion time of the whole presentation. This scheduling problem is strongly NP-hard. In our paper we propose some lower bounds for this problem based on a restricted version of the original problem.

## 1 Introduction

We consider a quantity-based buffer-constrained flowshop problem that can be applied to schedule media objects for an auto-assembled multimedia presentation from online multimedia databases. The retrieved media objects are presented sequentially without a predefined order in a given presentation. Each object has a download time and a playback time. The presentation of an object cannot be started earlier than the completion of its loading. To avoid the possible presentation lag that occurs while waiting for downloading of the media objects, the player is designed to ”prefetch” the objects before their presentation due times. To provide better quality of service (QOS) of the presentations, one main concern in practice is to schedule the objects to minimize the completion time of the entire presentation, i.e., the completion time of the presentation of the last media object. This technique has significant value in digital library/museum applications [5],[6],[2].

For applications in which portable devices such as PDAs are used for the playback of the media objects, one must consider the maximum buffer size allowed for keeping the prefetched objects concerned. Under a network environment with a rather stable transmission rate (for example, in a case where the network bandwidth is mainly limited by a slow ”last-mile” connection), the download time of an object is proportional to its file size. Therefore, the buffer quantity consumed by a buffered job is linearly proportional to its loading time.

Without lost of generality we suppose that the download time of an object is equal to its file size. Indeed, we can scale file size such that a unit of size is loaded per a unit of time. We also suppose that the download and presentation processes are non-resumable.

Two different prefetch models for loading the objects to the buffer are considered in scheduling literature. In [4] the authors suppose that the download of an incoming media object is not possible unless the free buffer space is sufficient for keeping the incoming object. In other worlds, if the size of the incoming job is greater than the size of free space available in the buffer, we have to wait until the completion of the presentation(s) of one or more buffered jobs.

In our paper we consider another prefetch model. We assume that the total size of the buffered and partially-buffered jobs at every instant does not exceed the size of the buffer. This is $NP$-hard [3]. Some polynomially solvable cases and variable neighborhood search metaheuristic for large scale instances were proposed in [3].

The paper is organized as follows. Section 2 introduces the notations and the mathematical model. Section 3 describes the lower bounds. The first one based on the linear integer formulation. The second one uses the Johnson rules for the problem without buffer. Computational results are discussed in Section 4. In final Section 5, we give some conclusions and directions for future research.

## 2 Mathematical model

Proposed model for these media object sequence optimization problem can be considered as a two-machine buffer-constrained flowshop problem by associating the download time and the playback time of each object with the processing times of some job on the first machine and on the second machine, correspondingly.

Let us consider a set of media objects $N = \{1,\ldots,n\}$. Each object $i$ have a download time $a_i$, a playback time $b_i$. The size of buffer we denote as $\Omega$. We associate the download time $a_i$ and the playback time $b_i$ of an object $i$ with the processing times of job $i$ on machine $A$ and on machine $B$ correspondingly. Each job cannot start on machine $B$ before it has been finished on machine $A$. The download of each job is assumed to be non-resumable.

Let a sequence $\sigma$ determines the order of jobs on machine $A$. For job $i$ we introduce variables $x_{ij} \in \{0,1\}$, $j = 1,\ldots,n$ which are equal to 1 if it is the $j$-th job in $\sigma$ and 0 otherwise. Since there exists an one-to-one correspondence between each job and its sequence position we obtain the following constraints:

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1,\ldots,n, \tag{1}$$

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1,\ldots,n. \tag{2}$$

Let variables $p_j^a$ and $p_j^b$ specify the processing times of the $j$-th job in $\sigma$ on the first and the second machine, correspondingly. We have

$$p_j^a = \sum_{i=1}^{n} a_i x_{ij}, \quad j = 1,\ldots,n, \tag{3}$$

$$p_j^b = \sum_{i=1}^{n} b_i x_{ij}, \quad j = 1,\ldots,n. \tag{4}$$

Finally, variables $s_j^a$ and $s_j^b$ specify the starting times of the $j$-th job in $\sigma$ on the first and the second machine, correspondingly. For these variable we have [3]:

$$s_1^a = 0, \tag{5}$$

$$s_{j+1}^a \geq s_j^a + p_j^a, \quad j = 1,\ldots,n-1, \tag{6}$$

$$s_{j+1}^a \geq s_j^b + p_j^b + p_j^a - \Omega, \quad j = 1,\ldots,n-1, \tag{7}$$

$$s_{j+1}^b \geq s_j^b + p_j^b, \quad j = 1,\ldots,n-1, \tag{8}$$

$$s_j^b \geq s_j^a + p_j^a \quad j = 1, \ldots, n. \tag{9}$$

It is clear that the makespan of the schedule is equal to $s_n^b + p_n^b$. Thus we can formulate this problem as follows: minimize $F(\sigma) = s_n^b + p_n^b$ to constraints (1)–(9). Denote this problem as *AP*-problem.

The ILP formulation allows us to use general-purpose ILP solvers to search an optimal solution of the *AP*-problem. Recent research shows that the general-purpose ILP solvers are very competitive with the best special-purpose exact algorithms for some hard combinatorial optimization problem. We use CPLEX solver and call it from GAMS (http://www.gams.com) to solve instances with 20 jobs. However, for some instances it requires more than the whole day of computation.

### 2.1 Restricted problem

Below we present a trick which can help to improve the behaviour of exact methods. Since each job should be loaded in the buffer, the file size of a job cannot exceed the size of the buffer. Thus we have $a_i \leq \Omega$ for all $i \in N$. In this section we consider an even stronger constraint on the input data of the media scheduling problem. Let us assume that all jobs satisfy the following condition

$$a_i + b_i \leq \Omega \text{ for all } i \in N. \tag{10}$$

We denote the *AP*-problem with constraint (10) by the *RAP*-problem. It can be shown [3] that the *AP*-problem linearly reduces to the *RAP*-problem. Let $I$ be an instance of the *AP*-problem with $n$ jobs. Let $a_i$, $b_i$ be given for each job $i$ and let the size of the buffer be equal to $\Omega$. We construct an instance $\bar{I}$ of the *RAP*-problem with $n$ jobs, and $\bar{a}_i = a_i$, and $\bar{b}_i = \min(b_i, \Omega - a_i)$. Let $\delta_i = b_i - \bar{b}_i$. The buffer in the instance $\bar{I}$ has the same size as in the instance $I$. It is easy to see that $\bar{a}_i + \bar{b}_i \leq \Omega$ for all $i$, and thus we get an instance of the *RAP*-problem. Then for the objective function $\bar{F}(\sigma)$ of *RAP*-problem we have $F(\sigma) = \bar{F}(\sigma) + \sum_{i=1}^n \delta_i$ where $\sigma$ is arbitrary permutation of the jobs.

## 3 Lower bounds

Consider some lower bounds for the *AP*-problem and the *RAP*-problem.

### 3.1 Lower bound based on linear programming

Let *LP* denote the linear program corresponding to ILP (1)–(9) for the *AP*-problem in which we relax the integrality constraints $x_{ij} \in \{0,1\}$ by setting $x_{ij} \in [0,1]$. Denote $\overline{LP}$ the linear program for corresponding *RAP*-problem. These linear programs can be solved in polynomial time. Thus, we obtain a lower bound $LB_{lp}$ for the *AP*-problem and a lower bound $\overline{LB}_{lp}$ for the *RAP*-problem.

**Theorem 1.** $LB_{lp} \leq \overline{LB}_{lp} + \sum_{i=1}^n \delta_i$.

*Proof.* Let $\bar{x}_{ij}$ be an optimal solution for the *RAP*-problem. Using the values of $\bar{x}_{ij}$ we can determine remaining variables $\bar{p}_j^a$, $\bar{p}_j^b$, $\bar{s}_j^a$ and $\bar{s}_j^b$ for the *RAP*-problem. The solution $\bar{x}_{ij}$ is a feasible solution for the *AP*-problem. Observe that $p_j^a = \bar{p}_j^a$, $p_j^b = \sum_{i=1}^n b_i \bar{x}_{ij} = \sum_{i=1}^n (\bar{b}_i + \delta_i)\bar{x}_{ij} = \bar{p}_j^b + \sum_{i=1}^n \delta_i \bar{x}_{ij}$. Then we define $\chi_j = \sum_{i=1}^n \delta_i \bar{x}_{ij}$. Note that $\sum_{j=1}^n \chi_j = \sum_{i,j=1}^n \delta_i \bar{x}_{ij} = \sum_{i=1}^n \delta_i$.

We claim that $s_j^a \leq \bar{s}_j^a + \sum_{k=1}^{j-1} \chi_k$ and $s_j^b \leq \bar{s}_j^b + \sum_{k=1}^{j-1} \chi_k$ for arbitrary $j = 1, \ldots, n$. This statement will be shown by induction for $j = 1, \ldots, n$. For $j = 1$ it is true. Suppose the claim holds for some $j$ and $j < n$. We have to show that it still holds for $j+1$.

By condition (1)–(9) we have

$$s^a_{j+1} = \max \left\{ \begin{array}{l} s^a_j + p^a_j \\ s^b_j + \overline{p}^b_j + p^a_j - \Omega + \chi_j \end{array} \right\} \leq \max \left\{ \begin{array}{l} s^a_j + p^a_j, \\ s^b_j + \overline{p}^b_j + p^a_j - \Omega \end{array} \right\} + \chi_j \leq$$

by induction hypothesis

$$\leq \max \left\{ \begin{array}{l} \overline{s}^a_j + p^a_j + \sum_{k=1}^{j-1} \chi_k \\ \overline{s}^b_j + \overline{p}^b_j + p^a_j - \Omega + \sum_{k=1}^{j-1} \chi_k \end{array} \right\} + \chi_j =$$

$$\max \left\{ \begin{array}{l} \overline{s}^a_j + p^a_j \\ \overline{s}^b_j + \overline{p}^b_j + p^a_j - \Omega \end{array} \right\} + \sum_{k=1}^{j} \chi_k.$$

So, $s^a_{j+1} \leq \overline{s}^a_{j+1} + \sum_{k=1}^{j} \chi_k$.

It remains to show that it holds for $s^b_{j+1}$:

$$s^b_{j+1} = \max \left\{ \begin{array}{l} s^a_{j+1} + p^a_{j+1} \\ s^b_j + p^b_j \end{array} \right\} \leq \max \left\{ \begin{array}{l} \overline{s}^a_{j+1} + p^a_{j+1} + \sum_{k=1}^{j} \chi_k \\ \overline{s}^b_j + \sum_{k=1}^{j-1} \chi_k + \overline{p}^b_j + \chi_j \end{array} \right\} =$$

$$= \max \left\{ \begin{array}{l} \overline{s}^a_{j+1} + p^a_{j+1} \\ \overline{s}^b_j + \overline{p}^b_j \end{array} \right\} + \sum_{k=1}^{j} \chi_k = \overline{s}^b_{j+1} + \sum_{k=1}^{j} \chi_k.$$

We conclude that $\overline{LB}_{lp} + \sum_{i=1}^{n} \delta_i = \overline{F}(\overline{x}_{ij}) + \sum_{i=1}^{n} \delta_i = \overline{s}^b_n + \overline{p}^b_n + \sum_{k=1}^{n} \chi_k \geq s^b_n + p^b_n = F(\overline{x}_{ij}) \geq LB_{lp}$. □

### 3.2 Johnson's lower bound

The problem without buffer limitation is equivalent to the well-known Johnson problem $F2||C_{max}$. Recall that, for the Johnson two-machine problem, there always exists an optimal solution which is a permutation schedule without idles on the first machine. In one of the first paper in the theory of scheduling, Johnson [1] demonstrates that $F2||C_{max}$ can be solved in $O(n \log n)$ time by the following sequencing rule: first schedule the jobs with $a_j \leq b_j$ in order of nondecreasing $a_j$, and then schedule the remaining jobs in order of nonincreasing $b_j$. This order of jobs is called a Johnson's permutation. Let $s^a_j(I, \sigma)$ be a starting time of the $j$-th job on the first machine in $\sigma$ for instance $I$. For each permutation $\sigma$ the starting times of jobs can be obtained by the following formulae:

$$s^a_{j+1}(I, \sigma) = s^a_j(I, \sigma) + a_{\sigma(j)},$$

$$s^b_{j+1}(I, \sigma) = \max \left\{ \begin{array}{l} s^a_{j+1}(I, \sigma) + a_{\sigma(j+1)} \\ s^b_j(I, \sigma) + b_{\sigma(j)} \end{array} \right\}.$$

The value of optimal solution is a lower bound for $AP$-problem. Denote $LB_J$ the lower bound obtained by Johnson algorithm for $AP$-problem, and $\overline{LB}_J$ the lower bound obtained by Johnson algorithm for $RAP$-problem.

**Theorem 2.** $LB_J \leq \overline{LB}_J + \sum_i \delta_i$.

*Proof.* Let $I$ be an instance of the *AP*-problem, let $\overline{I}$ be an instance of the *RAP*-problem and let $\sigma$ be a Johnson's permutation for $\overline{I}$. Show that $s^b_j(I,\sigma) \le s^b_j(\overline{I},\sigma) + \sum_{k=1}^{j-1} \delta_{\sigma(k)}$ for every $j$. This statement will be shown by induction for $j = 1,\ldots,n$. For $j = 1$ it is true. Suppose the claim holds for some $j$ and $j < n$. We have to show that it still holds for $j+1$. Indeed we have:

$$s^b_{j+1}(I,\sigma) = \max\left\{ \begin{array}{l} s^a_{j+1}(I,\sigma) + a_{\sigma(j+1)} \\ s^b_j(I,\sigma) + b_{\sigma(j)} \end{array} \right\} \le$$

$$\le \max\left\{ \begin{array}{l} s^a_{j+1}(I,\sigma) + a_{\sigma(j+1)} \\ s^b_j(\overline{I},\sigma) + \sum_{k=1}^{j} \delta_{\sigma(k)} + \overline{b}_{\sigma(j)} \end{array} \right\} \le$$

$$\le \max\left\{ \begin{array}{l} s^a_{j+1}(I,\sigma) + a_{\sigma(j+1)} \\ s^b_j(\overline{I},\sigma) + \overline{b}_{\sigma(j)} \end{array} \right\} + \sum_{k=1}^{j} \delta_{\sigma(k)} \le s^b_{j+1}(\overline{I},\sigma) + \sum_{k=1}^{j} \delta_{\sigma(k)}.$$

Since a length of an optimal schedule for $I$ isn't greater than the length of $\sigma$, we have $LB_J \le \overline{LB}_J + \sum_i \delta_i$. □

## 4 Computational results

Theorem 1 and Theorem 2 show that the lower bounds of the restricted problem are not less than the lower bounds of the initial problem. The following computational experiments show the growing of the bound. In the first computational experiment, we consider the instances with $n = 20, 30, 50, 100$. Processing times $a_i$ and $b_i$ are randomly generated as integer values in the range from 20 to 40 with uniform distribution. For each $n$ we generate 50 instances of the *AP*-problem. For each instance we consider different values of the buffer $\Omega = 40, 50, 55, 58, 60, 62, 70, 80$.

Figure 1: Relative deviation $\varepsilon$ as a function of $\Omega$

We studied a dependence of $\varepsilon = \frac{\overline{LB}_{lp} + \sum_{i=1}^{n} \delta_i - LB_{lp}}{LB_{lp}} 100\%$ on the buffer size. Our computational results are presented on Figure 1. If $\Omega \ge 80$ then the relative deviation is equal to 0 because the restricted problem coincides with the initial problem. The maximum of the relative deviation between $LB_{lp}$ and $\overline{LB}_{lp}$ is reached for $\Omega = 58$. Thus, computational results show that the lower bound based on the linear program of the *RAP*-problem is better than the lower bound based on the linear program of the *AP*-problem.

In the second experiment we compare our lower bounds for the restricted problem with an upper bound obtained by variable neighborhood search (VNS) [3]. Table 1 shows how often in 50 instances the corresponding inequality holds. On average $\overline{LB}_J$ is rather better than $\overline{LB}_{lp}$. However, differences between values of these lower bounds are not exceed 0,5%.

We observe that the lower bounds are equal to the upper bound for $n \ge 50$. Thus we can make a surprising conclusion that "big" instances are more easy than "small" instances. Such effect is a result of our way to generate of instances. Since $a_i$ and $b_i$ randomly takes on integer values in the range from 20 to 40, each instance has many distinct values. Thus VNS easy finds an optimal solution.

## 5 Conclusion

This study considers a two-machine flowshop problem with processing time-depend buffer constraints for multimedia applications. We present new results for active prefetch model that further exploits the

Table 1: Comparison of upper and lower bounds for different values of buffer

| n | | 40 | 50 | 55 | 58 | 60 | 62 | 65 | 70 | 80 |
|---|---|---|---|---|---|---|---|---|---|---|
| 20 | $\overline{LB}_{lp} \neq \overline{LB}_J$ | 0 | 0 | 0 | 0 | 0 | 3 | 4 | 3 | 4 |
| | $VNS \neq \overline{LB}_J$ | 0 | 0 | 0 | 2 | 5 | 7 | 2 | 0 | 0 |
| | $VNS \neq \overline{LB}_{lp}$ | 0 | 0 | 0 | 2 | 5 | 8 | 6 | 3 | 4 |
| 30 | $\overline{LB}_{lp} \neq \overline{LB}_J$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | $VNS \neq \overline{LB}_{lp}$ | 0 | 0 | 0 | 0 | 2 | 2 | 1 | 0 | 0 |
| 50 | $\overline{LB}_{lp} \neq \overline{LB}_J$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | $VNS \neq \overline{LB}_{lp}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

available free space by advancing the download by a maximal duration with which the buffer does not overflow. We developed a simple lower bound based on Johnson's algorithm and a lower bound based on linear program. Also we show that the lower bounds of the restricted problem are not worse than the lower bounds of the initial problem. The difference between lower bounds of original problem and restricted problem can attain 9%. Our computational result show that random generated instances when the download time and playback time are taken independently from a given interval, are not difficult for the VNS metaheuristic. As the rule new lower bounds coincide with upper bounds if the number of media objects is big enough. For small instance VNS finds optimal solutions again but our lower bounds show small average deviation from the optimum.

## 6 Acknowledgments

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# Optimal pricing and replenishment policy of a two warehouse inventory model with non instantaneous deteriorating items

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**Abstract**

In this paper, the optimal pricing and replenishment policy of a two warehouse inventory model with non instantaneous deteriorating items is studied. In the model, more goods than can be stored in the owned warehouse may be purchased and the excess quantities are stored in a rented warehouse. Shortages are allowed and partially backlogged at a rate, which is any non-increasing function of the waiting time up to the next replenishment. This two-warehouse inventory problem is studied under the replenishment policy starting with no shortages and ending with shortages. For this model, we suggest a simple solution procedure to determine the optimal replenishment policy and some numerical examples are provided for illustration purposes.

**KEYWORDS**

Inventory; Non-instantaneous deteriorating items; Two-Warehouse; Shortages

## 1. INTRODUCTION

In studies of inventory models, unlimited warehouse capacity is often assumed. However, in busy marketplaces, such as super markets, corporation markets, etc., the storage area for items may be limited. When an attractive price discount for bulk purchase is available or the cost of procuring goods is higher than the other inventory related costs or demand for items is very high or when the item under consideration is a seasonal product such as the yield of a harvest or there are some problems in frequent procurement, the procurement of a large amount of items at a time is decided. These items cannot be accommodated in the existing store house (the Own Warehouse, denoted as OW). In this situation, in order to store the excess items, an additional warehouse (the rented warehouse, denoted as RW), which may be located at a short distance from OW, is hired on a rental basis. It is generally assumed that the holding cost in RW is higher than that in OW due to the additional cost of maintenance, material handling, etc. To reduce the inventory costs, it will be cost-effective to consume the goods of RW at the earliest.

In addition, for certain types of commodities, such as medicine, volatile liquids, blood, foodstuffs, deterioration is usually observed during their normal storage period (Shah, 2006). Two excellent surveys on trends in modeling of continuously deteriorating inventory are those by Raafat (1991), and Goyal and Giri (2001). Assuming constant demand rate, Sarma (1987) studied an inventory model for a single deteriorating item with shortages and two levels of storage. Pakkala and Achary (1992) further considered the two-warehouse model for deteriorating items with finite replenishment rate and shortages. In the previous two models, the demand was assumed to be constant and the cost of transporting items from RW to OW was not taken into account. Goswami and Chaudhuri (1992) developed the model with or without shortages by assuming that the demand varies over time with linearly increasing trend and that the transportation cost from RW to OW depends on the quantity being transported. Benkherouf (1997) developed a two-warehouse model for deteriorating items with time varying demand rate and shortages during the infinite period. Kar et al. (2001) developed a two-storage inventory model with linearly time dependent demand and complete backlogging over finite time horizon. Recently Yang (2006) considered a two-warehouse inventory problem for deteriorating items with constant demand rate under inflation and Hsieh et al. (2008) developed an

inventory model for deteriorating items with two warehouses using the net present value of the total cost. In most of the models mentioned above, it is generally assumed that that the items that enter the inventory warehouses are subject to deterioration and also that different warehouses have different deterioration rates, since the deterioration depends on preserving facilities and environmental conditions available in a warehouse. But there are a lot of items that start to deteriorate after some time period. These items are characterized as non-instantaneous deteriorating items (Wu et al., 2006).

Pricing strategy is one of the major policies for sellers or retailers to obtain its maximum profit. Price has a direct impact on demand, and models with price dependent demand occupy a prominent place in the inventory literature. Wee (1995) studied the joint pricing and replenishment policy for a deteriorating inventory with price elastic demand rate that decline over time. Abad (1996) considered the dynamic pricing and lot-sizing problem of a perishable good under partial backlogging of demand. He assumed that the fraction of shortages backordered is variable and is a decreasing function of the waiting time. Wee and Law (2001) developed an inventory model for deteriorating items with price-dependent demand in which time-value of money is also taken into account. Abad (2003) presented the joint pricing and lot-sizing under conditions of perishability, finite production and partial backlogging. Chang et al. (2006) introduced a deteriorating inventory model with price-time-dependent demand and partial backlogging. Dye (2007) proposed the joint pricing and ordering policies for a deteriorating inventory with price-dependent demand and Valliathal and Uthayakumar (2010) studied the joint pricing and lot-sizing for non-instantaneous deteriorating items with shortages.

In this paper, we extend the inventory model for non-instantaneous deteriorating items by Valliathal and Uthayakumar (2010), coordinating replenishment and pricing policy in a two warehouses inventory system. The main goal is to determine the optimal pricing and ordering polices such that the total profit per unit time has a maximum value. Furthermore, we provide an easy-to-use algorithm to find the optimal solution and some numerical examples are provided to illustrate the theoretical results. The rest of the paper is organised as follows. In the next section, we provide the assumptions for the proposed inventory model and the notation to be used throughout the entire article. In Sections 3 and 4, we give the mathematical formulation of the model. In section 5 a solution procedure to determine the optimal policy is given. Finally, a numerical example and a single parameter sensitivity analysis are provided to test and verify the theoretical results in Section 6.

## 2. NOTATIONS AND ASSUMPTIONS

To develop the mathematical model, the following notations and assumptions are used.

### 2.1 Notations

$X_1$ : The time at which the inventory level of the RW reaches zero

$W$ : The capacity of OW

$T$ : The length of the replenishment cycle

$t_d$ : The length of time during which no deterioration occurs, $0 \le t_d \le T$

$t_1$ : The time at which the shortage starts, $0 \le t_1 \le T$

$\theta_1$ : The deterioration rate in RW

$\theta_2$ : The deterioration rate in OW, $0 < \theta_1 < \theta_2 < 1$

$A$ : The set up cost per cycle

$c$ : The purchase cost per unit

$p$ : The selling price per unit, where $p > c$

$h_1$ : The holding cost in RW per unit per unit time

$h_2$ : The holding cost in OW per unit per unit time $(h_2 < h_1)$

$s$ : The shortage cost per unit per unit time

$\pi$ : The opportunity cost due to lost sales per unit
$K$ : The deterioration cost per unit
$c_t$ : The transportation cost per unit

## 2.2 Assumptions

- The planning horizon of the inventory problem is infinite
- Lead time is zero and replenishment rate is infinite.
- The demand rate function is determenistic and is a known function of selling price and time, $R(p,t) = g(p)f(t)$, where $f(t)$ is any non-negative, continuous, decreasing and log-concave function of time in the interval $[0,T]$ and $g(p)$ is any non-negative, continuous, convex decreasing function of the selling price $p$
- Shortages are allowed and partially backlogged. The fraction of shortages backlogged is a non-increasing function $B(x)$, where $x$ is the waiting time up to the next replenishment, and $0 \le B(x) \le 1$ with $B(0) = 1$. To guarantee the existence of an optimal solution, we assume that $B(x) + TB'(x) \ge 0$, where $B'(x)$ is the derivative of $B(x)$. When $B(x) = 1$ (or 0) it corresponds to a complete backlogging (or complete lost sales) model.
- During the fixed period $[0,t_d]$ no deterioration occurs. After $t_d$ the items deteriorate at constant rate $\theta_1$ in the RW and constant rate $\theta_2$ in the OW, where $\theta_2 > \theta_1$.
- There is no repair or replenishment of the deteriorated items during the replenishment cycle
- The OW has a fixed capacity of $W$ units and the RW has unlimited capacity.
- The holding costs in RW are higher than those in OW, i.e. $h_1 > h_2$.

# 3. MODEL FORMULATION

For the development of the model, we assume that a company purchases $Q$ (Q>W) units out of which $W$ units are kept in OW and the excess quantities are kept in RW. Initially, demand is met by using the stocks of RW during the time period $[0,X_1]$ until the stock level drops to zero at time $t = X_1$. During time period $[X_1,T]$ the stocks of OW are used to meet further demand. At $t = t_1$ the inventory level of the OW falls to zero and during the period $[t_1,T]$, shortages occur which are partially backlogged. Since no deterioration occurs during the time period $[0,t_d]$, the inventory gradually depletes due to demand only. After the time point $t_d$, deterioration takes place and the inventory decreases due to both demand and deterioration. Hence, for the development of the model three cases arise: (A) $0 \le t_d \le X_1$, (B) $X_1 \le t_d \le t_1$ and (C) $t_1 \le t_d$.

## 3.1 Case A. ($0 \le t_d \le X_1$)

From the above assumptions, we know that the inventory levels in the RW, $I_R(t)$ and in the OW, $I_O(t)$ respectively, at time *t* satisfy the following differential equation:

$$\frac{dI_R(t)}{dt} = -g(p)f(t),\ 0 \le t \le t_d \tag{1}$$

$$\frac{dI_R(t)}{dt} = -\theta_1 I_R(t) - g(p)f(t),\ t_d \le t \le X_1,\ I_R(X_1) = 0 \tag{2}$$

$$\frac{dI_O(t)}{dt} = -\theta_2 I_O(t),\ t_d \le t \le X_1,\ I_O(0) = I_O(t_d) = W \tag{3}$$

$$\frac{dI_O(t)}{dt} = -\theta_2 I_O(t) - g(p)f(t),\ X_1 \le t \le t_1,\ I_O(t_1) = 0 \tag{4}$$

$$\frac{dI_O(t)}{dt} = -g(p)f(t)B(T-t),\ t_1 \le t \le T,\ I_O(t_1) = 0 \tag{5}$$

Solving the above differential equations we obtain the system's inventory level as:

$$I(t) = \begin{cases} I_R(t) = e^{-\theta_1 t_d}\int_{t_d}^{X_1} e^{\theta_1 u} g(p)f(u)du + \int_0^{t_d} g(p)f(u)du - \int_0^t g(p)f(u)du,\ 0 \le t \le t_d \\ I_R(t) = e^{-\theta_1 t}\int_t^{X_1} e^{\theta_1 u} g(p)f(u)du,\ t_d \le t \le X_1 \\ I_O(t) = W,\ 0 \le t \le t_d \\ I_O(t) = We^{-\theta_2(t-t_d)},\ t_d \le t \le X_1 \\ I_O(t) = e^{-\theta_2 t}\int_t^{t_1} e^{\theta_2 u} g(p)f(u)du,\ X_1 \le t \le t_1 \\ I_O(t) = -\int_{t_1}^t g(p)f(u)B(T-u)du,\ t_1 \le t \le T \end{cases} \tag{6}$$

Since $I_O(X_1^-) = I_O(X_1^+)$ we get:

$$W = \int_{X_1}^{t_1} e^{\theta_2(u-t_d)} g(p)f(u)du \tag{7}$$

### 3.1.1 Profit function – Case A

The total profit per unit time during $[0,T]$ consists of the following elements:

The set up cost per cycle $A$.

The holding cost in RW during the time period $[0, X_1]$: $HC_R = \int_0^{t_d} h_1 I_R(t)dt + \int_{t_d}^{X_1} h_1 I_R(t)dt$ (8)

The holding cost in OW during the time period $[0, t_1]$: $HC_O = \int_0^{t_d} h_2 I_O(t)dt + \int_{t_d}^{X_1} h_2 I_O(t)dt + \int_{X_1}^{t_1} h_2 I_O(t)dt$ (9)

The shortage cost during the time period $[t_1, T]$: $SC = s\int_{t_1}^{T}\int_{t_1}^{t} g(p)f(u)B(T-u)dudt$ (10)

The opportunity cost due to lost sales: $OC = \pi\int_{t_1}^{T} g(p)f(t)\left(1 - B(T-t)\right)dt$ (11)

The deterioration cost: $DC = K \cdot \left(I_R(0) + W - \int_0^{t_1} g(p)f(u)du\right)$ (12)

The transportation cost: $TC = c_t\int_0^{X_1} g(p)f(t)dt$ (13)

The purchase cost: $PC = c\left[I_R(0) + W - I_O(T)\right]$ (14)

The sales revenue over the period $[0,T]$: $SR = p\left[\int_0^{t_1} g(p)f(t)dt + \int_{t_1}^{T} g(p)f(t)B(T-t)dt\right]$ (15)

Therefore, the profit per unit time during [0,T]:

$$TP_1(t_1, X_1, T, p) = \frac{1}{T}\{SR - A - HC_R - HC_O - SC - OC - DC - TC - PC\} \tag{16}$$

## 3.2 Case B. ($X_1 \le t_d \le t_1$)

In this case the inventory level in the RW, $I_R(t)$ and in the OW, $I_O(t)$ respectively, at time $t$ satisfy the following differential equation:

$$\frac{dI_R(t)}{dt} = -g(p)f(t),\ 0 \le t \le X_1,\ I_R(X_1) = 0 \tag{17}$$

$$\frac{dI_O(t)}{dt} = -g(p)f(t),\ X_1 \le t \le t_d,\ I_O(X_1) = W \tag{18}$$

$$\frac{dI_O(t)}{dt} = -\theta_2 I_O(t) - g(p)f(t),\ t_d \le t \le t_1,\ I_O(t_1) = 0 \tag{19}$$

$$\frac{dI_O(t)}{dt} = -g(p)f(t)B(T-t),\ t_1 \le t \le T,\ I_O(t_1) = 0 \tag{20}$$

Solving the above differential equations we obtain the system's inventory level as:

$$I(t) = \begin{cases} I_R(t) = \int_0^{X_1} g(p)f(u)du - \int_0^t g(p)f(u)du,\ 0 \le t \le X_1 \\ I_O(t) = W,\ 0 \le t \le X_1 \\ I_O(t) = W - \int_{X_1}^t g(p)f(u)du,\ X_1 \le t \le t_d \\ I_O(t) = e^{-\theta_2 t}\int_t^{t_1} e^{\theta_2 u} g(p)f(u)du,\ t_d \le t \le t_1 \\ I_O(t) = -\int_{t_1}^t g(p)f(u)B(T-u)du,\ t_1 \le t \le T \end{cases} \tag{21}$$

Since $I_O(t_d^-) = I_O(t_d^+)$ we get:

$$W = \int_{X_1}^{t_d} g(p)f(t)dt + e^{-\theta_2 t_d}\int_{t_d}^{t_1} e^{\theta_2 t} g(p)f(t)dt \tag{22}$$

### 3.2.1 Profit function – Case B

As in Case A, the total profit per unit time during $[0,T]$ consists of the following elements:

The set up cost per cycle $A$.

The holding cost in RW per cycle: $HC_R = \int_0^{X_1} h_1 I_R(t)dt$ (23)

The holding cost in OW per cycle which is given by: $HC_O = \int_0^{X_1} h_2 I_O(t)dt + \int_{X_1}^{t_d} h_2 I_O(t)dt + \int_{t_d}^{t_1} h_2 I_O(t)dt$ (24)

And SC, OC, DC, TC, PC, SR which are defined the same as in Case A.

Therefore, the profit per unit time during [0,T] is given by:

$$TP_2(t_1, X_1, T, p) = \frac{1}{T}\{SR - A - HC_R - HC_O - SC - OC - DC - TC - PC\} \tag{25}$$

## 3.3 Case C. ($t_1 \le t_d$)

Finaly, in this case, the inventory level in the RW, $I_R(t)$ and in the OW, $I_O(t)$ respectively, at time *t* satisfy the following differential equation:

$$\frac{dI_R(t)}{dt} = -g(p)f(t),\ 0 \le t \le X_1,\ I_R(X_1) = 0 \tag{26}$$

$$\frac{dI_O(t)}{dt} = -g(p)f(t),\ X_1 \le t \le t_1,\ I_O(t_1) = 0 \tag{27}$$

$$\frac{dI_O(t)}{dt} = -g(p)f(t)B(T-t),\ t_1 \le t \le T,\ I_O(t_1) = 0 \tag{28}$$

Solving the above differential equations we obtain the system's inventory level as:

$$I(t) = \begin{cases} I_R(t) = \int_0^{X_1} g(p)f(u)du - \int_0^t g(p)f(u)du,\ 0 \le t \le X_1 \\ I_O(t) = W,\ 0 \le t \le X_1 \\ I_O(t) = \int_t^{t_1} g(p)f(u)du,\ X_1 \le t \le t_1 \\ I_O(t) = -\int_{t_1}^t g(p)f(u)B(T-u)du,\ t_1 \le t \le T \end{cases} \tag{29}$$

Since $I_O(X_1) = W$ we get:

$$W = \int_{X_1}^{t_1} g(p) f(u) du \tag{30}$$

### 3.3.1 Profit function – Case C

The total profit per unit time during $[0,T]$ for this case consists of the following elements:

The set up cost per cycle $A$.

The holding cost in RW per cycle: $HC_R = \int_0^{X_1} h_1 I_R(t) dt$ (31)

The holding cost in OW per cycle which is given by: $HC_O = \int_0^{X_1} h_2 I_O(t) dt + \int_{X_1}^{t_1} h_2 I_O(t) dt$ (32)

And SC, OC, TC, PC, SR which are defined the same as in Case A.

Therefore, the profit per unit time during [0,T] is given by:

$$TP_3(t_1, X_1, T, p) = \frac{1}{T}\{SR - A - HC_R - HC_O - SC - OC - TC - PC\} \tag{33}$$

## 4. THEORETICAL RESULTS AND OPTIMAL SOLUTION

To find the optimal solution for the problem, we maximize $TP_1$, $TP_2$ and $TP_3$, separately and then compare them to obtain the optimal values of the decision variables.

## 4.1 Maximization of $TP_1(t_1, X_1, T, p)$

Since $W = \int_{X_1}^{t_1} e^{\theta_2(t-t_d)} g(p) f(t) dt$ (i.e. Eq. 7), we can solve with respect to $X_1$. Hence, we can consider $X_1 = X_1(t_1)$ a function of $t_1$. Consequently, the profit function becomes a function of three variables $TP_1(t_1, X_1(t_1), T, p) \equiv TP_1(t_1, T, p)$. Since $g(p)$ is a decreasing, convex and positive function of $p$ then $p \in [p_{\min}, p_{\max}]$. So by starting with $p_{\min}$ and using a step of increase say $p_{in}$ we find the optimal $t_1$ and $T$ for any given $p \in [p_{\min}, p_{\max}]$. It is obvious that we need to solve a finite number of maximization problems. Thus let $(t_1(p), T(p)) = \arg\max_{t_1, T} TP_1(t_1, T, p)$ denote the optimal $(t_1, T)$ corresponding to a given $p$. From the first order conditions for a maximum $(t_1(p), T(p))$ follows as the solution of the equations:

$$TP_1(t_1, T, p) =$$

$$(p + \pi - c)\int_{t_1}^{T} g(p) f(t) \frac{\partial B(T-t)}{\partial T} dt + (p-c) g(p) f(T) - s\int_{t_1}^{T}\int_{t_1}^{t} g(p) f(u) \frac{\partial B(T-u)}{\partial T} du dt - s\int_{t_1}^{T} g(p) f(t) B(T-t) dt \tag{34}$$

$$(p - c + \pi)(1 - B(T - t_1)) + s(T - t_1) B(T - t_1)$$

$$= (c + K) e^{\theta_1(X_1 - t_d) + \theta_2(t_1 - X_1)} + \frac{h_2}{\theta_2}\left(e^{\theta_2(t_1 - X_1)} - 1\right) + \frac{h_1}{\theta_1} e^{\theta_2(t_1 - X_1)}\left(e^{\theta_1(X_1 - t_d)} - 1\right) + c_t e^{\theta_2(t_1 - X_1)} + h_1 t_d e^{\theta_1(X_1 - t_d) + \theta_2(t_1 - X_1)} \tag{35}$$

In the above we make use of the relation:

$$\frac{\partial X_1}{\partial t_1} = e^{\theta_2(t_1 - X_1)} \frac{f(t_1)}{f(X_1)} \tag{36}$$

We can prove (to save space the proof is omitted) that

**Proposition 1**. For any given $p \in [p_{\min}, p_{\max}]$, if $\frac{\partial X_1}{\partial t_1} < 1$, there exist unique $(t_1(p), T(p))$.

Now let $p^* = \arg\max_p \{TP_1(t_1(p), T(p), p), p \in [p_{\min}, p_{\max}]\}$ denote the value of $p$ which gives the maximum $TP_1(t_1(p), T(p), p), p \in [p_{\min}, p_{\max}]$ the optimal values of $(t_1, T)$ are $(t_1^*, T^*) = (t_1(p^*), T(p^*))$.

The optimal value of $X_1$, i.e. $X_1^*$, can then be obtained by setting $t_1 = t_1^*$ in Eq. 7.

The maximum profit $TP_1\left(t_1^*, X_1^*, T^*, p^*\right)$ can be calculated by substituting $\left(t_1^*, X_1^*, T^*, p^*\right)$ in Eq. 16.
The optimal order quantity is $Q^* = I_R(0) + W - I_O\left(T\right)$, where $I_R(0)$ and $I_O\left(T\right)$ can be calculated by using Eq. 6.

## 4.2 Maximization of $TP_2\left(t_1, X_1, T, p\right)$

Since $W = \int_{X_1}^{t_d} g(p)f(t)dt + e^{-\theta_2 t_d}\int_{t_d}^{t_1} e^{\theta_2 t} g(p)f(t)dt$ (i.e. Eq. 22), we can solve with respect to $X_1$. Hence, we can consider $X_1 = X_1\left(t_1\right)$ a function of $t_1$. Consequently, the profit function becomes again a function of three variables $TP_2\left(t_1, X_1\left(t_1\right), T, p\right) \equiv TP_2(t_1, T, p)$. As in section 4.1, by starting with $p_{\min}$ and using a step of increase say $p_{in}$ we find the optimal $t_1$ and $T$ for any given $p \in [p_{\min}, p_{\max}]$. From the first order conditions for a maximum $(t_1(p), T(p))$ follows as the solution of the equations:

$TP_2\left(t_1, T, p\right) =$

$$\left(p + \pi - c\right)\int_{t_1}^{T} g(p)f(t)\frac{\partial B(T-t)}{\partial T}dt + + (p-c)g(p)f(T) - s\int_{t_1}^{T}\int_{t_1}^{t} g(p)f(u)\frac{\partial B(T-u)}{\partial T}dudt - s\int_{t_1}^{T} g(p)f(t)B(T-t)dt \quad (37)$$

$$\left(p - c + \pi\right)\left(1 - B(T-t_1)\right) + s\left(T-t_1\right)B(T-t_1) = \left(c + \frac{h_2}{\theta_2} + K\right)\left(e^{\theta_2(t_1-t_d)} - 1\right) + \left(h_1 X_1 + h_2\left(t_d - X_1\right) + c_t\right)e^{\theta_2(t_1-t_d)} \quad (38)$$

In the above we make use of the relation:

$$\frac{\partial X_1}{\partial t_1} = e^{\theta_2(t_1-t_d)}\frac{f(t_1)}{f(X_1)} \quad (39)$$

We can prove (to save space the proof is omitted) that

**Proposition 2**. For any given $p \in [p_{\min}, p_{\max}]$, there exist unique $(t_1(p), T(p))$.

Now let $p^* = \arg\max_p \{TP_2\left(t_1(p), T(p), p\right), p \in [p_{\min}, p_{\max}]\}$ denote the value of $p$ which gives the maximum $TP_2\left(t_1(p), T(p), p\right), p \in [p_{\min}, p_{\max}]$ the optimal values of $(t_1, T)$ are $(t_1^*, T^*) = (t_1(p^*), T(p^*))$.

The optimal value of $X_1$, i.e. $X_1^*$, can then be obtained by setting $t_1 = t_1^*$ in Eq. 22.

The maximum profit $TP_2\left(t_1^*, X_1^*, T^*, p^*\right)$ can be calculated by substituting $\left(t_1^*, X_1^*, T^*, p^*\right)$ in Eq.25.

The optimal order quantity is $Q^* = I_R(0) + W - I_O\left(T\right)$, where $I_R(0)$ and $I_O\left(T\right)$ can be calculated by using Eq. 21.

## 4.3 Maximization of $TP_3\left(t_1, X_1, T, p\right)$

Since $W = \int_{X_1}^{t_1} g(p)f(u)du$ (i.e. Eq. 30), we can solve with respect to $X_1$. Hence, we can consider $X_1 = X_1\left(t_1\right)$ a function of $t_1$. Again, the profit function becomes a function of three variables $TP_3\left(t_1, X_1\left(t_1\right), T, p\right) \equiv TP_3\left(t_1, T, p\right)$. As in the previous cases, by starting with $p_{\min}$ and using a step of increase say $p_{in}$ we find the optimal $t_1$ and $T$ for any given $p \in [p_{\min}, p_{\max}]$. From the first order conditions for a maximum $(t_1(p), T(p))$ follows as the solution of the equations:

$TP_3\left(t_1, T, p\right) =$

$$\left(p + \pi - c\right)\int_{t_1}^{T} g(p)f(t)\frac{\partial B(T-t)}{\partial T}dt + + (p-c)g(p)f(T) - s\int_{t_1}^{T}\int_{t_1}^{t} g(p)f(u)\frac{\partial B(T-u)}{\partial T}dudt - s\int_{t_1}^{T} g(p)f(t)B(T-t)dt \quad (40)$$

$$\left(p - c + \pi\right)\left(1 - B(T-t_1)\right) + s\left(T-t_1\right)B(T-t_1) = h_1 X_1 + h_2\left(t_d - X_1\right) + c_t \quad (41)$$

In the above we make use of the relation:

$$\frac{\partial X_1}{\partial t_1} = \frac{f(t_1)}{f(X_1)} \quad (42)$$

We can prove (to save space the proof is omitted) that

**Proposition 3**. For any given $p \in [p_{\min}, p_{\max}]$, there exist unique $(t_1(p), T(p))$.

Now let $p^* = \arg\max_p \{TP_3(t_1(p), T(p), p), p \in [p_{\min}, p_{\max}]\}$ denote the value of $p$ which gives the maximum $TP_3(t_1(p), T(p), p), p \in [p_{\min}, p_{\max}]$ the optimal values of $(t_1, T)$ are $(t_1^*, T^*) = (t_1(p^*), T(p^*))$.

The optimal value of $X_1$, i.e. $X_1^*$, can then be obtained by setting $t_1 = t_1^*$ in Eq. 30.

The maximum profit $TP_3(t_1^*, X_1^*, T^*, p^*)$ can be calculated by substituting $(t_1^*, X_1^*, T^*, p^*)$ in Eq.33.

The optimal order quantity is $Q^* = I_R(0) + W - I_O(T)$, where $I_R(0)$ and $I_O(T)$ can be calculated by using Eq. 29.

## 5. SOLUTION PROCEDURE

In this section we present the procedure for the determination of the optimal replenishment and pricing for the proposed inventory model.

For a given capacity $W$ of the OW, first we use the one warehouse model developed by Valliathal and Uthayakumar (2010) to calculate the maximum inventory level. If the maximum inventory level is less or equal than the warehouse capacity (i.e. $W$ ) then the optimal solution has been obtained. Otherwise, the two warehouse model proposed here is used. The optimization procedure is summarized as follows:

**Step 1** Maximize separately the $TP_1(t_1, X_1, T, p)$, $TP_2(t_1, X_1, T, p)$, $TP_3(t_1, X_1, T, p)$ as shown in sections 4.1-4.3.

**Step 2** Compare the values of $TP_1(t_1^*, X_1^*, T^*, p^*)$, $TP_2(t_1^*, X_1^*, T^*, p^*)$, $TP_3(t_1^*, X_1^*, T^*, p^*)$ found in Step 1 to obtain the optimal solution for the problem

## 6. NUMERICAL EXAMPLES

In this section we give a numerical example to illustrate the theoretical results. Let $A = 50$, $c = 2$, $s = 5$, $t_d = 1.5$, $f(t) = e^{-0.2t}$, $\pi = 1.5$, $h_1 = 0.8$, $h_2 = 0.6$, $\theta_1 = 0.06$, $\theta_2 = 0.08$, $K = 0.06$, $c_t = 0.5$, $B(x) = e^{-0.02x}$, $g(p) = 30 - 0.5p$, $W$ =10. By using the model developed by Valliathal and Uthayakumar (2010) the optimal maximum inventory level is $Q^* = 13.7789 > W$. Consequently, the two warehouse model must be used. By using the above solution procedure we obtain the computational results as shown in Table 1.

Table 1. Optimal Solutions for Cases A-C

| | $p^*$ | $t_1^*$ | $T^*$ | $X_1^*$ | $Q^*$ | $TP^*$ |
|---|---|---|---|---|---|---|
| Case A | 30.5458 | 2.471 | 2.471 | 1.5 | 29.085 | 298.13 |
| Case B | 31.5122 | 1.5 | 1.5 | 0.6322 | 18.4588 | 321.41 |
| Case C | 31.364 | 0.88 | 1.069 | 0.11 | 13.774 | 327.133 |

Consequently, by comparing the results in Table 1, the optimal solution to the problem is:

$(T^*, t_1^*, X_1^*, p^*) = (1.069, 0.88, 0.11, 31.364)$, $Q^* = 13.774$, $TP^* = 327.133$

Table 2. Single parameter Sensitivity analyses

| Parameter | Value | $p^*$ | $t_1^*$ | $T^*$ | $X_1^*$ | $Q^*$ | TP |
|---|---|---|---|---|---|---|---|
| $t_d$ | 0.75 | 31.36 | 0.87 | 1.065 | 0.104 | 11.47 | 327.111 |
| | 2 | 31.36 | 0.88 | 1.069 | 0.11 | 11.57 | 327.133 |
| $W$ | 5 | 31.39 | 0.89 | 1.09 | 0.49 | 16.68 | 324.546 |
| | 15 | 31.1585 | 1.05261 | 1.14798 | 0 | 13.7789 | 328.042 |

# 7. CONCLUSION

Recently, Valliathal and Uthayakumar (2010) formulated an inventory model for non-instantaneous deteriorating items with partial backlogging. In this paper, we have extended their model to a two-warehouse inventory system, assuming that the OW has a fixed capacity and the RW has unlimited capacity. Depending on the time point at which the deterioration of the items beggins three cases arise which are examined seperately. The optimal solution is obtained by maximizing each case and comparing the results. Conducting a simple sensitivity analyses to the most important parameters ($W$ and $t_d$) we could observe that as $t_d$ increases the optimal price remains unchangeable while all other variables and total profit increase. Also the optimal price decreases as the capacity of OW increases while all other variables and total profit increase.

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# Maximum Induced Matchings in Grids

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**Abstract**

An induced matching in a graph is a matching such that no two edges are joined by an edge of $G$. For a connected graph $G$, denote by $i\mu(G)$ the maximum cardinality of an induced matching in $G$. In this paper we study the problem of finding a maximum induced matching in grid graphs with $n$ lines and $m$ columns - $G_{n,m}$, and determine the exact value for $i\mu(G_{n,m})$ when $n$ or $m$ are even.

**Keywords** maximum induced matching, strong matching, grid graph, subgrid

## 1 Introduction

Let $G = (V, E)$ be a connected graph. A *matching* in $G$ is a subset of edges $M \subseteq E$ such that no two edges of $M$ are adjacent. A matching $M$ in $G$ is called *induced* if no two edges of $M$ are joined by an edge of $G$. Denote by $i\mu(G)$ the maximum cardinality of an induced matching in $G$. A *maximum induced matching* in $G$ is an induced matching with $i\mu(G)$ edges [6, 2]. The problem of finding a maximum induced matching of a graph (MIM) was introduced as a variation of the maximum matching problem, motivated by the "risk-free" marriage problem. Induced matchings have many applications, for example in networking ([5, 1]) and discrete mathematics, since finding large induced matchings is a subtask of finding a strong edge-colouring in a graph ([4]). MIM is proved to be NP-complete even for bipartite graphs of maximum degree four [6] and for $k$-regular bipartite graph for any $k \geq 3$ ([3]).

Denote by $G_{n,m}$ the grid graph with $n$ lines and $m$ columns. It is known that the problem of finding a maximum induced matching in subgrids is NP-hard [1]. In this paper we study the maximum induced problem in grids and determine the exact value for $i\mu(G_{n,m})$ when $n$ or $m$ are even.

## 2 Maximum induced matching in grids

Let $G_{n,m}$ be the grid graph with $n$ lines and $m$ columns and $M$ a matching in $G_{n,m}$. A vertex $v \in V$ is called *$M$-saturated* if $v \in V(M)$ and *$M$-unsaturated* otherwise. If $H$ is an induced subgraph in $G_{n,m}$, we denote by $s_M(H) = |V(M) \cap V(H)|$ the number of $M$-saturated vertices of $H$. Also, we denote by $s\mu(G_{n,m})$ the maximum number of vertices that can be saturated in a subgrid isomorphic to $G_{n,m}$ of a grid by an induced matching of the grid:

$$s\mu(G_{n,m}) = \max\{s_M(H) | H \cong G_{n,m} \text{ subgrid in a grid } G_{p,q},\ M \text{ induced matching in } G_{p,q}\}.$$

Then $s\mu(G_{n,m}) = s\mu(G_{m,n})$ and $i\mu(G_{m,n}) = i\mu(G_{n,m}) \leq \left\lfloor \frac{s\mu(G_{n,m})}{2} \right\rfloor$.

Let $H_1, \ldots, H_p$ be subgraphs in $G_{n,m}$. We say that $(H_1, \ldots, H_p)$ is a partition of $G_{n,m}$ if the subgraphs are vertex-disjoint and $V(G_{n,m}) = \bigcup_{i=1}^{p} V(H_i)$.

Partition $(H_1, \ldots, H_p)$ is called a $(k_1 G_{n_1,m_1}, k_2 G_{n_2,m_2}, \ldots, k_q G_{n_q,m_q})$-partition of $G_{n,m}$ if $k_i$ subgraphs from partition are isomorphic to $G_{n_i,m_i}$, for $1 \leq i \leq q$.

**Lemma 2.1.** *The following properties hold:*
*a)* $s\mu(G_{3,3}) = 5$ *and equality holds only for two configurations (modulo a rotation), shown in Figure 1 (a).*
*b)* $s\mu(G_{3,1}) = 2$*; c)* $s\mu(G_{3,2}) = 4$*; d)* $s\mu(G_{3,4}) = 6$*; e)* $s\mu(G_{2,2}) = 2$.

*Proof.* As illustrated in Figure 1, we have $s\mu(G_{3,3}) \geq 5$, $s\mu(G_{3,1}) \geq 2$, $s\mu(G_{3,2}) \geq 4$, $s\mu(G_{3,4}) \geq 6$, $s\mu(G_{2,2}) \geq 2$. Next we prove the reverse inequalities.

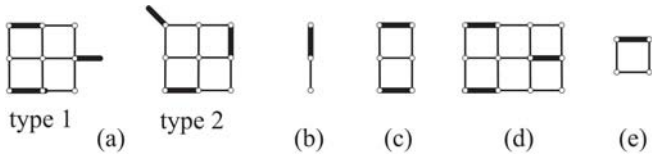


Figure 1

Consider a grid $G$ and $M$ an induced matching in $G$. Let $H$ be a subgrid isomorphic to $G_{p,q}$ in this grid. It is easy to see that, by definition of an induced matching, if $u, v, w$ are 3 distinct vertices such that $uv, vw \in E(H)$ then $|\{u, v, w\} \cap V(M)| \leq 2$. Using this remark affirmation e) follows. Moreover, if $H \cong G_{3,p}$, then at most 2 vertices from each column of $H$ belong to $V(M)$, hence $s_M(H) \leq 2p$. For $p = 1$ and $p = 2$ affirmations b) and c) follow.

For a column $i$ of $H$ denote by $s_i$ the number of $M$-saturated vertices from column $i$.

If $p = 3$, denote by $y_i^1, y_i^2, y_i^3$ the vertices from column $i$ of graph $H \cong G_{3,3}$. For every column $i$ we have $s_i \leq 2$. If there exists a column $j$ with $s_j \leq 1$, then $s_M(H) = s_1 + s_2 + s_3 \leq 5$. Assume now that $s_i = 2$ for every $1 \leq i \leq 3$. Consider the $M$-saturated vertices from column 1. If $y_1^1$ and $y_1^2$ are $M$-saturated, then $y_1^1y_1^2 \in M$ and both vertices $y_2^1$, $y_2^2$ are $M$-unsaturated, hence $s_2 \leq 1$, contradiction. If $y_1^1$ and $y_1^3$ are $M$-saturated then, since on column 2 there are 2 $M$-saturated vertices, it follows that $y_1^1y_2^1 \in M$ and $y_1^3y_2^3 \in M$. Then $y_3^1$ and $y_3^3$ are $M$-unsaturated, contradiction with the assumption that $s_3 = 2$. Hence there exists a column $j \in \{1, 2, 3\}$ with $s_j \leq 1$, and we obtain $s_M(H) \leq 5$. It follows that $s\mu(G_{3,3}) \leq 5$ and the equality holds only for two type of configurations (modulo a rotation), shown in Figure 1 (a).

Moreover, it follows that in a subgrid isomorphic to $G_{3,p}$ we can not have 3 consecutive columns, with 2 $M$-saturated vertices each.

Consider $p = 4$. If there exist two distinct columns $j$ and $k$ such that $s_j \leq 1$ and $s_k \leq 1$ then $s_M(H) = s_1 + s_2 + s_3 + s_4 \leq 2 + 2 + 1 + 1 = 6$. Assume now that there exists at most one column $j$ with $s_j \leq 1$. Since we proved that in $H$ we can not have 3 consecutive column such that each column has 2 saturated vertices, it suffice to consider the case when $s_1 = s_2 = 2$, $s_3 = 1$ and $s_4 = 2$. Then we must have $y_1^1y_2^1 \in M$ and $y_1^3y_2^3 \in M$. It follows that $y_3^1$ and $y_3^3$ are $M$-unsaturated, hence $y_3^2$ is $M$-saturated. Then $y_3^2y_4^2 \in M$ and we obtain $s_4 = 1$, contradiction. It follows that $s\mu(G_{3,4}) \leq 6$. □

**Theorem 2.2.** *Let* $m, n \geq 2$ *be two positive integers, with* $m$ *even. Then*

$$s\mu(G_{n,m}) = \begin{cases} \frac{mn+2}{2}, & \text{if } n \text{ is odd and } m = 4k+2 \\ \frac{mn}{2}, & \text{otherwise} \end{cases} \quad \text{and} \quad i\mu(G_{n,m}) = \left\lceil \frac{mn}{4} \right\rceil.$$

*Proof.* We have $i\mu(G_{n,m}) \leq \frac{s\mu(G_{n,m})}{2}$.
**Case 1.** If $n$ is even then we consider a $(\frac{mn}{4}G_{2,2})$-partition of $G_{n,m}$ (for example see Figure 2 (a)). Using Lemma 2.1 for every subgraph $H \cong G_{2,2}$ from the partition we obtain $s\mu(G_{n,m}) \leq 2 \cdot \frac{mn}{4} = \frac{mn}{2}$. The matching $M'$ described in Figure 2 (a) is an induced matching in $G_{n,m}$ which saturates $\frac{mn}{2}$ vertices, hence $i\mu(G_{n,m}) = \frac{s\mu(G_{n,m})}{2} = \frac{mn}{4}$.

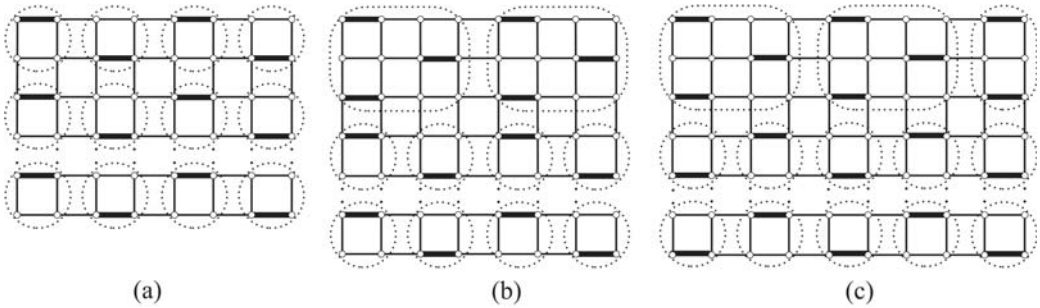


Figure 2: Maximum induced matchings

**Case** 2. If $n$ is odd and $m = 4k$ there exists a $(\frac{m}{4}G_{3,4}, \frac{m(n-3)}{4}G_{2,2})$-partition of $G_{n,m}$ (for example Figure 2 (b)). Then from Lemma 2.1 for subgraphs isomorphic to $G_{2,2}$ and $G_{3,4}$, we have $s\mu(G_{n,m}) \leq 6 \cdot \frac{m}{4} + 2 \cdot \frac{m(n-3)}{4} = \frac{mn}{2}$. The matching $M'$ described in Figure 2 (b) is an induced matching which saturates $\frac{mn}{2}$ vertices, hence again we obtain $i\mu(G_{n,m}) = \frac{s\mu(G_{n,m})}{2} = \frac{mn}{4}$.

**Case** 3. If $n$ is odd and $m = 4k + 2$, then we consider a $(\frac{m-2}{4}G_{3,4}, G_{3,2}, \frac{m(n-3)}{4}G_{2,2})$-partition of $G_{n,m}$ (for example Figure 2 (c)). Using again Lemma 2.1 we have $s\mu(G_{n,m}) \leq 6 \cdot \frac{m-2}{4} + 4 + 2 \cdot \frac{m(n-3)}{4} = \frac{mn+2}{2}$. An induced matching which saturates $\frac{mn+2}{2}$ vertices is described in Figure 2 (c), hence in this case we have $i\mu(G_{n,m}) = \frac{s\mu(G_{n,m})}{2} = \frac{mn+2}{4} = \left\lceil \frac{mn}{4} \right\rceil$. □

**Theorem 2.3.** *If $m \geq 3$ is an odd positive integer, then*

$$s\mu(G_{3,m}) = \frac{3m+1}{2} \quad \text{and} \quad i\mu(G_{3,m}) = \begin{cases} \frac{3(m-1)+2}{4}, & \text{if } m = 4k+3 \\ \frac{3(m-1)}{4} + 1, & \text{otherwise.} \end{cases}$$

*Proof.* If $m = 4k + 1$ then there exists a $(\frac{m-1}{4}G_{3,4}, G_{3,1})$-partition of $G_{3,m}$ (for example see Figure 3 (a)). By Lemma 2.1 it follows that $s\mu(G_{3,m}) \leq 6 \cdot \frac{m-1}{4} + 2 = \frac{3m+1}{2}$. An induced matching in $G_{3,m}$ that saturates $\frac{3m+1}{2}$ vertices is described in Figure 3 (a).

We obtain $i\mu(G_{3,m}) = \frac{s\mu(G_{3,m})}{2} = \frac{3m+1}{4}$.

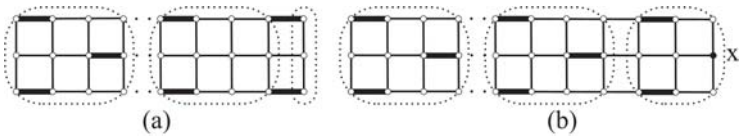


Figure 3: $G_{3,m}$

If $m = 4k + 3$ then we consider a $(\frac{m-3}{4}G_{3,4}, G_{3,3})$-partition of $G_{3,m}$ (for example see Figure 3 (b)). Again, by Lemma 2.1, it follows that an induced matching in a grid can saturate at most $6 \cdot \frac{m-3}{4} + 5 = \frac{3m+1}{2}$ vertices of a subgrid isomorphic to $G_{3,m}$. Since this bound is odd and a matching in a graph saturates an even number of vertices, it follows that $i\mu(G_{3,m}) \leq \frac{s\mu(G_{3,m})-1}{2} \leq \frac{3m-1}{4}$. An induced matching in $G_{3,m}$ that saturates $\frac{3m-1}{2}$ vertices is described in Figure 3 (b). Moreover, if we consider $G_{3,m}$ as subgrid in a grid, this matching can be easily extended such that vertex $x$ is also saturated. Hence we obtain $s\mu(G_{3,m}) = \frac{3m+1}{2}$ and $i\mu(G_{3,m}) = \frac{3m-1}{4}$. □

**Theorem 2.4.** *Let $n, m$ be two odd integers with $m \geq n \geq 5$. Then*

$$i\mu(G_{n,m}) \geq \begin{cases} \frac{n(m-1)}{4} + 1 & \text{if } m = 4k+1 \\ \frac{n(m-1)+2}{4} & \text{if } m = 4k+3 \text{ and } n < \frac{m+5}{2} \\ \frac{n(m-1)+2}{4} + 1 & \text{if } m = 4k+3 \text{ and } \frac{m+5}{2} \leq n. \end{cases}$$

*Proof.* If $m = 4k+1$, let $M$ be the matching described in Figure 4 (a). By considering the $(\frac{m-1}{4}G_{3,4}, G_{3,1}, \frac{(n-3)(m-3)}{4}G_{2,2}, \frac{n-3}{2}G_{2,3})$-partition of $G_{n,m}$ suggested in the figure, it is easy to see that the number of $M$-saturated vertices is

$$s_M(G_{n,m}) = 6 \cdot \frac{m-1}{4} + 2 + 2 \cdot \frac{m-3}{2} \cdot \frac{n-3}{2} + 2 \cdot \frac{n-3}{2} = \frac{n(m-1)+4}{2}$$

Hence $i\mu(G_{n,m}) \geq |M| = \dfrac{s_M(G_{n,m})}{2} = \dfrac{n(m-1)}{4} + 1.$

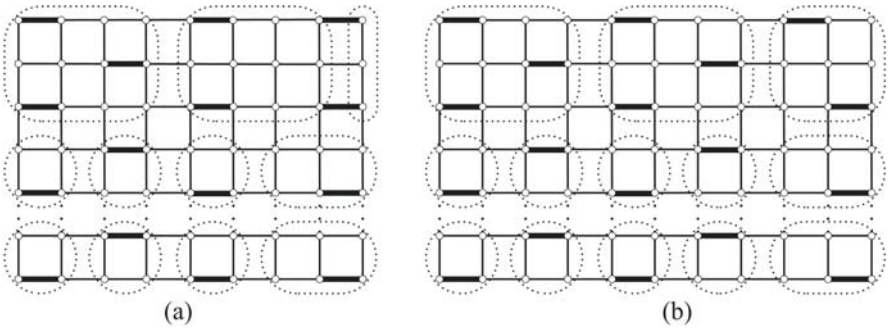


Figure 4: Maximum induced matchings

Assume now that $m = 4k+3$. If $n < \frac{m+5}{2}$, let $M$ be the matching described in Figure 4 (b). By considering the $(\frac{m-3}{4}G_{3,4}, G_{3,2}, \frac{(n-3)(m-3)}{4}G_{2,2}, \frac{n-3}{2}G_{2,3})$-partition of $G_{n,m}$ suggested in the figure, we obtain that the number of $M$-saturated vertices is

$$s_M(G_{n,m}) = 6 \cdot \frac{m-3}{4} + 4 + 2 \cdot \frac{n-3}{2} + 2 \cdot \frac{m-3}{2} \cdot \frac{n-3}{2} = \frac{n(m-1)+2}{2}$$

Hence $i\mu(G_{n,m}) \geq |M| = \dfrac{s_M(G_{n,m})}{2} = \dfrac{n(m-1)+2}{4}$.

If $n \geq \left\lceil \frac{m+5}{2} \right\rceil$ let $M$ be the matching described in Figure 5. By considering the $(G_{\frac{m+5}{2}}, (n - \frac{m+5}{2} - 1)G_{2,m}, G_{1,m})$-partition of $G_{n,m}$ shown in figure and counting the number of saturated vertices from each subgraph of partition we obtain

$$|M| = \frac{s_M(G_{n,m})}{2} = \frac{(m-1)(m+5)}{8} + 1 + \frac{1}{2}\left(n - \frac{m+5}{2} - 1\right)\frac{m-1}{2} + \frac{m+1}{4} = \frac{n(m-1)+6}{4}.$$

□

**Theorem 2.5.** *Let $m \geq 5$ be an odd positive integer. Then*

*a)* $s\mu(G_{5,m}) = \dfrac{5m+1}{2}$ *and* $i\mu(G_{5,m}) = \begin{cases} \frac{5(m-1)+2}{4}, & \text{if } m = 4k+3 \\ \frac{5(m-1)}{4} + 1, & \text{otherwise.} \end{cases}$

*b) If* $m = 4k+1$ *then* $s\mu(G_{7,m}) = \dfrac{7m-1}{2}$ *and* $i\mu(G_{7,m}) = \dfrac{7(m-1)}{4} + 1.$

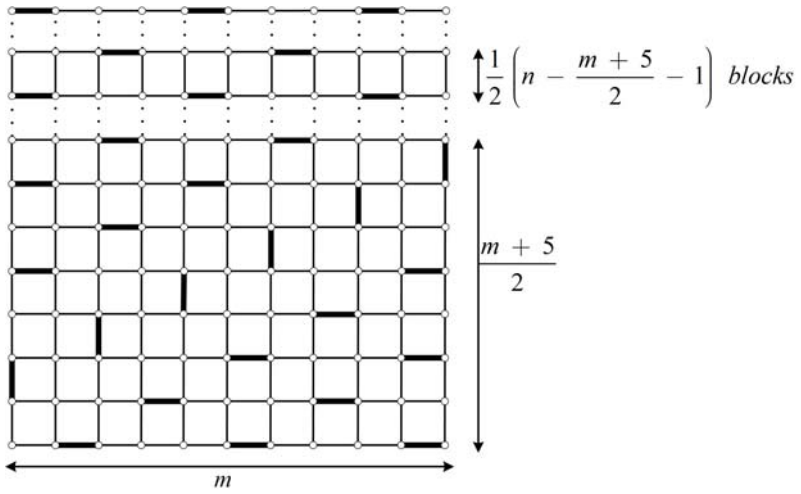


Figure 5

*c) If* $m = 4k+3$ *then* $s\mu(G_{7,m}) = \dfrac{7m+3}{2}$, $i\mu(G_{7,7}) = 12$ *and* $\dfrac{7(m-1)+2}{4} \leq i\mu(G_{7,m}) \leq \dfrac{7(m-1)+2}{4} + 1$.

*Proof.* Let $G = G_{p,q}$ be a grid and $M$ an induced matching in $G$. Let $H$ be a subgraph isomorphic to $G_{n,m}$ in $G$. Denote by $y_j^i$ the vertex from line $i$ and column $j$ in $H$ and by $G_{j:k}^{i:s}$ the subgrid induced in $H$ by vertices $y_b^a$ with $i \leq a \leq s$, $j \leq b \leq k$.

a) Let $n = 5$. We consider two $(G_{2,m}, G_{3,m})$-partitions of $H$: first partition determined by subgraphs $H_1 = G_{1:m}^{1:2} \cong G_{2,m}$ induced by the first two lines, and $H_2 = G_{1:m}^{3:5} \cong G_{3,m}$ induced by the remaining 3 lines, and second determined by subgraphs $H_1' = G_{1:m}^{4:5} \cong G_{2,m}$ induced by the last two lines, and $H_2' = G_{1:m}^{1:3} \cong G_{3,m}$ . We have $s_M(H) \leq s_M(H_1) + s_M(H_2)$. By Theorems 2.2 and 2.3, $s_M(H_1) \leq m+1$ and $s_M(H_2) \leq \frac{3m+1}{2}$.

**Case 1.** $m = 4k+1$

Consider first $m = 5$.

Assume that $s_M(H_1) = m+1 = 6$ and $s_M(H_2) = \frac{3m+1}{2} = 8$. We consider the $(G_{2,2}, G_{2,3})$-partition of $H_1$ determined by $G_{1:2}^{1:2}$ and $G_{3:5}^{1:2}$. In order to have $s_M(H_1) = 6 = s\mu(G_{2,5})$, we must have $s_M(G_{3:5}^{1:2}) = s\mu(G_{2,3}) = 4$ and $s_M(G_{1:2}^{1:2}) = s\mu(G_{2,2}) = 2$. It follows that $y_3^1y_3^2, y_5^1y_5^2 \in M$ and then $y_1^1y_1^2 \in M$. Moreover, since $s_M(G_{1:4}^{3:5}) \leq s\mu(G_{3,4}) = 6$ and $s_M(G_{5:5}^{3:5}) \leq s\mu(G_{3,1}) = 2$, in order to have $s_M(H_2) = 8$ we must have $s_M(G_{1:4}^{3:5}) = 6$ and $s_M(G_{5:5}^{3:5}) = 2$. It follows that $y_5^4y_5^5 \in M$. But then vertices $y_1^3, y_3^3, y_4^4, y_4^5$ are unsaturated, and it is easy to see that in this situation $s_M(G_{1:4}^{3:5}) < 6$, contradiction.

It follows that $s_M(H) \leq s_M(H_1) + s_M(H_2) - 1 = 13$ and then $s\mu(G_{5,5}) \leq 13$. The matching illustrated in Figure 6 saturates 13 vertices in a subgrid isomorphic to $G_{5,5}$, hence $s\mu(G_{5,5}) = 13$.

If $m > 5$, since $m-5$ is even, by Theorem 2.2 we have

$$s\mu(G_{5,m}) \leq s\mu(G_{5,5}) + s\mu(G_{5,m-5}) = 13 + \frac{5(m-5)}{2} = \frac{5m+1}{2}.$$

As illustrated in Figure 6 there exists an induced matching which saturates $\frac{5m+1}{2}$ vertices (having only vertical edges), hence $s\mu(G_{5,m}) = \frac{5m+1}{2}$.

Then we also have $i\mu(G_{5,m}) \leq \left\lfloor \frac{s\mu(G_{5,m})}{2} \right\rfloor = \left\lfloor \frac{5m+1}{4} \right\rfloor = \frac{5m-1}{4}$. By Theorem 2.4 we obtain $i\mu(G_{5,m}) \geq \frac{5(m-1)}{4} + 1 = \frac{5m-1}{4}$, hence $i\mu(G_{5,m}) = \frac{5m-1}{4}$.

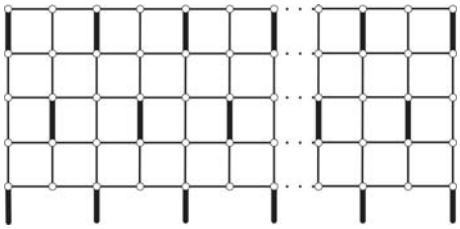

Figure 6

**Case** 2. $m = 4k + 3$

Consider first $m = 7$.

We will prove that $s_M(H) \leq 18$ and determine the configurations for which equality holds.

If $s_M(H_1) < 7$ or ($s_M(H_1) = 7$ and $s_M(H_2) < 11$) then $s_M(H) \leq 17$. Then it suffices to consider the following situations.

**2.a.** $s_M(H_1) = m + 1 = 8$ or $s_M(H'_1) = 8$.

By symmetry, it suffices to assume $s_M(H_1) = 8$. Considering a $(2G_{2,2}, G_{2,3})$-partition of $H_1$, as in previous case, it follows that we must have $y_1^1y_1^2, y_3^1y_3^2, y_5^1y_5^2, y_7^1y_7^2 \in M$.

Assume $s_M(H_2) = \frac{3m+1}{2} = 11$. Since $s_M(G_{1:3}^{3:5}) \leq s\mu(G_{3,3}) = 5$ and $s_M(G_{4:7}^{3:5}) \leq s\mu(G_{3,4}) = 6$, in order to have $s_M(H_2) = 11$ we must have $s_M(G_{1:3}^{3:5}) = 5$. That is impossible since vertices $y_1^3, y_3^3$ are unsaturated (Lemma 2.1).

It follows that $s_M(H_2) \leq 10$, hence $s_M(H) \leq 18$. Moreover, if we have $s_M(H_2) = 10$, then we must have $s_M(G_{1:3}^{3:5}) = 4$ and $s_M(G_{4:7}^{3:5}) = 6$ and this is possible only in situation illustrated in Figure 7 ($c_1$).

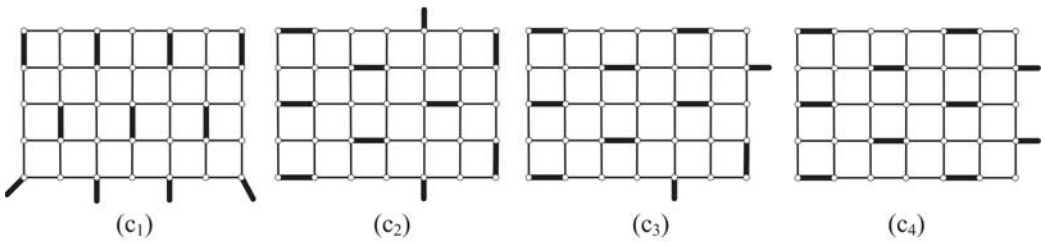


Figure 7: Possible configurations for $s\mu(G_{5,7})$

**2.b.** $s_M(H_1) = s_M(H'_1) = 7$, $s_M(H_2) = s_M(H'_2) = 11$.

Considering $(G_{3,3}, G_{3,4})$ partitions of $H_2$ and $H'_2$, by Lemma 2.1 and Theorem 2.3 it follows that $s_M(G_{1:3}^{3:5}) = 5$ and $s_M(G_{4:7}^{3:5}) = 6$.

By symmetry, using similar arguments, we obtain $s_M(G_{5:7}^{3:5}) = s_M(G_{1:3}^{1:3}) = s_M(G_{5:7}^{1:3}) = 5$ and $s_M(G_{1:4}^{3:5}) = s_M(G_{4:7}^{1:3}) = s_M(G_{1:4}^{1:3}) = 6$.

Moreover, by Theorem 2.3 we have $s_M(G_{1:3}^{1:5}) \leq 8$, hence $s_M(G_{1:3}^{1:2}) = 3$. Using the same arguments we obtain that $s_M(G_{5:7}^{1:2}) = s_M(G_{1:3}^{4:5}) = s_M(G_{5:7}^{4:5}) = 3$.

It follows that $s_M(G_{3:3}^{4:5}) = s_M(G_{3:3}^{1:2}) = s_M(H_1) - 6 = 1$ and $s_M(G_{1:3}^{3:3}) = s_M(G_{5:7}^{3:3}) = 2$.

Then $y_3^1y_3^2, y_3^4y_3^5, y_5^1y_5^2, y_5^4y_5^5 \notin M$. Moreover, since subgrids $G_{1:3}^{3:5}, G_{5:7}^{3:5}, G_{1:3}^{1:3}, G_{5:7}^{1:3}$ are isomorphic to $G_{3,3}$, by Lemma 2.1, there exist only two type of configurations possible for each of these subgrids (modulo a rotation), as shown in Figure 1 (a). It is easy to see that not all configurations can be of type 2 (since $y_3^1y_3^2, y_3^4y_3^5, y_5^1y_5^2, y_5^4y_5^5 \notin M$). Then, by symmetry, it suffices to consider the situations when $G_{1:3}^{3:5}$ has a configuration of type 1. By a simple analysis of these situations, it follows that there are only 3 possible configurations for edges of $M$ such that the

determined number of saturated vertices are reached, shown in Figure 7 $(c_2)$, $(c_3)$, $(c_4)$.

In all situations we obtain that $s_M(H) \leq 18$, hence $s\mu(G_{5,7}) \leq 18$.

Assume now that $m > 7$. Then, since $m-5$ is even, using Theorem 2.2 we obtain

$$s\mu(G_{5,m}) \leq s\mu(G_{5,7}) + s\mu(G_{5,m-7}) = 18 + \frac{5(m-7)}{2} = \frac{5m+1}{2}.$$

As illustrated in Figure 6 there exists an induced matching which saturates $\frac{5m+1}{2}$ vertices, hence $s\mu(G_{5,m}) = \frac{5m+1}{2}$.

If $H = G$ (i.e. $p = 5$ and $q = m$) we have

$$s_M(H) \leq s_M(G_{1:7}^{1:5}) + s_M(G_{8:m}^{1:5}) \leq 18 + s\mu(G_{5,m-7}) = \frac{5m+1}{2},$$

and equality can hold only if configuration $(c_4)$ occurs for $G_{1:7}^{1:5}$ and $s_M(G_{8:m}^{1:5}) = \frac{5(m-7)}{2}$.

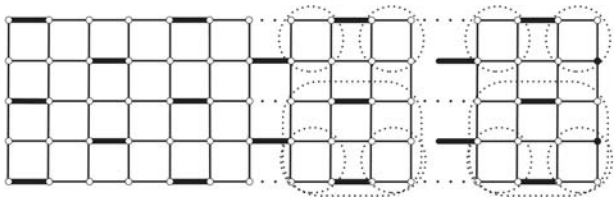

Figure 8

But, if we consider partitions of $G_{8:m}^{1:5}$ into subgrids isomorphic to $G_{2,2}$ and $G_{3,4}$, as shown in Figure 8, then each subgrid isomorphic to $G_{2,2}$ must have 2 saturated vertices and each subgrid isomorphic to $G_{3,4}$ must have 6 saturated vertices. It follows that all edges of $M$ must be horizontal and then $s_M(G_{m-1:m}^{1:2}) < 2$, contradiction. Hence $s_M(H) \leq \frac{5m+1}{2} - 1 = \frac{5m-1}{2}$. If $M$ is a maximum induced matching in $G$ we obtain $i\mu(G_{5,m}) \leq \left\lfloor \frac{s_M(H)}{2} \right\rfloor \leq \left\lfloor \frac{5m-1}{4} \right\rfloor = \frac{5m-3}{4}$. From Theorem 2.4 we have that $i\mu(G_{5,m}) \geq \frac{5m-3}{4}$, hence equality holds.

b) If $n = 7$ and $m = 4k+1 \geq 7$ then consider $G_1 \cong G_{1:5}^{1:7}$ and $G_2 \cong G_{6:m}^{1:7}$ a $(G_{7,5}, G_{7,m-5})$-partition of $G_{7,m}$. Since $m-5$ is a multiple of 4, $s_M(G) \leq s\mu(G_{7,5}) + s\mu(G_{7,m-5}) = 18 + \frac{7(m-5)}{2}$ and it is easy to see that equality holds for a matching with all edges horizontal, as suggested for $m = 5$ (Figure 8). If $H = G$ equality can hold only if configuration $(c_1)$ (rotated) occurs for $G_1$ and $s_M(G_2) = \frac{7(m-5)}{2}$. Using the same arguments as for $n = p = 5$, if we consider a partition of $G_2$ into subgrids isomorphic to $G_{2,2}$ and $G_{3,4}$, then each subgrid isomorphic to $G_{2,2}$ must have 2 saturated vertices and each subgrid isomorphic to $G_{3,4}$ must have 6 saturated vertices. It follows that all edges of $M$ must be horizontal and then $s_M(G_{m-1:m}^{1:2}) < 2$, contradiction. Hence $s_M(G) \leq 17 + \frac{7(m-5)}{2} = \frac{7m-1}{2}$. Then $i\mu(G_{7,m}) \leq \left\lfloor \frac{7m-1}{4} \right\rfloor = \frac{7m-3}{4}$. By Theorem 2.4 we have $i\mu(G_{7,m}) \geq \frac{7(m-1)}{4} + 1 = \frac{7m-3}{4}$, hence $i\mu(G_{7,m}) = \frac{7m-3}{4}$, for $m = 4k+1$.

If $m = 4k+3$, using similar arguments, we obtain $s\mu(G_{7,m}) = 18 + \frac{7(m-5)+2}{2} = \frac{7m+3}{2}$ and $i\mu(G_{7,m}) \leq \left\lfloor \frac{s\mu(G_{7,m})-1}{2} \right\rfloor = \frac{7m-1}{4}$. By Theorem 2.4, the result follows. □

**Theorem 2.6.** *Let $n, m \geq 2$ be two odd integers. Then*

$$i\mu(G_{n,m}) \leq \left\lfloor \frac{nm+1}{4} \right\rfloor.$$

*Proof.* If $n = 4k + 3$ consider a $(G_{3,m}, \frac{n-3}{4}G_{4,m})$-partition of $G_{n,m}$.

Then $i\mu(G_{n,m}) \leq \frac{s\mu(G_{3,m})}{2} + \frac{n-3}{4} \cdot \frac{s\mu(G_{4,m})}{2}$. By Theorems 2.3 and 2.2 we obtain $i\mu(G_{n,m}) \leq \frac{3m+1}{4} + \frac{n-3}{4} \cdot \frac{4m}{4} = \frac{nm+1}{4}$.

If $n = 4k + 1$ consider a $(G_{5,m}, \frac{n-5}{4}G_{4,m})$-partition of $G_{n,m}$. Then $i\mu(G_{n,m}) \leq \frac{s\mu(G_{5,m})}{2} + \frac{n-5}{4} \cdot \frac{4m}{4}$, and by Theorems 2.5 and 2.2 we obtain $i\mu(G_{n,m}) \leq \frac{5m+1}{4} + \frac{n-5}{4} \cdot \frac{4m}{4} = \frac{nm+1}{4}$.

□

**Conjecture.** The lower bounds from Theorem 2.4 are actually the exact values for $i\mu(G_{n,m})$.

**Conclusion.** The main results obtained for $i\mu(G_{n,m})$, where $2 \leq n \leq m$ are summarized in the table below.

| | $m$ even | $m = 4k + 1$ | $m = 4k + 3$ |
|---|---|---|---|
| $n$ even | $\frac{mn}{4}$ | $\lceil \frac{mn}{4} \rceil$ | $\lceil \frac{mn}{4} \rceil$ |
| $n = 3$ | $\lceil \frac{mn}{4} \rceil$ | $\frac{3(m-1)}{4} + 1$ | $\frac{3(m-1)+2}{4}$ |
| $n = 5$ | $\lceil \frac{mn}{4} \rceil$ | $\frac{5(m-1)}{4} + 1$ | $\frac{5(m-1)+2}{4}$ |
| $n = 7$ | $\lceil \frac{mn}{4} \rceil$ | $\frac{7(m-1)}{4} + 1$ | $\in \left\{ \frac{7(m-1)+2}{4}, \frac{7(m-1)+2}{4} + 1 \right\}$ |
| $n \geq 9$ odd, $n < \frac{m+5}{2}$ | $\lceil \frac{mn}{4} \rceil$ | $\geq \frac{n(m-1)}{4} + 1$ | $\geq \frac{n(m-1)+2}{4}$ |
| $n \geq 9$ odd, $n \geq \frac{m+5}{2}$ | $\lceil \frac{mn}{4} \rceil$ | $\geq \frac{n(m-1)}{4} + 1$ | $\geq \frac{n(m-1)+2}{4} + 1$ |

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# A multi-GPU implementation and performance model for the standard simplex method

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**Abstract**

The standard simplex method is a well-known optimization algorithm for solving linear programming models in operations research. It is part of software often employed by businesses for solving scheduling or assignment problems. But their always increasing complexity and size drives the demand for more computational power. In the past few years, GPUs have gained a lot of popularity as they offer an opportunity to accelerate many algorithms. In this paper we present a mono and a multi-GPU implementation of the standard simplex method, which is based on CUDA. Measurements show that it outperforms the *CLP* solver on large enough problems. We also derive a performance model and establish its accurateness. To our knowledge, only the revised simplex method has so far been implemented on a GPU.

## 1 Introduction

The simplex method is a well-known optimization algorithm for solving linear programming (LP) models in operations research. It is part of software often employed by businesses for finding solutions to problems such as airline scheduling problems. The original standard simplex method was proposed by Dantzig in 1947 [1]. A more efficient method named the revised simplex, was later developed. Nowadays its sequential implementation can be found in almost all commercial LP solvers. But the always increasing complexity and size of LP problems from the industry, drives the demand for more computational power. Indeed, current implementations of the revised simplex strive to produce the expected results, if any. In this context, parallelisation is the natural idea to investigate [6]. Already in 1996, Thomadakis et al. [10] implemented the standard method on a massively parallel computer and obtained an increase in performances when solving dense or large problems.

A few years back, in order to meet the demand for processing power, graphics card vendors made their graphical processing units (GPU) available for general-purpose computing. Since then GPUs have gained a lot of popularity as they offer an opportunity to accelerate algorithms having an architecture well-adapted to the GPU model. The simplex method falls into this category. Indeed, GPUs exhibit a massively parallel architecture optimized for matrix processing. To our knowledge, only the revised simplex method has so far been implemented on a GPU [3], showing encouraging results on small to mid-size LP problems.

In this paper, we present a single and a multi-GPU implementation of the standard simplex method, based on the CUDA technology of NVIDIA. We also derive the associated performance model and establish its accurateness. There are many technicalities involved in CUDA programming, in particular regarding the management of tasks and memories on the GPU. Thus fine tuning is indispensable to avoid a breakdown on performance. With respect to the algorithm, special attention must be given to numerical stability.

The paper is organized as follows. First, we give a short description of the standard simplex method, also introducing the heuristics we used. Then we cover the basics of GPU architecture. Next, we focus on the single GPU implementation and then explain how we split the LP model

in the multiple GPU version. Two sections are then devoted to describing performance models and comparing our implementations with the *CLP* solver. Finally, we summarize the results and consider new perspectives. Note that a more detailed account of this work can be found in [7].

## 2 The standard simplex method

The simplex method [1] applies to linear programming models. For the sake of simplicity, let us consider the canonical form of such problems :

$$\begin{array}{ll} \text{maximize} & z = \mathbf{c^T} \cdot \mathbf{x} \\ \text{subject to} & \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0} \end{array} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{A}$ is the $m \times n$ constraint matrix. The objective function $z = \mathbf{c^T} \cdot \mathbf{x}$ is the product of the cost vector $\mathbf{c}$ and the unknown variables $\mathbf{x}$. An element $\mathbf{x}$ is called a solution, which is said to be a feasible if it satisfies the linear constraints imposed by $\mathbf{A}$ and the bound $\mathbf{b}$.

The first step of the simplex method is to reformulate the model. So called *slack variables* $x_{n+i}$ are added to the canonical form in order to replace the inequalities by equalities :

$$x_{n+i} = b_i - \sum_{j=1}^{n} A_{ij} x_j \quad (i = 1, 2, ..., m) \tag{2}$$

The resulting problem is called the *augmented form* in which the variables are divided into two disjoint sets: basic and nonbasic variables. Basic variables, which form the basis of the problem, are on the r.h.s. of the equations; nonbasic variables, which form the core, appear on the l.h.s.

The simplex algorithm searches for the optimal solution through an iterative process. A typical iteration consists of three operations : selecting the entering variable, choosing the leaving variable and pivoting. The entering variable is a nonbasic variable whose increase will lead to an augmentation of the objective value $z$. This variable must be selected with care in order to yield a substantial leap towards the optimal solution. The leaving variable is the basic variable which first violates its constraint as the entering variable increases. The choice of the leaving variable must guarantee that the solution remains feasible. Once both variables are defined, the pivoting operation makes these variables switch sets: the entering variable enters the basis, taking the place of the leaving variable which becomes nonbasic.

However, specific heuristics or methods are needed in order to improve the performance and stability of this algorithm. Our implementations use the *two-phase simplex* [1] to get hold of an initial feasible solution. Various methods for choosing the entering variable offer a compromise between computation cost and the number of iterations needed to solve the problem. Among these, the *steepest edge* [9] greatly reduces the number of iterations required. Moreover, its implementation fully benefits from the special architecture of a GPU. Stability and robustness of the algorithm depend considerably on the choice of the leaving variable. With respect to this, the *expand* method [5] proves to be very useful, particularly in helping to avoid cycles.

## 3 GPU architecture : CUDA

Since our programs are designed to run on a given type of GPU, we will focus on the parallel computing architecture developed by NVIDIA: Compute Unified Device Architecture (CUDA [8]). This architecture falls into the *Single Instruction Multiple Data* (SIMD) category. However, it is classified by NVIDIA under the concept of *Single Instruction Multiple Threads* (SIMT). The

key feature of this architecture is to exploit data parallelism and to offer fast context switching for the threads. When used to process large matrices, recent generation GPUs exhibit TFLOPS performances on single precision operations.

A GPU appears as a device, usually connected to the motherboard via the PCI bus, which is controlled by the CPU. The global memory of the GPU is used to share data between both processing units. Moreover, it is possible to have several GPUs in a computer. In this case, each GPU must be managed by its own CPU thread. In a standard application, the CPU manages the data and the global algorithm, while the GPU does the computing, as in a master-slave paradigm. Due to the nature of the GPU architecture, conditional branching instructions are usually delegated to the CPU to avoid sequential execution of the GPU threads.

The GPU architecture is organized around two important notions : work decomposition and memory levels. Work decomposition, describes how a computation can be decomposed using multiple threads. The parallel processing is related to the architecture of the computing units on a GPU. The latter contains several *streaming multiprocessors* (SM). Each of them has a fixed number of processing units that run instructions in parallel from a single instruction unit. Thus, each thread on a multiprocessor must execute the same instruction to ensure true parallelism.

The hierarchy of processing units leads to decomposing a computation over a multitude of threads, each of them doing mainly the same work. Consider for example the case of a matrix addition: $\mathbf{A}=\mathbf{B}+\mathbf{C}$. Each thread could load a single element $B_{ij}$ and $C_{ij}$ from the matrices, sum both values and store the result into $A_{ij}$. Threads are grouped into blocks which will be dispatched to the SMs. Inside a block, threads are grouped into so-called *warps*. Blocks are further organized into a grid of blocks, so as to provide a unique ID for each thread. The ID of a thread consists of the position of the thread in its block and the position of the latter on the grid. This ID is of crucial importance to access individual data during the computations.

There are different memory levels. A thread has its own registers on which the instructions are applied. Data accesses on registers are the fastest in this architecture. However, registers are shared inside a block, thus leading to a limited number of registers per thread. At another level, each block has its own shared memory accessible only by its own threads. This memory is nearly as fast as the registers when correctly accessed. More importantly, shared memory is the safest and fastest way to communicate between threads. Since SMs are not synchronized instruction-wise, threads on different blocks cannot communicate directly. Finally, the main memory of the GPU (GRAM) is a global memory. It is within the scope of all threads as well as the CPU. But access to this memory is costly since, in addition to the read/write instruction, it takes 400-600 cycles before the data is available from the GRAM.

A GPU architecture uses pipelines to improve performances. There is a fixed depth ($D$) pipeline depending on the number of processors per SM, because a *warp* consists of $D$ batches whose threads run concurrently on the SM. Another pipeline is used to hide the latency due to GRAM accesses. The scheduler swaps *warps* waiting for data, with *warps* ready to be executed. The more warps, and thus threads, per block, the more efficiently this latency can be hidden.

# 4 Implementations

## 4.1 Single GPU

The first step performed by our implementation is done sequentially by the CPU. It consists in reading a LP model from a standard MPS file and transforming it to its augmented form. This modified problem is then entirely loaded in the GRAM. The next steps are listed in algorithm 1.

**Algorithm 1** Simplex method

**Require:** $p_{GPU}$ {Data structure containing the problem on the GPU}
**Ensure:** $feasible, infeasible$ {Feasibility of the problem}
1: **while** exists($x_{in}$) **do**
2: $x_{in} \leftarrow$ find_entering($p_{GPU}$), $x_{out} \leftarrow$ find_leavingVar($x_{in}$, $p_{GPU}$) {GPU}
3: **if** not exists($x_{out}$) **return** $infeasible$
4: pivoting($x_{in}$, $x_{out}$,$p_{GPU}$) {GPU}
5: **end while**
6: **return** $feasible$

At each iteration, the CPU launches the *kernels*[1] and checks the results returned. The first *kernel* searches for the entering variable using the *steepest edge* method. The complexity of this method is $\mathcal{O}(mn)$ and benefits thoroughly from the massively parallel architecture of the GPU. This *kernel* uses the problem loaded in the GRAM and returns only the selected entering variable. For the *expand* method used for choosing the leaving variable, it is required to use 2 or 3 *kernels* because of the different phases of this heuristic. Each phase has complexity $\mathcal{O}(m)$ and returns only one result. Finally, the pivoting operation requires a small constant amount of GRAM accesses done by the CPU in a preliminary stage. Then the pivoting is done by a *kernel* available in CUBLAS, an optimized CUDA library for basic linear algebra. Once the final optimal solution is found, the CPU has to retrieve the problem data.

This algorithm ensures that the communications between CPU and GPU are minimized. Besides loading the $m \times n$ matrix representing the problem into the GRAM and retrieving it into the CPU memory at the end of the computation, all other communications take a small and constant amount of time and can thus be neglected when dealing with large problems.

The problem is stored in the GRAM in a way to optimize data access. GPUs are able to retrieve more than a single data in one instruction whenever addresses are coalesced[2]. For example, an SM is able to read single precision variables for up to 16 threads in a single instruction, if the addresses are coalesced. Such accesses substantially increase the GRAM throughput.

## 4.2 Multi-GPU

The multi-GPU implementation uses the same algorithm as in section 4.1. However, some communications have to be added so as to synchronize the GPUs. The communication pattern is dependent on the way our LP problem is scattered across the GPUs. We split the constraint matrix vertically (see figure 1) in order to minimize communications.

After reading and preparing the problem, the CPU creates CPU threads that will manage the GPUs. Then, each of them initializes its GPU and loads a sub-matrix in its GRAM. During an iteration, each CPU thread launches concurrently the *steepest edge kernel* to obtain a local entering variable. The first communication intervenes here: each GPU must share with the others its local result. They then reduce the local result to elect the same best entering variable. Due to the selected splitting, only the CPU thread that holds the entering variable, launches the *expand kernels* to select the leaving variable. The column of this variable, called *pivot column*, is needed by all the other GPUs to execute the pivoting step. Thus this column, along with the leaving variable, is broadcasted. Finally, each GPU prepares and launches the pivoting *kernel* with no further communications. Note that GPUs communicate together via the CPU threads.

[1] CUDA functions are called *kernels*.
[2] CUDA concept for successive data addresses.

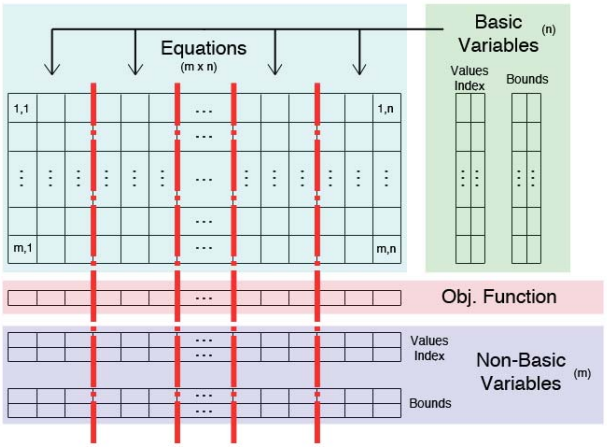


Figure 1: Vertical splitting of the LP model

# 5 Performance models

## 5.1 Modelling a *kernel*

CUDA *kernels* require a different kind of modelling than usually encountered in parallelism. Indeed, we have to take into account the GPU architecture described in section 3. The key idea is to capture in the model the decomposition of computations into threads, warps and blocks. One must also pay a particular attention to GRAM accesses and to how the pipelines reduce the associated latency. Our *kernel* model is inspired by [4].

The first step is to estimate the work done by a single thread of a *kernel* in terms of cycles. There are two different types of cycle. The first one is related to arithmetic instructions. Each instruction requires a different number of cycles in function of the operand and the operator. The second type is linked to the GRAM accesses. Since these accesses are non-blocking operation, they have to be counted apart from the arithmetic instructions.

The total work $C_T$ done by a thread can be defined either as the sum or as the maximum of the memory access cycles and the arithmetic instructions cycles. Summing both types of cycles means we consider that latency cannot be used to hide arithmetic instructions. The maximum variant represents the opposite situation where arithmetic instructions and memory accesses are concurrent. Then only the biggest of the two represents the total work of a thread. This could occur for example when the latency is entirely hidden by the pipeline.

We must now find out how many threads are run by a processor. A *kernel* decomposes into blocks of threads. Thus each *streaming multiprocessor* (SM) must run $N_B$ blocks. Each block itself has $N_W$ *warps* consisting of $N_T$ threads. An SM has $N_P$ processors which execute batches of threads in a pipeline of depth $D$. Thus the total work $C_{Proc}$ done by a processor is given by

$$C_{Proc} = N_B \cdot N_W \cdot N_T \cdot C_T \cdot \frac{1}{N_P \cdot D} \tag{3}$$

Finally, the execution time of a *kernel* is obtained by dividing $C_{Proc}$ by the processor frequency.

## 5.2 A *kernel* example : *Steepest Edge*

The selection of the entering variable is done by the *steepest edge* method. This method requires the computation of the Euclidean norm of each column $\eta_j$ of the constraint matrix. The coef-

ficients $c_j$ of the variables in the objective function, are then divided by the norm $\|\eta_j\|$. The selected variable is the one having the smallest *steepest edge* ratio

$$se_l = \frac{c_l}{\|\eta_l\|} = \min_{j=1..n} \left( \frac{c_j}{\sqrt{1+\sum_{i=1}^{m} x_{ij}^2}} \right) \tag{4}$$

The processing of a column is done in a single block. Each thread of a block has to compute $N_{el} = \frac{m}{N_W \cdot N_T}$ coefficients of the column. This first computation consists of $N_{el}$ multiplications and additions. The resulting partial sum of squared variables must then be reduced on a single thread of the block. This requires $N_{red} = \log_2(N_W \cdot N_T)$ additions. Since the shared memory is used optimally, there are no added communications. Finally, the coefficient $c_j$ must be divided by the result of the reduction.

Each block of the *kernel* computes $N_{col} = \frac{n}{N_B \cdot N_{SM}}$ columns, where $N_{SM}$ is the number of SMs per GPU. After processing a column, a block keeps only the minimum of its previously computed $se_j$ ratios. Thus the number of arithmetic instruction cycles for a given thread is given by

$$C_{Arithm} = N_{col} \cdot (N_{el} \cdot (C_{add} + C_{mul}) + N_{red} \cdot C_{add} + C_{div} + C_{cmp}) \tag{5}$$

where $c_{ins}$ denotes the number of cycles to execute instruction *ins*.

Each thread has to load $N_{el}$ variables to compute its partial sum of squared variables. The thread computing the division also loads the coefficient $c_j$. This must be done for the $N_{col}$ columns that a block has to deal with. We must also take into account that the scheduler hides some latency by swapping the *warps*, so the total latency $C_{latency}$ must be divided by the number of *warps* $N_W$. Thus the number of cycles relative to memory accesses is given by:

$$C_{Accesses} = \frac{N_{col} \cdot (N_{el}+1) \cdot C_{latency}}{N_W} \tag{6}$$

At the end of the execution of this *kernel*, each block stores in the GRAM its local minimum $se_l$. The CPU will then have to retrieve the $N_B \cdot N_{SM}$ local minimums and reduce them. It is then profitable to minimize the number $N_B$ of blocks per SM. With a maximum of two blocks per SM, the cost of this final CPU operation can be neglected when $m$ and $n$ are big.

We now take the maximimum or sum $C_{Arithm}$ and $C_{Accesses}$ to obtain $C_T$. The result of equ. (3) divided by the processor frequency yields the time $T_{KSteepestEdge}$ of the *steepest edge kernel*.

### 5.3 Single and multi-GPU implementation models

As seen in section 4.1, the single GPU implementation requires only a few CPU-GPU communications. Since each of these communications are constant and small, they will be neglected in the model. For the sake of simplicity, we will consider the second phase of the *two-phase simplex* where we apply iteratively the three main operations: selecting the entering variable, choosing the leaving variable and pivoting. Each of these operations is computed as a *kernel*. The time for an iteration $T_{iteration}$ then amounts to the sum of all three *kernel* times

$$T_{iteration} = T_{KSteepestEdge} + T_{KExpand} + T_{KPivoting} \tag{7}$$

The times $T_{KExpand}$ and $T_{KPivoting}$ for the *expand* and pivoting *kernels* are obtained in a similar way as for the *steepest edge kernel* described in the previous section.

With the estimated time per iteration $T_{iteration}$, we can express the total time for solving a problem as $T_{prob} = T_{init} + r \cdot T_{iteration}$ where $r$ is the number of iterations.

The model for a multi-GPU implementation is quite similar since each GPU basically executes the same *kernels* than the ones mentioned above. But there are two important differences: 1) the size of the local matrix processed by a GPU is $\frac{m \cdot n}{k}$, where $k$ is the number of GPUs used; 2) we have to deal with the inter-GPU communications.

As seen in section 4.2, GPUs must share their locally computed entering variable so as to select one of them. Similarly the GPU choosing the leaving variable must share the *pivoting column* with the others. This depends on the number of equations $m$ (also the size of a column) and the number of GPUs. The time of an iteration in the multi-GPU case is obtained by adding the communication times in equation (7).

# 6 Measurements

*IBM CPLEX* and *gurobi* are among the best performance proprietary solvers, whereas *GLPK*, *lpsolve* and *CLP* are well-known open source solvers. We chose *CLP* for our tests because of its excellent performances and its affiliation to the COIN-OR project. *CLP* is a sequential implementation of the revised simplex method.

We first successfully checked that our implementations were as functional as *CLP* by using the *NETLIB* [2] repository. This dataset usually serves as a benchmark for LP solvers. It has a vast variety of real and specific problems for testing stability and robustness of an implementation.

To test the performances of our implementations, a large range of problem size and density (percentage of non-zero coefficients) is needed. As none of the existing datasets provided the needed diversity of problems, we used a problem generator. It is then possible to create problems of specific size and density. Unfortunately, all the problems generated fall into the same category. Hence, generated problems of the same size and density will be solved within approximately the same number of iterations and the optimum of the objective function will only slightly fluctuate.

Our test environment is composed of a CPU server (2 Intel Xeon X5570, 2.93GHz, with 24GB DDR3 RAM) and a GPU computing system (*NVIDIA tesla S1070*). This system connects 4 GPUs to the server. Each GPU has 4GB GDDR3 graphics memory, 30 streaming multiprocessors, each holding 8 processors (1.4GHz). Such a GPU offers at peak performance up to 1TFLOPS for single precision floating point, 80GFLOPS for double precision floating point and 100GB/s memory bandwidth between threads (registers) and GRAM.

We measured the execution time for each implementation on problems of different size with density below 5%. As shown in figure 2, for large enough problems ($m > 750$ and $n > 3750$) our implementations outperform substantially *CLP*. A time limit of 30 minutes was imposed which explains why there are no measurements for *CLP* above $m = 1250$. However, on small problems *CLP* remains more efficient (see inset in fig. 2). This is mainly due to the initialisation time of a GPU which includes the time required to setup the *kernel* context.

Implementations of the revised simplex method, like *CLP*, are highly influenced by the problem density. The denser the problem, the slower *CLP* runs. Thus we mainly compared performances on problems of density lower than 5%. However, to take a deeper look into how these implementations behave under various densities, we measured the execution time on a fixed size problem ($500 \times 2000$) but with increasing density. The execution time of our simplex implementation remained constant, while that of the *CLP* grew continuously. In this experiment *CLP* was outperformed when the problem density rose above 5% (see fig. 3). Moreover, we noticed that the problem density often increased significantly throughout a run.

Finally, we validated our performance models by computing the correlation between measurements and models for 1 to 4 GPUs, which was above 0.999. For big problems ($m > 5000$ and $n > 25000$), the speedup predicted by the model is nearly optimal in the range up to 40 GPUs.

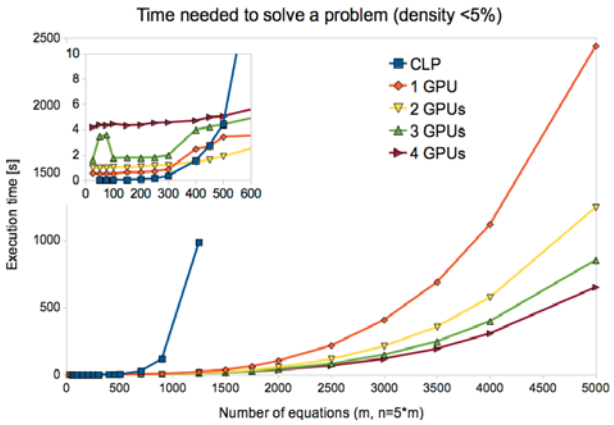


Figure 2: Implementations execution time as a function of the problem size

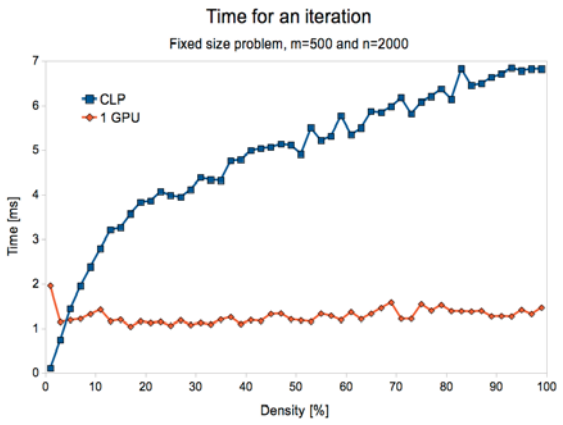


Figure 3: Time of an iteration as a function of the problem density

# 7 Conclusion

In this paper, we presented robust and efficient single and multi GPU implementations of the standard simplex method. These implementations outperformed a well-known open source linear programming solver, *CLP*, on mid-size to big and/or dense problems. Moreover, we derived accurate performance models for these implementations. These models actually predict a nearly optimal speedup while using up to 40 GPUs to solve large problems.

As GPUs are constantly evolving, we can expect for such implementations an improvement of performances in the near future. The latest GPU generation of NVIDIA is up to 5 times faster on double precision instructions than the one used in this work. Finally, we believe that interior point methods could also benefit from the massively parallel architecture of GPUs.

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# Duality for multiple objective fractional programming with generalized type-I univexity

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**Abstract**

In this paper, a multiobjective fractional subset programming problem (Problem (P)) is considered. A new class of $(\mathscr{F},b,\phi,\rho,\theta)$-type-I univex function is introduced and a general dual model for (P) is presented. Based on these functions, weak, strong and converse duality theorems are derived. Almost all results presented in the literature were obtained under the assumption that the function $\mathscr{F}$ is sublinear in the third argument. Here, our results hold assuming only the convexity of this one.

## 1 Introduction

In this paper, we shall present a semiparametric dual model for the following multiobjective fractional subset programming problem

$$\text{(P)} \qquad \min\ \varphi(S)=\left(\frac{F_1(S)}{G_1(S)},\frac{F_2(S)}{G_2(S)},...,\frac{F_p(S)}{G_p(S)}\right)$$

subject to

$$H_j(S)\leqq 0,\ \ j\in\underline{q}=\{1,2,...,q\},\ S\in A^n,$$

where $A^n$ is the $n$-fold product of the $\sigma$-algebra $A$ of subsets of a given set $X$, $F_i:A^n\to\mathbb{R},\ G_i:A^n\to\mathbb{R},\ i\in\underline{p}=\{1,2,...,p\}$, and $H_j:A^n\to\mathbb{R},\ j\in\underline{q}$, such that for each $i\in\underline{p},\ G_i(S)>0$, for all $S\in\mathscr{P}$. We denoted by $\mathscr{P}=\{S\in A^n:H_j(S)\leqq 0,\ j\in\underline{q}\}$ the set of all feasible solutions for (P).

For any vectors $x,y\in\mathbb{R}^n$,we use the following conventions:$x<y$ iff $x_i<y_i,\ i\in\underline{n}=\{1,2,...n\}$; $x\leqq y$ iff $x_i\leqq y_i,\ i\in\underline{n}=\{1,2,...n\}$; $x\leq y$ iff $x\leqq y,$ but $x\neq y$; $x\not\leq y$ means the negation of $x\leq y$.

**Definition 1.1.** *A feasible solution $S^0\in\mathscr{P}$ is said to be an efficient solution to (P) if there is no other $S\in\mathscr{P}$ such that*

$$\left(\frac{F_1(S)}{G_1(S)},\frac{F_2(S)}{G_2(S)},...,\frac{F_p(S)}{G_p(S)}\right)\leq\left(\frac{F_1(S^0)}{G_1(S^0)},\frac{F_2(S^0)}{G_2(S^0)},...,\frac{F_p(S^0)}{G_p(S^0)}\right).$$

Following the introduction of the notion of convexity for set functions by Morris [8] and its extension for $n$-set functions by Corley [4], various generalizations of convexity for set and $n$-set functions were proposed in Lee [5], Lin [6], Preda [9, 11], Stancu-Minasian and Preda [15], Zalmai [17, 18]. More precisely, quasiconvexity and pseudoconvexity for set functions were defined in Lee [5], and in Lin [6] for $n$-set functions, $(\mathscr{F},\rho)$-convexity in Preda [9, 10], $(\rho,b)$-vexity in Preda [11], $(\mathscr{F},\alpha,\rho,\theta)$-$V$-convexity in Zalmai [17], $(\mathscr{F},b,\phi,\rho,\theta)$-univexity in Zalmai [18] and $(\mathscr{F},\rho,\sigma,\theta)$-$V$-type-I in Mishra [7]. Also, in Bector *et al.* [2], some types of generalized convexity and optimality and duality results for a

multiobjective programming problem involving $n$-set functions were given. For more information about fractional programming problems, the reader may consult the recently research bibliography compiled by Stancu-Minasian [14].

For formulating and proving various duality results, we shall introduce the class of generalized convex $n$-set functions called $(\mathscr{F},b,\phi,\rho,\theta)$-type-I univex functions. Until now, $\mathscr{F}$ was assumed to be a sublinear function in the third argument. In our approach, we suppose that $\mathscr{F}$ is a convex function in the third argument. The idea of replacing a sublinear function with a convex function in certain types of sufficiency and duality results is not new (see, for instance Bătătorescu *et al.* [1], Chinchuluun *et al.* [3], Preda *et al.* [12, 13] and Yuan *et al.* [16]).

## 2 Definitions and preliminaries

Let $(X,A,\mu)$ be a finite atomless measure space with $L_1(X,A,\mu)$ separable, and let $d$ be the pseudometric on $A^n$ defined by

$$d(R,S) := \left[\sum_{k=1}^{n} \mu^2 (R_k \Delta S_k)\right]^{1/2},$$

where $R=(R_1,\cdots,R_n)$, $S=(S_1,\cdots,S_n)\in A^n$ and $\Delta$ stands for the symmetric difference. Thus, $(A^n,d)$ is a pseudometric space.

For $h\in L_1(X,A,\mu)$ and $T\in A$, the integral $\int_T h\mathrm{d}\mu$ is denoted by $\langle h,\chi_T\rangle$, where $\chi_T\in L_\infty(X,A,\mu)$ is the indicator (characteristic) function of $T$.

**Definition 2.1.** [8] *A function $F:A\to\mathbb{R}$ is said to be differentiable at $S^*\in A$ if there exist $DF(S^*)\in L_1(X,A,\mu)$, called the derivative of $F$ at $S^*$, and $V_F:A\times A\to\mathbb{R}$ such that*

$$F(S)=F(S^*)+\langle DF(S^*),\chi_S-\chi_{S^*}\rangle+V_F(S,S^*)$$

*for each $S\in A$, where $V_F(S,S^*)$ is $o(d(S,S^*))$, that is, $\lim\limits_{d(S,S^*)\to 0}\dfrac{V_F(S,S^*)}{d(S,S^*)}=0$.*

**Definition 2.2.** [4] *A function $G:A^n\to\mathbb{R}$ is said to have a partial derivative at $S^*=(S_1^*,...,S_n^*)\in A^n$ with respect to its $i$-th argument if the function $F(S_i)=G(S_1^*,\cdots,S_{i-1}^*,S_i,S_{i+1}^*,\cdots,S_n^*)$ has derivative $DF(S_i^*)$, $i\in\underline{n}=\{1,2,...,n\}$.*

*We define $D_iG(S^*)=DF(S_i^*)$ and write $DG(S^*)=(D_1G(S^*),\cdots,D_nG(S^*))$.*

**Definition 2.3.** [4] *A function $G:A^n\to\mathbb{R}$ is said to be differentiable at $S^*$, if there exist $DG(S^*)$ and $W_G:A^n\times A^n\to\mathbb{R}$ such that*

$G(S)=G(S^*)+\sum\limits_{i=1}^{n}\left\langle D_iG(S^*),\chi_{S_i}-\chi_{S_i^*}\right\rangle+W_G(S,S^*)$, *where $W_G(S,S^*)$ is $o(d(S,S^*))$ for all $S\in A^n$.*

In the following we consider $\mathscr{F}:A^n\times A^n\times\mathbb{R}\to\mathbb{R}$. Also consider $F:A^n\to\mathbb{R}$ and $G:A^n\to\mathbb{R}$ two differentiable functions. The definitions below generalizes two definitions of Zalmai [18].

Let $\rho_i$ and $\rho_j$ real numbers and let $\rho=(\rho_i,\rho_j)$, $i\in I$, $j\in J$.

**Definition 2.4.** A pair $(F_i, G_j)\,(i \in I, j \in J)$ is said to be $(\mathscr{F}, b, \phi, \rho, \theta)$-pseudo quasi univex type-I at $S^* \in A^n$ according to the partition $\{I, J\}$ if there exist a function $b : A^n \times A^n \to \mathbb{R}_+$, a function $\theta : A^n \times A^n \to A^n \times A^n$ such that $S \neq S^* \Longrightarrow \theta(S, S^*) \neq (0,0)$, a function $\phi : \mathbb{R} \to \mathbb{R}$, and real numbers $\rho$ such that for all $S \in A^n$ the implications

$$\mathscr{F}(S, S^*; b(S, S^*) DF_i(S^*)) \geqq -\rho_i d^2(\theta(S, S^*)) \Longrightarrow \phi\,(F_i(S) - F_i(S^*)) \geqq 0, i \in I \tag{2.1}$$

and

$$\phi(-G_j(S^*)) \leqq 0 \Rightarrow \mathscr{F}(S, S^*; b(S, S^*) DG_j(S^*)) \leqq -\rho_j d^2(\theta(S, S^*)),\ j \in J \tag{2.2}$$

do hold.

If the second (implied) inequality in (2.1) is strict $(S \neq S^*)$, then we say that $(F_i, G_j)$ is $(\mathscr{F}, b, \phi, \rho, \theta)$-strictly pseudo quasi univex type-I at $S^* \in A^n$ according to the partition $\{I, J\}$.

**Definition 2.5.** *A pair $(F_i, G_j)\,(i \in I, j \in J)$ is said to be $(\mathscr{F}, b, \phi, \rho, \theta)$-quasi pseudo univex type-I at $S^* \in A^n$ according to the partition $\{I, J\}$ if there exist a function $b : A^n \times A^n \to \mathbb{R}_+$, a function $\theta : A^n \times A^n \to A^n \times A^n$ such that $S \neq S^* \Longrightarrow \theta(S, S^*) \neq (0,0)$, a function $\phi : \mathbb{R} \to \mathbb{R}$, and real numbers $\rho$ such that for all $S \in A^n$ the implications*

$$\phi\,(F_i(S) - F_i(S^*)) \leqq 0 \Rightarrow \mathscr{F}(S, S^*; b(S, S^*) DF_i(S^*)) \leqq -\rho_i d^2(\theta(S, S^*)),\ i \in I \tag{2.3}$$

*and*

$$\mathscr{F}(S, S^*; b(S, S^*) DG_j(S^*)) \geqq -\rho_j d^2(\theta(S, S^*)) \Longrightarrow \phi\,(-G_j(S^*)) \geqq 0, j \in J \tag{2.4}$$

*do hold.*

If the first (implied) inequality in (2.3) is strict $(S \neq S^*)$, and the second (implied) inequality in (2.4) is strict $(S \neq S^*)$, then we say that $(F_i, G_j)$ is $(\mathscr{F}, b, \phi, \rho, \theta)$-prestrictly quasi strictly pseudo univex type-I at $S^* \in A^n$ according to the partition $\{I, J\}$.

**Definition 2.6.** *A pair $(F_i, G_j)\,(i \in I, j \in J)$ is said to be $(\mathscr{F}, b, \phi, \rho, \theta)$-quasi quasi univex type-I at $S^* \in A^n$ according to the partition $\{I, J\}$ if there exist a function $b : A^n \times A^n \to \mathbb{R}_+$, a function $\theta : A^n \times A^n \to A^n \times A^n$ such that $S \neq S^* \Longrightarrow \theta(S, S^*) \neq (0,0)$, a function $\phi : \mathbb{R} \to \mathbb{R}$, and real numbers $\rho$ such that for all $S \in A^n$ the implications*

$$\phi\,(F_i(S) - F_i(S^*)) \leqq 0 \Rightarrow \mathscr{F}(S, S^*; b(S, S^*) DF_i(S^*)) \leqq -\rho_i d^2(\theta(S, S^*)),\ i \in I \tag{2.5}$$

*and*

$$\phi(-G_j(S^*)) \leqq 0 \Rightarrow \mathscr{F}(S, S^*; b(S, S^*) DG_j(S^*)) \leqq -\rho_j d^2(\theta(S, S^*)),\ j \in J \tag{2.6}$$

*do hold.*

If the first (implied) inequality in (2.5) is strict $(S \neq S^*)$, then we say that $(F_i, G_j)$ is $(\mathscr{F}, b, \phi, \rho, \theta)$-prestrictly quasi quasi univex type-I at $S^* \in A^n$ according to the partition $\{I, J\}$.

**Definition 2.7.** *A pair $(F_i, G_j)\,(i \in I, j \in J)$ is said to be $(\mathscr{F}, b, \phi, \rho, \theta)$-pseudo pseudo univex type-I at $S^* \in A^n$ according to the partition $\{I, J\}$ if there exist a function $b : A^n \times A^n \to \mathbb{R}_+$, a function $\theta : A^n \times A^n \to A^n \times A^n$ such that $S \neq S^* \Longrightarrow \theta(S, S^*) \neq (0,0)$, a function $\phi : \mathbb{R} \to \mathbb{R}$, and real numbers $\rho$ such that for all $S \in A^n$ the implications*

$$\mathscr{F}(S,S^*;b(S,S^*)DF_i(S^*)) \geqq -\rho_i d^2(\theta(S,S^*)) \Longrightarrow \phi\left(F_i(S)-F_i(S^*)\right) \geqq 0, i \in I \tag{2.7}$$

*and*

$$\mathscr{F}(S,S^*;b(S,S^*)DG_j(S^*)) \geqq -\rho_j d^2(\theta(S,S^*)) \Longrightarrow \phi\left(-G_j(S^*)\right) \geqq 0, j \in J \tag{2.8}$$

*do hold.*

If the second (implied) inequality in (2.7) and (2.8) is strict ($S \neq S^*$), then we say that $(F_i, G_j)$ is $(\mathscr{F}, b, \phi, \rho, \theta)$-strictly pseudo strictly pseudo univex type-I at $S^* \in A^n$ according to the partition $\{I,J\}$.

In [17], Zalmai state for Problem (P) the following necessary conditions for efficiency.

**Theorem 2.1.** *Assume that $F_i,\ G_i,\ i \in \underline{p}$, and $H_j,\ j \in \underline{q}$, are differentiable at $S^* \in A^n$, and that for each $i \in \underline{p}$, there exists $\widehat{S}_i \in A^n$ such that*

$$H_j(S^*) + \sum_{k=1}^{n} \left\langle D_k H_j(S^*), \chi_{\widehat{S}_k} - \chi_{S_k^*} \right\rangle < 0,\ j \in \underline{q},$$

*and for each $l \in \underline{p} \backslash \{i\}$,*

$$\sum_{k=1}^{n} \left\langle G_i(S^*) D_k F_l(S^*) - F_i(S^*) D_k G_l(S^*), \chi_{\widehat{S}_k} - \chi_{S_k^*} \right\rangle < 0.$$

*If $S^*$ is an efficient solution to (P), then there exist $u^* \in U = \{u \in \mathbb{R}^p : u > 0,\ \sum_{i=1}^{p} u_i = 1\}$ and $v^* \in \mathbb{R}_+^q$ such that*

$$\sum_{k=1}^{n} \langle \sum_{i=1}^{p} u_i^* \left[G_i(S^*) D_k F_i(S^*) - F_i(S^*) D_k G_i(S^*)\right] + + \sum_{j=1}^{q} v_j^* D_k H_j(S^*), \chi_{S_k} - \chi_{S_k^*} \rangle \geqq 0,$$

*for all $S \in A^n,\ v_j^* H_j(S^*) = 0,\ j \in \underline{q}$. We denoted by $\mathbb{R}_+^q$ the positive orthant of the q-dimensional space $\mathbb{R}^q$, i.e. $\mathbb{R}_+^q = \{x = (x_{1,\ldots,}x_n) \in \mathbb{R}^q : x_j \geqq 0,\ j \in \underline{q}\}$.*

*We shall call an efficient solution $S^*$ to (P) satisfying the first two conditions for some $\widehat{S}_i$, $i \in \underline{p}$ a regular efficient solution.*

## 3 A Zalmai's Semiparametric Duality Model

In this section we present a general duality model for (P). Here we use two partitions of the index sets $\underline{q}$ and respectively $\underline{p}$.

Let $\{I_0, I_1, ..., I_k\}$ be a partition of the index set $\underline{p}$ and $\{J_0, J_1, ..., J_m\}$ be a partition of the index set $\underline{q}$ such that $K = \{0,1,...,k\} \subset M = \{0,1,...,m\}$, and let the function $\Omega_t(S,\cdot,u,v) : A^n \to \mathbb{R}$ be defined for fixed $S, u$ and $v$, and $t \in K$ by

$$\Omega_t(S,T,u,v) = \sum_{i \in I_t} u_i \left[F_i(S)G_i(T) - F_i(T)G_i(S)\right] + \sum_{j \in J_t} v_j H_j(T)$$

We associate to the problem (P) the dual problem

$$\text{(D)} \qquad \max \delta(T,u,v) = \left(\frac{F_1(T)}{G_1(T)}, \frac{F_2(T)}{G_2(T)}, ..., \frac{F_p(T)}{G_p(T)}\right),$$

subject to

$$\mathscr{F}\left(S,T;b(S,T)\left\{\sum_{i=1}^{p}u_i\left[G_i(T)DF_i(T)-F_i(T)DG_i(T)\right]+\sum_{j=1}^{q}v_jDH_j(T)\right\}\right)\geqq 0,\ \forall\, S\in A^n \tag{3.1}$$

$$\sum_{j\in J_t}v_jH_j(T)\geqq 0,\ t\in M,\ T\in A^n,\ u\in U,\ v\in\mathbb{R}^q_+. \tag{3.2}$$

In the following we consider $\mathscr{F}(S,T;\cdot):L_1(X,A,\mu)\to\mathbb{R}$ a convex function and $\Lambda_t(\cdot,v^*):A^n\to\mathbb{R}$, $\Lambda_t(T,v^*)=\sum_{j\in J_t}v_j^*H_j(T)$, $t\in M$.

**Theorem 3.1** (Weak duality). *Let S and $(T,u,v)$ be arbitrary feasible solutions to (P) and (D), respectively, and assume that any one of the following sets of hypotheses is satisfied:*

(a) (i) $(2k\Omega_t(\cdot,T,u,v);\ 2(m-k)\Lambda_t(\cdot,v))$ is $(\mathscr{F},b,\phi_t,\rho_t,\theta)$-strictly pseudo quasi univex type-I at the point $T$, according to the partition $\{K,M\backslash K\}$, $\phi_t$ is increasing and $\phi_t(0)=0$ for each $t\in M$;

(ii) $\frac{1}{k}\sum_{t\in K}\rho_t+\sum_{t\in M\backslash K}\frac{\rho_t}{m-k}\geqq 0$;

(b) (i) $(2k\Omega_t(\cdot,T,u,v);\ 2(m-k)\Lambda_t(\cdot,v))$ is $(\mathscr{F},b,\phi_t,\rho_t,\theta)$- prestrictly quasi strictly pseudo univex type-I at the point $T$, according to the partition $\{K,M\backslash K\}$, $\phi_t$ is increasing and $\phi_t(0)=0$ for each $t\in M$;

(ii) $\frac{1}{k}\sum_{t\in K}\rho_t+\sum_{t\in M\backslash K}\frac{\rho_t}{m-k}\geqq 0$;

(c) (i) $(2k\Omega_t(\cdot,T,u,v);2(m-k)\Lambda_t(\cdot,v))$ is $(\mathscr{F},b,\phi_t,\rho_t,\theta)$-prestrictly quasi quasi univex-type-I at the point $T$, according to the partition $\{K,M\backslash K\}$,$\phi_t$ is increasing and $\phi_t(0)=0$ for each $t\in M$;

(ii) $\frac{1}{k}\sum_{t\in K}\rho_t+\sum_{t\in M\backslash K}\frac{\rho_t}{m-k}>0$;

(d) (i) $(3k_1\Omega_t(\cdot,T,u,v);\ 3(m-k)\Lambda_t(\cdot,v))$ is $(\mathscr{F},b,\phi_t,\rho_t,\theta)$-strictly pseudo quasi univex-type-I at the point $T$ according to the partition $\{K_1,M\backslash K\}$,
$(3k_2\Omega_t(\cdot,T,u,v)\ ;3(m-k)\Lambda_t(\cdot,v)$ is $(\mathscr{F},b,\phi_t,\rho_t,\theta)$-prestrictly quasi quasi univex type-I at the point $T$ according to the partition $\{K_2,M\backslash K\}$, $\phi_t$ is increasing and $\phi_t(0)=0$ for each $t\in M$, where $\{K_1,K_2\}$ is a partition of $K$, $K_1\neq\emptyset$, $k_1=|K_1|$ and $K_2\neq\emptyset$, $k_2=|K_2|$;

(ii) $\frac{1}{k_1}\sum_{t\in K_1}\rho_t+\frac{1}{k_2}\sum_{t\in K_2}\rho_t+\sum_{t\in M\backslash K}\frac{\rho_t}{m-k}\geqq 0$;

(e) (i) $(3k\Omega_t(\cdot,T,u,v);3(m_1-k_1)\Lambda_t(\cdot,v))$ is $(\mathscr{F},b,\phi_t,\rho_t,\theta)$-prestrictly quasi strictly pseudo univex-type-I at the point $T$, according to the partition $\{K,(M\backslash K)_1\}$,
$(3k\Omega_t(\cdot,T,u,v);3(m_2-k_2)\Lambda_t(\cdot,v))$ is $(\mathscr{F},b,\phi_t,\rho_t,\theta)$- prestrictly quasi quasi univex-type-I at the point $T$, according to the partition $\{K,(M\backslash K)_2\}$, $\phi_t$ is increasing and $\phi_t(0)=0$,for each $t\in M$, where $\{(M\backslash K)_1,(M\backslash K)_2\}$ is a partition of $M\backslash K$, $(M\backslash K)_1\neq\emptyset$, $m_1=|(M\backslash K_1)|$ and $(M\backslash K)_2\neq\emptyset$, $m_2=|(M\backslash K)_2|$;

(ii) $\frac{1}{k}\sum_{t\in K}\rho_t+\sum_{t\in(M\backslash K)_1}\frac{\rho_t}{m_1-k_1}+\sum_{t\in(M\backslash K)_2}\frac{\rho_t}{m_2-k_2}\geqq 0$;

(f) (i) $(4k_1\Omega_t(\cdot,T,u,v);4(m_1-k_1)\ \Lambda_t(\cdot,v))$ is $(\mathscr{F},b,\phi_t,\rho_t,\theta)$-strictly pseudo strictly pseudo univex-type-I at the point $T$, according to the partition $\{K_1,(M\backslash K)_1\}$,
$(4k_2\Omega_t(\cdot,u,v);4(m_2-k_2)\Lambda_t(\cdot,v))$ is $(\mathscr{F},b,\phi_t,\rho_t,\theta)$-prestrictly quasi qusasi univex-type-I at the point $T$, according to the partition $\{K_2,(M\backslash K)_2\}$, $\phi_t$ is increasing and $\phi_t(0)=0$ for each $t\in M$, where $\{K_1,K_2\}$ is a partition of $K$, $K_1\neq\emptyset$, $k_1=|K_1|$ and $K_2\neq\emptyset$, $k_2=|K_2|$;$\{(M\backslash K)_1,(M\backslash K)_2\}$ is a partition of $M\backslash K$, $(M\backslash K)_1\neq\emptyset$, $m_1=|(M\backslash K_1)|$ and $(M\backslash K)_2\neq\emptyset$, $m_2=|(M\backslash K)_2|$;

(ii) $\frac{1}{k_1}\sum_{t\in K_1}\rho_t+\frac{1}{k_2}\sum_{t\in K_2}\rho_t+\sum_{t\in(M\backslash K)_1}\frac{\rho_t}{m_1-k_1}+\sum_{t\in(M\backslash K)_2}\frac{\rho_t}{m_2-k_2}\geqq 0$;

(iii) $K_1 \neq \emptyset$ or $(M\backslash K)_1 \neq \emptyset$ or

$$\frac{1}{k_1}\sum_{t\in K_1}\rho_t + \frac{1}{k_2}\sum_{t\in K_2}\rho_t + \sum_{t\in(M\backslash K)_1}\frac{\rho_t}{m_1-k_1} + \sum_{t\in(M\backslash K)_2}\frac{\rho_t}{m_2-k_2} > 0.$$

Then $\varphi(S) \not\leqq \delta(T,u,v)$.

**Proof.** (a) Suppose to the contrary that $\varphi(S) \leq \delta(T,u,v)$. This implies that

$$G_i(T)F_i(S) - F_i(T)G_i(S) \leqq 0, \tag{3.3$_i$}$$

for each $i \in \underline{p}$, with strict inequality for at least one subscript $l \in \underline{p}$.

From the inequalities (3.2), (3.3$_i$), nonnegativity of $u$ and primal feasibility of $S$ we deduce that

$$2k\Omega_t(S,T,u,v) \leqq 2k\Omega_t(T,T,u,v) \text{ for each } t \in K,$$

with strict inequality holding for at least one $t \in K$ since $u > 0$.

It follows from the properties of $\phi_t$ ($\phi_t$ is increasing and $\phi_t(0) = 0$), that for each $t \in M$,

$$\phi_t\left(2k\Omega_t(S,T,u,v) - 2k\Omega_t(T,T,u,v)\right) \leqq 0, \tag{3.4}$$

Since for each $t \in M\backslash K$,

$$0 \leqq 2(m-k)\Lambda_t(T,v),$$

it follows from the properties of $\phi_t$ that

$$\phi_t\left(-2(m-k)\Lambda_t(T,v)\right) \leqq 0. \tag{3.5}$$

From (3.4) and (3.5) and assumption (i), we deduce that

$$\mathscr{F}\left(S,T;2kb(S,T)\left\{\sum_{i\in I_t}u_i\left[G_i(T)DF_i(T) - F_i(T)DG_i(T)\right] + \sum_{j\in J_t}v_jDH_j(T)\right\}\right) < \\ < -\rho_t d^2(\theta(S,T)),\ t \in K. \tag{3.6}$$

and

$$\mathscr{F}(S,T;b(S,T)2(m-k)\sum_{j\in J_t}v_jDH_j(T) \leqq -\rho_t d^2(\theta(S,T)), t \in M\backslash K. \tag{3.7}$$

From (3.6), summing after $t \in K$ we obtain

$$\sum_{t\in K}\mathscr{F}\left(S,T;kb(S,T)\left\{\sum_{i\in I_t}2u_i\left[G_i(T)DF_i(T) - F_i(T)DG_i(T)\right] + \ +\sum_{j\in J_t}2v_jDH_j(T)\right\}\right) < -\sum_{t\in K}\rho_t d^2(\theta(S,T)).$$

But from the convexity of $\mathscr{F}(S,T;\cdot)$ we have

$$\mathscr{F}\left(S,T;b(S,T)\left\{\sum_{t\in K}\sum_{i\in I_t}2u_i\left[G_i(T)DF_i(T) - F_i(T)DG_i(T)\right] + \ +\sum_{t\in K}\sum_{j\in J_t}2v_jDH_j(T)\right\}\right) =$$

$$\mathscr{F}\left(S,T;b(S,T)\frac{1}{k}\sum_{t\in K}2k\left\{\sum_{i\in I_t}u_i\left[G_i(T)DF_i(T) - F_i(T)DG_i(T)\right] + \ +\sum_{j\in J_t}v_jDH_j(T)\right\}\right) \leqq$$

$$\frac{1}{k}\sum_{t\in K}\mathscr{F}\left(S,T;b(S,T)2k\left\{\sum_{i\in I_t}u_i\left[G_i(T)DF_i(T)-F_i(T)DG_i(T)\right]+\ +\sum_{j\in J_t}v_jDH_j(T)\right\}\right)<-\frac{1}{k}\sum_{t\in K}\rho_t d^2(\theta(S,T)),$$

i.e.

$$\mathscr{F}\left(S,T;b(S,T)\left\{\sum_{t\in K}\sum_{i\in I_t}2u_i\left[G_i(T)DF_i(T)-F_i(T)DG_i(T)\right]+\ +\sum_{t\in K}\sum_{j\in J_t}2v_jDH_j(T)\right\}\right)<$$

$$<-\frac{1}{k}\sum_{t\in K}\rho_t d^2(\theta(S,T)). \tag{3.8}$$

From (3.7), summing after $t\in M\backslash K$ we obtain

$$\sum_{t\in M\backslash K}\mathscr{F}(S,T;b(S,T)2\,(m-k)\sum_{j\in J_t}v_jDH_j(T)\leqq-\sum_{t\in M\backslash K}\rho_t d^2(\theta(S,T))$$

But from the convexity of $\mathscr{F}\,(S,T;\cdot)$ we have

$$\mathscr{F}(S,T;b(S,T)2\sum_{t\in M\backslash K}\sum_{j\in J_t}v_jDH_j(T))=\mathscr{F}(S,T;b(S,T)\sum_{t\in M\backslash K}\frac{1}{m-k}2\,(m-k)\sum_{j\in J_t}v_jDH_j(T))\leqq$$

$$\frac{1}{m-k}\sum_{t\in M\backslash K}\mathscr{F}(S,T;b(S,T)2\,(m-k)\sum_{j\in J_t}v_jDH_j(T))\leqq-\sum_{t\in M\backslash K}\frac{\rho_t}{m-k}d^2(\theta(S,T)),$$

i.e.

$$\mathscr{F}(S,T;b(S,T)2\sum_{t\in M\backslash K}\sum_{j\in J_t}v_jDH_j(T))\leqq-\sum_{t\in M\backslash K}\frac{\rho_t}{m-k}d^2(\theta(S,T)) \tag{3.9}$$

Now, using (3.1), the convexity of $\mathscr{F}\,(S,T;\cdot)$,(3.8) and (3.9), we obtain

$$0\leqq\mathscr{F}\left(S,T;b(S,T)\left\{\sum_{t\in K}\sum_{i\in I_t}2u_i\left[G_i(T)DF_i(T)-F_i(T)DG_i(T)\right]+\sum_{t\in K}\sum_{j\in J_t}2v_jDH_j(T)\right\}\right)$$

$$+\mathscr{F}(S,T;b(S,T)2\sum_{t\in M\backslash K}\sum_{j\in J_t}v_jDH_j(T))<$$

$$-\frac{1}{k}\sum_{t\in K}\rho_t d^2(\theta(S,T))-\sum_{t\in M\backslash K}\frac{\rho_t}{m-k}d^2(\theta(S,T))=-\left(\frac{1}{k}\sum_{t\in K}\rho_t+\sum_{t\in M\backslash K}\frac{\rho_t}{m-k}\right)d^2(\theta(S,T))$$

which contradicts (a) (iii). Therefore, we conclude that $\varphi(S)\nleq\delta(T,u,v)$.

The proofs of (b)-(f) can be obtained following similar arguments to that of part (a) and Zalmai [18].

**Theorem 3.2.** (Strong duality). *Let $S^*\in\mathscr{P}$ be an regular efficient solution of (P), let $\mathscr{F}\,(S,S^*;DF(S^*))=\sum_{k=1}^{n}\left\langle D_kF(S^*),\chi_{S_k}-\chi_{S_k^*}\right\rangle$ for any differentiable function $F:A^n\to\mathbb{R}$ and $S\in A^n$, and assume that any one of the sets of hypotheses specified in Theorem 3.1. holds for all feasible solutions of (D). Then there exist $u^*\in U$ and $v^*\in R_+^q$ such that $(S^*,u^*,v^*)$ is an efficient solution of (D) and $\varphi(S^*)=\delta\,(S^*,u^*,v^*)$.*

**Proof.** By Theorem 2.1. there exist $u^* \in U$ and $v^* \in \mathbb{R}^q_+$ such that $(S^*, u^*, v^*)$ is a feasible solution of (D) and $\varphi(S^*) = \delta(S^*, u^*, v^*)$. That $(S^*, u^*, v^*)$ is efficient for (D) follows from the corresponding parts of Theorem 3.1.

**Remark 3.1.** *Using the previous Theorems 3.1. and 3.2., and techniques from [9] and [17], we can obtain also a converse duality result.*

# 4 Conclusions

In this paper, we have introduced a new class of generalized $(\mathscr{F}, b, \phi, \rho, \theta)$-type-I univex functions. Based on these functions a semiparametric dual model for a multiobjective fractional subset programming problem was introduced. Weak, strong and convex duality theorems were derived.

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# A bilevel programming model for farm planning

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**Abstract**

Farm planning usually uses linear or nonlinear mathematical programming models, which generally achieve one or more goals of the decision maker. The purpose of this study is to develop a model for farm planning, which involves more than one decision makers and uses several conflicting criteria. To this end, the methodology of Bilevel Linear Programming (BLP) for farm planning was applied in order to achieve the optimal production plan. In order to determine the two different levels of BLP we assume that the first goal is the maximization of gross margin for the farmer ("leader") and the second goal is the minimization of fertilizers use which we assume that is represented by the Greek society ("follower"), through the protection of nitrate sensitive areas scheme of the Greek Rural Development Plan 2007-2013. Finally, the results obtained from BLP are compared with the results of the linear programming model.

**KEYWORDS**

bilevel linear programming, farm planning, agri-environmental schemes, nitrate sensitive areas

## 1. INTRODUCTION

Farm planning usually employs linear or nonlinear mathematical programming models by using one goal (usually maximization of gross margin) or more than one conflicting criteria (goals) (such as maximization of gross margin, minimization of risk, minimization of labour use, minimization of fertilizers use, etc.). In addition, the classical Mathematical Programming refers mainly to problems with one decision maker and for this reason the mathematical model has one objective function. Mathematical programming models have been widely used in the agricultural sector since Heady (1954) used a linear programming model to determine the land allocation between two crops. When the farmers oriented and give weight to more than one factor it could be developed a Goal Programming model. Goal programming includes a linear model (LP), in which the objective function represents a weighted level of goals (Romero & Rehman 1984) and focuses on the weights assigned to decision variables according to their relative importance (Baernett, Blake & McCarl 1982). Sumpsi et al. (1997) and Amador et al. (1998) developed a methodology for the analysis and simulation of agricultural systems based on multicriteria techniques. These authors propose weighted goal programming as a methodology for the analysis of decision making. This methodology has been successfully implemented on real agricultural systems in Greece (Manos et al. 2006), (Manos et al. 2008), (Manos et al. 2010). On the other hand, in the case of Multilevel Mathematical Programming more than one decision makers are involved in the decision making process, at different hierarchical levels, and the mathematical model must maintain the structure of their respective levels of hierarchy.

The main objective of this paper is to estimate the optimal farm plan in Nitrate Sensitive Area, employing a BLP model that can achieve the optimal production plan assuming two conflicting goals: the maximization of gross margin and the minimization of fertilizers' use. To this end, bilevel linear programming methodology is applied for farm planning and especially in finding the optimal production plan. Initially, the optimal production plan is found with the use of the linear programming model, which maximizes the gross margin. Then, we use the bilevel linear programming with two different levels of decision making. We assume that the first level belongs to the farmer and the second level to the government. The respective levels use the conflicting criteria of gross margin maximization for farmers, and minimization of the fertilizers use through the protection of nitrate sensitive areas scheme in the Greek Rural Development Plan 2007-2013 implemented by the Greek government. The implementation of the model is done on an average farm, which emerged after a survey questionnaire in 24 farms in Thessaloniki, Pella, Imathia, and Kilkis areas involved in

the action 2.1. “Protection of nitrate sensitive areas” measure 2.1.4 of the Greek Rural Development Plan (RDP) 2007-2013 “Alexandros Baltatzis”.

## 2. BILEVEL LINEAR PROGRAMMING

The classic Mathematical Programming model refers mainly to decision problems with one decision maker and for this reason the classic mathematical model has one objective function or in the case of Multiobjective Mathematical Programming more than one objective functions. All the objective functions are controlled by one decision maker which tries to balance his priorities between them. In the case of Multilevel Mathematical Programming more than one decision makers are involved at different hierarchical levels in the decision making process and the mathematical model must maintain the structure of the respective levels of hierarchy. Bilevel programming problems are hierarchical optimization problems in the sense that their constraints are defined in part by a second parametric optimization problem. So in this case the constraints of the problem involve other mathematical models that represent decision-makers who are lower in the hierarchy.

The bilevel linear programming is a Multilevel Mathematical Programming problem, with two levels. The first appearance of the mathematical model of bilevel linear programming in 1973 was a work of J. Bracken and J. McGill (1973), although the first who use the term bilevel, and multilevel programming was the W. Candler, R. Norton (1977). However, only in the early 80s due attention was given to these problems. The Stackelberg (1952) game is a problem of mathematical game theory identical to the bilevel programming problem. Taking motivation from Stackelberg many authors studied the bilevel linear programming intensively contributing to the Mathematical Programming scientific community.

In the case of bilevel linear programming, upper level corresponds to the “leader” and the lower to the “follower”. For $x \in X \subset R^n$, $y \in Y \subset R^m$, $F: X \times Y \rightarrow R^1$ the bilevel linear problem can be written in the following format:

$$\min_{x \in X} F(x, y) = c_1 x + d_1 y$$

$$\text{subject to } A_1 x + B_1 y \leq b_1$$

$$\min_{y \in Y} f(x, y) = c_2 x + d_2 y$$

$$\text{subject to } A_2 x + B_2 y \leq b_2$$

where $c_1, c_2 \in R^n$, $d_1, d_2 \in R^m$, $b_1 \in R^p$, $b_2 \in R^q$, $A_1 \in R^{p \times n}$, $B_1 \in R^{p \times m}$, $A_2 \in R^{q \times n}$, $B_2 \in R^{q \times m}$. Sets $x$ and $y$ are additional constraints on the variables e.g. the upper and lower limits.

The BLP Model consists of two subproblems: the higher level decision problem (or the leader’s problem) and the lower level decision problem (or the follower’s problem). The objective function for the higher level problem is F(x, y) and for the lower level is f(x, y). The two problems are connected in a way that the leader’s problem sets parameters influencing the follower’s problem and the leader’s problem, in turn, is affected by the outcome of the follower’s problem. The decision sequence is as follows: the leader minimizes his or her objective F(x, y) by select an optimal solution of x from the feasibility set X. Given the optimal solution of x, the follower will optimally make his objective f(x, y). In other words, the follower s uses only his or her local information to make decisions while the leader make use of the complete information including the follower’s possible reaction to the leader’s decision in the decision making process.

## 3. FARM PLANNING MODEL

Farm planning can be done under conditions of uncertainty and we assume that the farmer takes his decisions by situations or policies that may occur at a certain time (Manos 2009). Dempe in his book Foundations of Bilevel Programming (2002) presents a model of bilevel linear programming as the ideal model for problems of environmental conflict, where the “farmer” pollutes the environment during production period and the consumer wants a clean environment in which he acts and consumes the products of the farmer. In this

context presents applications of bilevel linear programming problems in environmental policy (Dempe 1996) approaching the problem that we study. For these reasons, we chose a bilevel linear programming model for farm planning.

In farm planning farmer is usually interested in to maximize the gross margin. On the other hand we feel that society cares for a clean environment in which operates and acts. These two criteria are conflicting, as the farmer tries to achieve the maximization of gross margin using plant protection products, pesticides and fertilizers and with their excessive use pollutes the environment. In order to achieve the perfect coexistence of these two conflicting goals we chose to use a bilevel linear programming model, where several decision makers at different hierarchical levels involved in decision making process. The model developed consists of two levels of which the first goal is pursued by the farmers and comprises the first level of the BLP and the second goal is pursued by the society, through the government, comprises the second level of the BLP.

We chose these two goals by following the next assumptions. We assume that farmers belong to the first level and they have only one goal, to maximize their gross margin. Therefore, in order to maximize their gross margin they produce crops that will generate the maximum economic benefit, without regard for environmental consequences from the cultivation of these products. On the other hand, society wants to reduce the environmental burden. The role of defending the demands of society is undertaken by the Greek government which is trying to protect the environment with the RDP Agri-Environmental schemes but also to meet the farmers' goal for the possible loss of income by specifying an amount of subsidies for those involved in the RDP. More specifically, we chose the action 2.1 "Protection of nitrate sensitive areas" of Rural Development Plan "Alexandros Baltatzis". With the implementation of this action several environmental goals are achieved by reducing the amount of fertilizers and water use and by following crop rotation and set aside. Therefore, we assume that this action meets the desires of the society to protect the environment. The program defines to the farmers the price levels of subsidies and the restrictions on land use, nitrate fertilizer and water use and sets two different scenarios of which the farmers have to follow. By this bilevel problem we want to achieve the optimal decisions by satisfying both sides without having to overcome some historical limits and guarantees. It is a quite flexible mathematical model because it can be seen in isolation from the perspective of society and from the farmers' side.

## 3.1 Goals

The bilevel linear programming is similar to the linear programming with the difference being the area of constraints, which include a limitation as a linear objective function, creating a nested optimization problem involving two problems at two levels. The bilevel linear programming model which is used for farm planning is a max-min linear problem. The general form in which the max-min bilevel linear programming problem can be written is:

$$max_x\left\{min_y\{cx + dy : Ax + By \leq b, and\ x, y \geq 0\}\right\}$$

where the goal of the second level is a conflicting goal of the first level:

$$max_x\{cx + dy\}$$

where y solves the:

$$max_y\{-cx - dy\}$$

subject to:

$$Ax + By \leq b$$
$$x, y \geq 0$$

The goals included in the bilevel linear programming model are:

1. Maximization of the farmer' gross margin and belongs to the first level ("leader"). The objective function to maximize gross margin which is included in the model is:
   $$\max GM = \Sigma\ GMi \times Xi$$
2. Minimizing the fertilizers' use and belongs to the choices of society since it is a social goal and belongs to the second level ("follower"). The objective function to minimize the fertilizers use:
   $$\min FER = \Sigma\ FERi \times Xi$$

In the case of max-min bilevel linear programming, the mathematical form of fertilizers use in the objective function of the second level ("follower") is transformed into:

$$\max FER = -(\Sigma\ FERi \times Xi)$$

The two goals mentioned above are sufficient to develop a model for farm planning which involves more than one decision maker and uses two conflicting criteria.

The model can be written in the following mathematical form:

$$max_{GM}\{\Sigma\ GMi\ x\ Xi\}$$

where y solves the:

$$max_{FER}\{-\Sigma\ FERi\ x\ Xi\}$$

subject to:

$$Ax + By \leq b$$
$$x, y \geq 0$$

## 3.2 Constraints and scenarios

The constraints arise from a questionnaire survey of farms, from farmer's data, but also from the conditions for participating in the action 2.1 "Protection of nitrate sensitive areas". Thus, we used constraints for the total cultivated land, for irrigated or non-irrigated land, for variable costs, for total labour, from Common Agricultural Policy constraints and for the market and other constraints. In addition, constraints for participating in the Action 2.1 "Protection of nitrate sensitive areas" are used.

There are two scenarios, pursuant to which we put constraints on the bilevel linear programming model.

**Scenario A**: "Combination of permanent Set aside and reduction in fertilizers use". Under this scenario the beneficiaries will have to meet the following standards: 1. Set aside. To set permanent set aside area of at least 25% of potentially irrigable land. 2. Fertilizers use. Reduce 30% of the total fertilizers use in relation to the fertilizing action of the respective programs to the remaining 75% of the area.

**Scenario B**: "Combination of rotation – reduction of fertilizers use and set aside". Under this scenario the beneficiaries would have to meet the following standards: 1. Rotation. Crop rotation of at least 20% of eligible irrigated area to the whole area and put in rotation at least once during the 5 years period. A crop rotation applicable only to rain fed crops. 2. Set aside. Set aside area corresponding to at least 5% of the whole area. 3. Fertilizers use. Reduce 30% of the total fertilizers use in relation to the fertilizing action of the respective farm plan, both at 75% of main crop and 20% of the crop rotation.

All this information has been included both in the simple LP model and in the BLP model.

# 4. APPLICATION OF THE FARM PLANNING MODEL

The model is applied to an agricultural area in Northern Greece, which belongs to the Nitrate Sensitive Areas Scheme of the Greek Rural Development Plan 2007-2013. Specifically, we based our analysis on data from a survey of a random sample of 24 farms in areas that participate in the Agri-Environmental schemes of the Greek R.D.P. 2007-2013 and particularly in its action 2.1 "Protection of nitrate sensitive areas". The survey was conducted in May and June 2009 with local visits and personal interviews with heads of the farms. The sample was selected among farms from the prefectures of Thessaloniki - Imathia - Pella and the prefecture of Kilkis. We selected 24 farms participating in the above action. We chose 12 farms from the prefecture of Thessaloniki, 4 from the prefecture of Kilkis, 4 from the prefecture of Pella and 4 from the prefecture of Imathia. The farms selection was done by random sampling. For the collection of technical and economic data of the farms we used a very detailed questionnaire. From the data of this survey resulted the average farm on which the farm planning model was implemented.

The existent production plan of the average farm, which emerged from our research, consists of arable crops and trees. Specifically, 6.5% of the total area is cultivated with soft wheat, 13.3% with hard wheat, 2.5% with barley, 6.9% with vetch, 49.2% with maize, 1.7% with sugarbeet, 0.7% with cotton, 15.7% with alfalfa, while the trees cultivated area covered by 0.5% cherries and 0.3% pears. Finally, the average farm has set aside 2.8% of the total cultivated area.

Table 1. Production plan of the average farm.

| Crops | ha | % |
|---|---|---|
| Soft wheat | 14,0 | 6,5 |
| Hard wheat | 28,8 | 13,3 |
| Barley | 5,3 | 2,5 |
| Vetch | 15,1 | 6,9 |
| Maize | 107,0 | 49,2 |
| Sugarbeet | 3,7 | 1,7 |
| Cotton | 1,5 | 0,7 |
| Alfalfa | 34,1 | 15,7 |
| Cherries | 1,0 | 0,5 |
| Pear | 0,6 | 0,3 |
| Set Aside | 6,2 | 2,8 |
| Total | 217,4 | 100,0 |

## 4.1 Application of the linear programming (LP) model

As we mentioned before, initially we used the linear programming model for farm planning. Simple linear programming model was applied in order to maximize total gross margin for the average farm by using the two scenarios of the action 2.1.

### 4.1.1 Results of linear programming for scenario A

By applying the linear programming model for Scenario A, resulted the optimum production plan. The results show that the farmers abandon the cultivation of cotton and sugarbeet, since it is not proposed to continue cultivate them in the optimal production plan. We also observe a decrease of -74.88% in the cultivated area of alfalfa and by -35.24% in the cultivated area of maize and -6.30% in the cultivated area of pears.

Table 2. Comparison between existent and optimal production plan

| Crops | Existent Production Plan | Optimal Production Plan | Difference |
|---|---|---|---|
| Soft Wheat | 6,5 | 9,10 | 40,94% |
| Hard Wheat | 13,3 | 19,95 | 50,39% |
| Barley | 2,5 | 3,75 | 52,77% |
| Vetch | 6,9 | 8,28 | 19,36% |
| Maize | 49,2 | 31,86 | -35,24% |
| Sugarbeet | 1,7 | 0,00 | -100,00% |
| Cotton | 0,7 | 0,00 | -100,00% |
| Alfalfa | 15,7 | 3,94 | -74,88% |
| Cherries | 0,5 | 0,75 | 56,17% |
| Pear | 0,3 | 0,25 | -6,30% |
| Maize Set Aside | | 7,97 | |
| Sugarbeet Set Aside | | 1,71 | |
| Cotton Set Aside | | 0,69 | |
| Alfalfa Set Aside | | 11,75 | |
| Total Set Aside | 2,8 | 22,12 | 677,58% |
| Total | 100,0 | 100,0 | |

In addition, it is proposed to increase by 56.17% the cultivation area of cherry trees, to increase by 52.77% the area of barley, 50.39% the area of hard wheat, 40.94% the land of soft wheat and 19.36% increase in the cultivation area of vetches. Finally, it is suggested to set aside the 22.12% of the total cultivates area of the average farm in order to follow the rules of the action 2.1 rules. The comparison between the existent production plan and the optimal production plan obtained by linear programming model under the Scenario A of the action 2.1 is presented in the following Table 2.

In order to analyze further the impacts from the implementation of the models we have computed some attributes such as total gross margin, fertilizers use, labour use and water demand. From the comparison of the existent and optimal production plan we observe that gross margin is increased by 1.17%. Also, we observe a reduction in fertilizers use by 30% which is in line with the conditions of participating in the action 2.1 "Protection of nitrate sensitive areas'. Regarding the labour use we observe a reduction of 39.69%, due to increased set aside and finally water demand decrease 46,18%.

Table 3. Comparison between existent and optimal production plan

| Comparison between existent and optimal production plan | |
|---|---|
| Gross margin | 1,17% |
| Fertilizers use | -30,00% |
| Labour use | -39,69% |
| Water demand | -46,18% |

### 4.1.2 Results of linear programming for scenario B

The results show that the farmer has to make a reduction of 46.34% in cultivated area of sugarbeet, 21.46% in maize, 6.30% in pears and 5.27% in cotton cultivation. There is an increase 56.17% in the cherries, 52.77% in barley, 50.39% in hard wheat, 40.94% in soft wheat and an increase of 19.36% in vetch. The participation of set aside in optimal production plan reduced compared with the existent production plan -3.03%. The comparison between the existent production plan and the optimal production plan resulted from Scenario B presented in Table 4.

Table 4. Comparison between existent and optimal production plan

| Crops | Existent production plan | Optimal production plan | Difference |
|---|---|---|---|
| Soft Wheat | 6,5 | 9,10 | 40,94% |
| Hard Wheat | 13,3 | 19,95 | 50,39% |
| Barley | 2,5 | 3,75 | 52,77% |
| Vetch | 6,9 | 8,28 | 19,36% |
| Maize | 49,2 | 38,65 | -21,46% |
| Sugarbeet | 1,7 | 0,92 | -46,34% |
| Cotton | 0,7 | 0,66 | -5,27% |
| Alfalfa | 15,7 | 14,94 | -4,75% |
| Cherries | 0,5 | 0,75 | 56,17% |
| Pear | 0,3 | 0,25 | -6,30% |
| Maize Set Aside | | 1,93 | |
| Sugarbeet Set Aside | | 0,05 | |
| Cotton Set Aside | | 0,03 | |
| Alfalfa Set Aside | | 0,75 | |
| Total Set Aside | 2,8 | 2,76 | -3,03% |
| Total | 100,0 | 100,0 | |

From the comparison of the existent and optimal production plan we observe that gross margin is increased by 10.87%. Also, we observe a reduction in fertilizers use by 30% which is in line with the conditions of participating in the action 2.1 "Protection of nitrate sensitive areas'. Regarding the labour use we observe a reduction of 13.25%, due to increased set aside and finally water demand decrease 16,77%.

Table 5. Comparison between existent and optimal production plan

| Comparison between existent and optimal production plan | |
|---|---|
| Gross margin | 10,87% |
| Fertilizers | -30,00% |
| Labour | -13,25% |
| Water demand | -16,77% |

## 4.2 Application of bilevel linear programming (BLP) model

During the simulation of mathematical model of bilevel linear programming for average farm, the first level "leader" is the farmer and his efforts to maximize his income. The "leader" is trying to maximize gross margin based on specific quantities of inputs (land, labour, capital) that already has. The second level "follower" is trying to minimize the fertilizers use. The level of "follower" desires to calculate the optimal crop plan for each product to have a minimal fertilizers use.

### 4.2.1 Results of bilevel linear programming for scenario A

After applying the bilevel linear programming model for Scenario A, resulted the optimal production plan shown in Table 6. The results of the bilevel linear programming for scenario A suggest the abandonment of sugar beet, cotton and alfalfa cultivations. There is a decrease of 35.66% in the cultivated area of maize, and 6.30% in the cultivated area of pears. In addition, there is an increase of 56.17% in the cultivated area of cherry trees, 52.77% in the area of barley, 50.39% in the area of hard wheat, 40.94% of soft wheat and an increase 19.36% in the cultivated area of vetch. The participation of set aside in optimal production plan increases as compared with the existent production plan by 26.25% of the total cultivated area of the average farm.

Table 6. Comparison between existent and optimal production plan

| Crops | Existent production plan | Optimal production plan | Difference |
|---|---|---|---|
| Soft Wheat | 6,5 | 9,10 | 40,94% |
| Hard Wheat | 13,3 | 19,95 | 50,39% |
| Barley | 2,5 | 3,75 | 52,77% |
| Vetch | 6,9 | 8,28 | 19,36% |
| Maize | 49,2 | 31,66 | -35,66% |
| Sugarbeet | 1,7 | 0,00 | -100,00% |
| Cotton | 0,7 | 0,00 | -100,00% |
| Alfalfa | 15,7 | 0,00 | -100,00% |
| Cherries | 0,5 | 0,75 | 56,17% |
| Pear | 0,3 | 0,25 | -6,30% |
| Maize Set Aside | | 8,16 | |
| Sugarbeet Set Aside | | 1,71 | |
| Cotton Set Aside | | 0,69 | |
| Alfalfa Set Aside | | 15,69 | |
| Total Set Aside | 2,8 | 26,25 | 822,94% |
| Total | 100,0 | 100,0 | |

From the comparison of the existent and optimal production plan we observe that gross margin is decreased by -3.90%. In addition, we observe a reduction in fertilizers' use by 33.97% which is in line with the conditions of participating in the action 2.1 "Protection of nitrate sensitive areas'. Regarding labour use we observe a reduction of 45.38%, due to increased set aside and finally water demand decrease 52.63%.

Table 7. Comparison between existent and optimal production plan

| Comparison between existent and optimal production plan | |
|---|---|
| Gross margin | -3,90% |
| Fertilizers Use | -33,97% |
| Labour | -45,38% |
| Water demand | -52,63% |

### 4.2.2 Results of bilevel linear programming for scenario B

The results of the bilevel programming for Scenario B suggest abandonment of the cultivations of sugar beet and pear. We also note that we have a reduction of 18.47% in the cultivated area of maize, 13.28% in the cultivated area of cherries, 5.27% in cotton and 4.75% reduction in the cultivation of alfalfa. On the other hand, there is an increase of 52.77% in barley, 50.39% in hard wheat, 40.94% in soft wheat and 19.36% in vetch cultivation. The participation of set aside in optimal production plan is reduced compared with the existent production plan, by -2.05%. The comparison between the existent production plan and the optimal production plan resulting under Scenario B presented in Table 8.

Table 8. Comparison between existent and optimal production plan

| Crops | Existent production plan | Optimal production plan | Difference |
|---|---|---|---|
| Soft Wheat | 6,5 | 9,10 | 40,94% |
| Hard Wheat | 13,3 | 19,95 | 50,39% |
| Barley | 2,5 | 3,75 | 52,77% |
| Vetch | 6,9 | 8,28 | 19,36% |
| Maize | 49,2 | 40,12 | -18,47% |
| Sugarbeet | 1,7 | 0,00 | -100,00% |
| Cotton | 0,7 | 0,33 | -52,43% |
| Alfalfa | 15,7 | 14,94 | -4,75% |
| Cherries | 0,5 | 0,75 | 56,17% |
| Pear | 0,3 | 0,00 | -100,00% |
| Maize Set Aside | | 2,01 | |
| Sugarbeet Set Aside | | 0,00 | |
| Cotton Set Aside | | 0,03 | |
| Alfalfa Set Aside | | 0,75 | |
| Total Set Aside | 2,8 | 2,79 | -2,05% |
| Total | 100,0 | 100,0 | |

From the comparison of the existent and optimal production plans we observe that gross margin is increased by 3.59%. In addition, we observe a reduction in fertilizers' use by 31.71% which is in line with the conditions of participating in the action 2.1 "Protection of nitrate sensitive areas'. Regarding labour use we observe a reduction of -14.66%, due to increased set aside and finally water demand decrease by 16,88%.

Table 9. Comparison between existent and optimal production plan

| Comparison between existent and optimal production plan | |
|---|---|
| Gross margin | 3,59% |
| Fertilizers use | -31,71% |
| Labour use | -14,66% |
| Water demand | -16,88% |

## 5. COMPARISON OF RESULTS

The following table shows the results of linear programming and the bilevel linear programming for both scenarios apply to the action 2.1 "Protection of nitrate sensitive areas".

Table 10. Comparison between results of scenario A and scenario B

| Comparison between results of scenario A and scenario B | | | | |
|---|---|---|---|---|
| | Linear Programming | | Bilevel Programming | |
| | A | B | A | B |
| Gross margin | 1,17% | 10,87% | -3,90% | 3,59% |
| Fertilizers use | -30,00% | -30,00% | -33,97% | -31,71% |
| Labour use | -39,69% | -13,25% | -45,38% | -14,66% |
| Water demand | -46,18% | -16,77% | -52,63% | -16,88% |

We can proceed in two different comparisons. First, is the comparison between scenarios A and B in order to see which of the scenarios is more appropriate for farmers who participate in the action 2.1 and the second comparison between the bilevel linear programming with linear programming to see which model achieves both the two conflicting goals we have set for farmers and for society.

The comparison between the two scenarios shows that the Scenario B, which includes a combination of rotation – reduction of fertilizers uses and set aside seems to give better results for the farmers than Scenario A. More specifically, by applying linear programming model, under scenario B gives an increased gross margin by 10.87% compared with the existent production plan, while Scenario A give only 1.17% increase compared with the existent production plan. Total fertilizers use in both scenarios decreased by 30% as set by the rules in participating into action 2.1. On the other hand, the bilevel linear programming shows that

Scenario B has better results for farmers, since the total gross margin is increased by 3.59% while fertilizers use is reduced by 31.71% compared to Scenario A showing total gross margin is reduced by 3.90% and gives higher reduction in fertilizers' use by 33.97%.

Regarding labour use, Scenario A achieves a 45.38% reduction and Scenario B a 14.66% reduction. The water demands for Scenario A are reduced by 52.63% and for Scenario B are reduced by 16.88%. To conclude, the BLP in Scenario B model achieved to increase the total gross margin of farmers but also achieved to reduce the fertilizers' use for society. The second comparison between linear and bilevel linear programming shows that linear programming has better economic results for farmers in the maximization of total gross margin. On the other hand does not achieve satisfactory social benefit in reducing the fertilizer use, as reserves to reduce fertilizer use only to the limits set by the constraints of the problem. The bilevel linear programming, by definition, is trying to achieve the perfect combination between two conflicting goals. Thus, the results show that BLP is reducing the fertilizers use more than 30% which was the limit set by the action 2.1, while maintaining a nearby flat compared with the existent production plan in gross margin for the farmers. Specifically, the reduction in total gross margin is 3.90% in Scenario A and increased 3.59% in Scenario B. The corresponding reduction in the fertilizers use is 33.97% in Scenario A and 31.71% in Scenario B.

Finally, regarding the total labour use we can see that both linear and bilevel linear programming in Scenario A shows a greater reduction than in Scenario B, since it uses set aside, compared with the rotation of the second scenario.

## 6. CONCLUSIONS

This paper presents a bilevel linear programming model for farm planning in agricultural areas that are sensitive to nitrates. The model developed consists of two levels of which the first goal is pursued by the farmers, and comprises the first level of BLP, and the second goal is pursued by the society, through the government, and comprises the second level of BLP. The first of the two hierarchical levels concerns the maximization of farm gross margin and the second level concerns the minimization of fertilizers use' that we assume that society cares for, since it results in a cleaner environment.

The results showed that the linear programming model gives better economic results for farmers, after achieving the unique goal, to maximize total gross margin. The drawbacks are that it ignores the social benefit, which we determined in this study as the environmental goal of reducing the fertilizers use. On the other hand, the bilevel linear programming under the second scenario, manages to combine the two conflicting goals set in the model at the two different levels of the farmers and of the society as expressed by the action 2.1 "Protection of nitrate sensitive areas". BLP model achieved to increase the total gross margin of the farmer but also achieved to reduce fertilizers use. Regarding the comparison between the two scenarios, proposed from the Nitrate Sensitive Areas Scheme of the Greek Rural Development Plan 2007-2013, we can conclude that Scenario B (combination of crop rotation – reduction of fertilizers use and set aside) has more economic benefits to farmers in relation to the Scenario A (combination of permanent Set aside and reduction of fertilizers use). Regarding the reduction of fertilizers' use for society, we can conclude that scenario A has better results but these are very close to those of Scenario B. On the other hand, regarding total labour use, which consists the second social goal, we observe that Scenario B results in a lower decrease in employment in comparison to Scenario A. This is due to the permanent set aside proposed by Scenario A which requires less working hours in comparison to the rotation proposed by Scenario B. Finally, the water demand has higher decrease when the Scenario A is implemented than Scenario B. This reduction is the result of the high percentage of set aside in the optimal production plan. Therefore, the results suggest that Scenario B manages to best combine the two conflicting goals set in the model at the two different levels, of farmers and of society.

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# A multi-level production scheduling model for two stage flowshop

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**Abstract**

Production scheduling is to simultaneously production planning and scheduling the processes of different batches. Also, each final product has its own structure shown with bill of material (BOM). This paper models a two stage flowshop production scheduling problem in the case of dealing with multi level product structure. In the first stage, end items in the BOM structure of different final products are produced in batches, and in the second stage, these produced items are assembled together. As the setups are sequence-dependent (SDS), the problem is NP-Hard. But, there are special characteristics in the problem that a pseudo polynomial branch and cut method is proposed to solve the problem with the objective of achieving optimal makespan.

**KEYWORDS***:* Production scheduling; Multi-level; Flowshop; Sequence-dependent setups (SDS); Makespan**.**

## 1. INTRODUCTION

Multi-level production planning involves in determining batch sizes of different items used in producing various final products. Scheduling is to decide about the timing of different processes over each item. In real cases, production managers should simultaneously take these two above decisions, known as production scheduling. This paper models a two stage flowshop production scheduling problem in the case of dealing with multi level product structures. In the first stage, end items in the BOM structure of different final products are produced in batches, and in the second stage, these produced items are assembled together. Also, this problem is not considered so comprehensively in the literature.

Allahverdi and Aydilek (2010) considered a two-machine flowshop scheduling problem to minimize maximum lateness where processing times were random variables with lower and upper bounds. Allahverdi (2006) addressed the two-machine flowshop scheduling problem where setup times were considered as separate from processing times and where the objective was to minimize total completion time and all setup and processing times on both machines were unknown variables between lower and upper bounds. Flowshop scheduling problem is solved with a number of solution approaches in the literature based on problem assumptions with different accuracy. Most common solution methods could be summarized as:

(a) heuristic or metaheuristic methods such as:
- genetic algorithms (Akrami *et al.* 2006, Fakhrzad and Zare 2009),
- tabu search (Akrami *et al.* 2006),
- threshold acceptance (Fleischmann and Meyr 1997),
- simulated annealing (Meyr 2000),
- particle swarm optimization (Wong *et al.* 2009),
- ant colony optimization (Marimuthu *et al.* 2009),
- hybrid metaheuristics (Edis and Ornek 2009, Jenabi *et al.* 2007, Marimuthu *et al.* 2008),
- experimental heuristics (Clark *et al.* 2009, Huang and Yao 2007).

(b) exact methods such as:
- branch and bound (Kurihara *et al.* 2009),
- dynamic programming (Jans and Degraeve 2004),
- golden section (Ouenniche and Boctor 2001),
- linear programming methods (Tang *et al.* 2000).

All studies aimed to optimize various types of objectives based on researchers' priorities in lot sizing or lot streaming. The main objectives of previous researches can be categorized as minimization of: (1) make span (Edis and Ornek 2009, Kim and Jeong 2009), (2) earliness and tardiness (Iranpoor *et al.* 2007, Li *et al.* 1998), (3) flow time (Marimuthu *et al.* 2008, Marimuthu *et al.* 2009), (4) total cost (Clark *et al.* 2009, Fakhrzad and Zare 2009). Remainder parts of this paper are organized as follows: section 2 provides problem formulation; section 3 explains the proposed cutting plane method. Section 4 draws concluding remarks and future researches.

## 2. PROBLEM FORMULATION

The problem is formulated in this section, based on the following assumptions:

1) These two machines are dependent. After production of machine 1, second machine must get the production, otherwise the production on machine 2 gets block and has to wait until second machine completes its.
2) There are different families of products organized by different types of parts. Products can have in common parts.
3) If the produced batch contains more than need of current family, remained items can be stored to be used in other families who needs such parts.
4) Number of times production can be run is a constant number; Machines must be run on constant number to produce all needed batches.
5) Each family of products has parts structure which defines needed parts and consumption factor. By these structures and consumption factors, demand for each part can be calculated easily.
6) There are two different types of batches. Several times producing batches, we call them "normal batches" for part j, and one time batches, "unusual batches" for part j. Number of parts produced in an unusual batch must be equal to or less than the number of parts produced in an usual batch.
7) Production of any family can be started if all parts demand in its parts structure have been produced completely.
8) When a family has been seated on machine 2, no other families can be seated till all the needed job be accomplished for seated family.
9) There are two different set-up times. First, if we were producing part type j and now we want to produce part type e, there is a defined set-up time shown by $S_{e,f}$. Second, if we want to produce a batch of part type j, we have to pay $\theta \times B_j$ to prepare first machine.
10) If we were producing family type i on the second machine and now we want to produce another type of families, we have to wait some time for setting up the machine shown by $S^2_{i,f}$ .
11) Each type of parts has different processing times on first machine, shown by $P^1_j$
12) Each type of families has different processing times on second machine according to its types of parts, shown by $P^2_{i,j}$.
13) Producing of any unusual batches of known part type can be started if all usual batches of that part type have been produced.
14) Production volumes for any parts or families are exactly equal to their demands; No Storage or shortage is acceptable.
15) Number of production batches is integer and no fractional numbers is available.
16) If family of type I is seated on machine 2, till satisfaction of its needs, first machine can produced no other family's demands.
17) There is no in common sub-products between families. But it is possible to have in common parts at the last level of BOM.
18) if the first machine starts to produce part type j in the $n^{th}$ order for being assembled by the second machine over a final product type i in $r^{th}$order, the amount of production of part type j should be at least as the total need of final product type i for part type j.

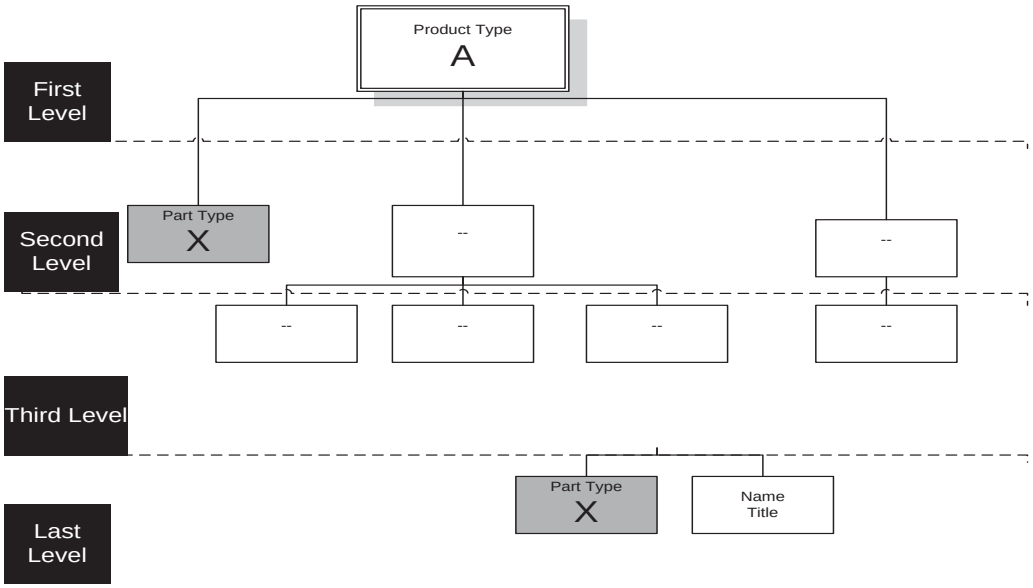


Figure1: The Real BOM of Product A.

By These assumptions we have such problem: 2 | SDS |$C_{max}$.

| Input Parameters | |
|---|---|
| $K$ | number of final product types |
| $i$ | index of final products ($i \in \{1,2,\ldots, K\}$) |
| $r$ | index of order in which final products are assembled on machine two ($r \in \{1,\ldots, K\}$) |
| $N$ | total number of batches that is allowed to be processed on machine one |
| $n$ | index of order for batches of all item types produced on machine one ($n \in \{1,\ldots, N\}$) |
| $j$ | index of item types |
| $A_i$ | number of levels in the BOM of final product type $i$ |
| $a_i$ | index of level of BOM for final product type $i$ ($a_i \in \{1,\ldots, A_i\}$) |
| $d_{i,j,a_i}$ | demand of sub-assembly $j$ in $a_i^{th}$ level of BOM of final product $i$ |
| $\theta_j$ | unit loading and unloading time of an item $j$ over machine one |
| $S^2_{i,0}$ | setup time of processing final product $i$ at the first order on machine two |
| $S^2_{i,f}$ | time of change in setup of machine two when switching from final product $f$ to $i$ |
| $S^1_{j,e}$ | time of change in setup of machine one when switching from item type $e$ to $j$ |
| $M$ | a big enough positive number |
| $P^2_{i,j}$ | unit assembly time for part $j$ on product type $i$ on machine two |
| $P^1_j$ | unit processing time for part $j$ on machine one |
| $a_{i,j}$ | $\begin{cases}1 & \text{if item type } j \text{ is used in the last level of BOM of final product type } i\\ 0 & \text{otherwise}\end{cases}$ |
| $J$ | total number of unique items used in the last level of BOM of all product types |
| **Decision Variables** | |
| $y_{i,r}$ | $\begin{cases}1 & \text{if montage of final product type } i \text{ could be started over machine two in order } r^{th}\\ 0 & \text{otherwise}\end{cases}$ |
| $O_{i,r}$ | $\begin{cases}1 & \text{if montage of final product type } i \text{ starts over machine two in order } r^{th}\\ 0 & \text{otherwise}\end{cases}$ |
| $C^2_i$ | completion time of all demands of final product type $i$ on machine two |

$C_{max}$ maximum completion time of production and montage of all product types on machine two

$B_j$ number of items produced in a usual batch of item type $j$

$w_j$ number of items produced in the final batch of item type $j$

$k_{i,j}$ number of times that production of item type $j$ for family $i$ must be run

$g_{j,r}$ $\begin{cases} 1 \\ 0 \end{cases}$ if the final batch of item $j$ starts for montage over a product in order $r^{th}$ on machine two; otherwise

$R_{j,n,r}$ $\begin{cases} 1 \\ 0 \end{cases}$ if a batch of items $j$ is produced in the $n^{th}$ order on machine one for montage over a product in order $r^{th}$ on machine two; otherwise

$l_{j,r}$ number of batches of item $j$ produced for montage over a product in order $r^{th}$ on machine two

$h_{j,e,n,r}$ $\begin{cases} 1 \\ 0 \end{cases}$ if batches produced in orders $n^{th}$ and $(n-1)^{th}$ on machine one are for item types $j$ and $e$ for montage over a product in order $r^{th}$ on machine two; otherwise

---

$$\min C_{max} = \max \{C_i^2\} \quad (1)$$

s.t.

$$O_{i,r} \le y_{i,r} \qquad \forall\, i,r \quad (2)$$

$$\sum_{r=1}^{K} O_{i,r} = 1 \qquad \forall\, i \quad (3)$$

$$\sum_{i=1}^{K} O_{i,r} = 1 \qquad \forall\, r \quad (4)$$

$$\sum_{i=1}^{K} d_{i,j,A_i} = w_j + B_j \sum_{r=1}^{K} l_{j,r} \qquad \forall\, j \in BOM_i \quad (5)$$

$$w_j g_{j,r} + B_j \sum_{t=1}^{r} l_{j,t} - \sum_{f \neq i} \sum_{t=1}^{r-1} d_{f,j,A}\, O_{f,t} \ge d_{i,j,A_i} - M(1 - y_{i,r}) \qquad \forall\, a_{i,j} a_{f,j} = 1 \quad (6)$$

$$w_j g_{j,r} + B_j \sum_{t=1}^{r} l_{j,t} - \sum_{f \neq i} \sum_{t=1}^{r-1} d_{f,j,A_f}\, O_{f,t} < d_{i,j,A_i} + M y_{i,r} \qquad \forall\, a_{i,j} a_{f,j} = 1 \quad (7)$$

$$B_j \sum_{t=1}^{r-1} l_{j,t} \ge \sum_{i=1}^{K} d_{i,j,A_i} - w_j - M(1 - g_{j,r}) \qquad \forall\, j \in BOM_i \quad (8)$$

$$B_j \sum_{t=1}^{r-1} l_{j,t} < \sum_{i=1}^{k} d_{i,j,A_i} - w_j + M g_{j,r} \qquad \forall\, j \in BOM_i \quad (9)$$

$$C_i^2 \ge \sum_{a_i=1}^{A_i} \sum_{j=1}^{N_a^i} P_{i,j,a_i}^2\, d_{i,j,a_i} + S_{i,0}^2 + \sum_{j \in BOM_i} P_j^1 l_{j,1} B_j + \sum_{j \in BOM_i} \sum_{e \neq j} \sum_{n=1}^{N} h_{j,e,n,1} S_{j,e}^1 + \sum_{j \in BOM_i} \theta_j l_{j,1} B_j - \mathrm{M}\,(1 - O_{i,1}) \qquad \forall\, i \quad (10)$$

$$C_i^2 \geq C_f^2 + \sum_{a=1}^{A_i} \sum_{j=1}^{N_a^i} P_{i,j,a_i}^2 d_{i,j,a_i} + S_{i,f}^2 + \sum_{j \in BOM_i} P_j^1 \left\{ g_{j,r} w_j + B_j \sum_{t=1}^{r} l_{j,t} - \sum_{f=1}^{K} \sum_{t=1}^{r-1} d_{f,j,A_i} O_{f,t} \right\} + \sum_{j \in BOM_i} \sum_{e \neq j} \sum_{n=1}^{N} h_{j,e,n,r} S_{j,e}^1 + \sum_{j \in BOM_i} \theta_j d_{i,j,A_i} - M\{2 - (O_{i,r} + O_{f,r-1})\} \quad \forall i, \forall r > 1 \quad (11)$$

$$\frac{R_{j,n,r} + R_{e,n-1,r} - 1.5}{2} \leq h_{j,e,n,r} \leq \frac{0.5 + R_{j,n,r} + R_{e,n-1,r}}{2} \quad \forall j,n,e,r \quad (12)$$

$$\sum_{i} \sum_{j \in BOM_i} k_{i,j} \leq N \quad (13)$$

$$\sum_{j=1}^{J} R_{j,n,r} = 1 \quad \forall n, \forall r \quad (14)$$

$$\sum_{r=1}^{K} \sum_{j=1}^{J} R_{j,n,r} = 1 \quad \forall n \quad (15)$$

$$\sum_{r=1}^{K} \sum_{n=1}^{N} R_{j,n,r} = \sum_{i=1}^{K} k_{i,j} \quad \forall j \in BOM_i \quad (16)$$

$$\sum_{n=1}^{N} R_{j,n,r} \geq \sum_{i=1}^{K} a_{i,j} O_{i,r} \quad \forall j, \forall r \quad (17)$$

$$y_{i,r}, O_{i,r}, R_{j,n,r}, g_{j,r}, h_{j,e,n,r} \in \{0,1\} \quad \forall i,j,e,n,r \quad (18)$$

$$B_j, w_j, k_{i,j}, l_{j,r} \in Z^+ \quad \forall i,j,r \quad (19)$$

$$C_i^2 \geq 0 \quad \forall i \quad (20)$$

In the model, objective function (1) aims at minimizing the makespan. Constraints (2) to (4) determine the sequence of different final product types over machine two. Constraint (2) along with (6) and (7) determine possible times for processing different product types in the terms of a sequence. Constraints (3) and (4) determine the actual position of every final product type among those possible places determined by (6) and (7). Constraint (5) relates demand volume of different items $j$ to usual batch size $B_j$ and a final batch size $w_j$. Constraints (8) and (9) determine the order in which a final batch for item $j$ with size $w_j$ should be produced. Constraints (10) and (11) determine the total completion time of production of all items needed in the BOM structure of final product type $i$ over machine one and assembling these items together to satisfy the demand volume of the final product type $i$ over machine two. It is clear that either (10) or (11) are binding in a feasible production schedule. In other words, constraint (10) determine the total completion time of satisfying demand volume of final product $i$ when $i$ is the first final product assembled over machine two. On the other hand, constraint (11) determine the total completion time of satisfying demand volume of final product $i$ when $i$ is not the first final product assembled over machine two. Constraint (12) determine if batches produced in orders $n^{th}$ and $(n-1)^{th}$ on machine one are for item types $j$ and $e$ for montage over a product in order $r^{th}$ on machine two (i.e. $h_{j,e,n,r}$=1). Constraint (12) along with introducing $h_{j,e,n,r}$ helps in

determining if there need to change the setup over machine one from producing item type *e* to item type *j* and considering $S^1_{j,e}$ amount of time for this change in the total completion time of final product type *i* in (10) and (11). This constraint is written for avoiding nonlinear term of $R_{j,n,r}\ R_{e,n-1,r}$ with the introduction of $h_{j,e,n,r}$. Constraint (13) dictates the total number of batches that are allowed to be processed on machine one. Constraints (14) to (17) together assign a batch of item type *j* to position n in the sequence on machine one while this batch will be used in the montage process a final product type in position *r* on machine two. Relations (18) to (20) state the type of decision variables in the model.

## 3. BRANCH AND CUT METHOD

In this section the proposed cutting plane method is described in detail. First, some lemmas and pruning rules are provided.

**Lemma 1.** Let $D_i$ be the approximate time needed for completion and montage of all orders for final product type *i*. Then, in the depth of *v* of the search tree with known values of $B_1$ to $B_v$, the lower bound of each $D_i$ (*i.e.*$D_i^{min} \le D_i$) could be calculated as

$$C_i^2 \ge \sum_{j \in BOM_i} d_{i,j,A} P_j^1 + \sum_{a=1}^{A_i} \sum_{j=1}^{N_d} P_{i,j,a}^2 d_{i,j,a} + S_2^* = D_i^{min} \forall\ i$$

**Rule 1 (a lower bound for $C_{max}$)**

Consider all orders of the same product type *i* as a job with duration of $D_i^{min}$. Sequencing these jobs according to *SPT* (*Shortest Processing Time*) dispatching rule results in a lower bound for $C_{max}$.

**Lemma 2.** Let $D_i$ be the approximate time needed for completion and montage of all orders for final product type *i*. Then, in the depth of *v* of the search tree with known values of $B_1$ to $B_v$, the upper bound of each $D_i$ (*i.e.*$D_i \le D_i^{max}$) could be calculated as

$$D_i^{max} = \max_{f \ne i}\{S_{i,f}^2\} + \sum_{a=1}^{A_i} \sum_{j=1}^{N_a^i} P_{i,j,a}^2 d_{i,j,a} + \sum_{\substack{j=1 \\ j \in BOM_i}}^{v} \left\lceil \frac{d_{i,j,A}}{B_j} \right\rceil B_j P_j^1 + \sum_{\substack{j=v+1 \\ j \in BOM_i}}^{J} d_{i,j,A} P_j^1$$

$$+ \sum_{\substack{j=1 \\ j \in BOM_i}}^{v} \left(\max_{j,e}\{S_{j,e}^1\}\right) \left\lceil \frac{d_{i,j,A}}{B_j} \right\rceil + \sum_{\substack{j=v+1 \\ j \in BOM_i}}^{J} \left(\max_{j,e}\{S_{j,e}^1\}\right) d_{i,j,A} + \sum_{\substack{j=1 \\ j \in BOM_i}}^{v} \theta_j \left\lceil \frac{d_{i,j,A}}{B_j} \right\rceil B_j$$

$$+ \sum_{\substack{j=v+1 \\ j \in BOM_i}}^{J} \theta_j d_{i,j,A}$$

**Rule 2 (an upper bound for $C_{max}$)**

Consider all orders of the same product type $i$ as a job with duration of $D_i^{max}$. In depth $v$ of the search tree with known values of $B_1$ to $B_v$, sequencing these jobs according to *LPT* (*Longest Processing Time*) dispatching rule results in an upper bound for $C_{max}$.

Also, the pseudo-code of this algorithm is as followings:

```
1.  Let j=1.
2.  branch the current node
3.  call pruning rules
4.  If current node is pruned then
5.      if there are undecided nodes then
6.          backtrack to a neighbor node
7.          go to line 3
8.      else
9.          report the optimal solution
10.         stop the algorithm
11.     end
12. else
13.     if j = J then
14.         call Simplex algorithm
15.         if solution space is  null then
16.             go to line 5
17.         else
18.             update upper / lower bounds
19.             if the optimal solution of the Simplex is integral then
20.                 go to line 5
21.             else
22.                 call Separation algorithm
23.                 add new cutting planes to constraints
24.                 delete redundant constraints / cuts
25.                 go to line 14
26.             end
27.         end
28.     else
29.         go to line 2
30.     end
31. end
```

## 4. CONCLUSIONS AND FUTURE RESEARCHES

In this study, we proposed a new mathematical formulation for production scheduling problem with sequence-dependent setup times/costs. This model is capable of determining lot sizing and lot streaming of multiple products in a flow shop. To the best of our knowledge, this model is the first that formulates both lot sizing and lot streaming concepts in a unique model for flow shop production environment. For future researches, the proposed model can be extended by taking into account the following assumptions which could make the model much closer to some of practical applications: (1) finite capacity inter-stage buffers could be utilized in real world applications, (2) machine blocking could be occurred because of different reasons like unpredictable machine breakdown, finite inter-stage buffer capacity and so forth, (3) other production environments such as job shop, open shop and so on could be investigated, (4) lot splitting could be assumed in the model.

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# A single machine production scheduling model with variable processing times

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**Abstract**

This paper models single machine production scheduling problem (SMPSP) with multi production in the case of dealing with compressible processing time. All processing times are unknown variables where the only known information is the lower and upper bounds processing times of each job. Processing time for each product is considered as a decision variable which can include different amounts of time caused by fluctuation in the batch size. Here the objective is to determine the quantity of batches and the size of each batch to minimize total weighted tardiness and earliness.

**KEYWORDS***:* Production scheduling; single machine; variable processing time; total weighted tardiness and earliness

## 1. INTRODUCTION

This paper focuses on modeling a single machine production scheduling problem (SMPSP) with variable processing times. The machine is available from the beginning of period. Each production has a deterministic and specific demand. In this problem, products are processed in batches and the period of production is limited and specific.

Manufacturing systems with different objectives require different optimization criteria regarding stock size, earliness and tardiness, mean lead time, idle times, due-date reliability, setup times, processing times, and makespan. Nowadays in order to eliminate hidden production and inventory problems which cause high costs for example, setup time, waiting time, delivery time, extra labor costs, rework, tied-up capital and order changes, manufacturers must consider scheduling problems involving not only the tardiness penalty, but also the earliness penalty. So here the objective is to determine the quantity of batches and the size of each batch provided that the total weighted earliness and tardiness be minimized. To the best of our knowledge, there is no previous work in the literature that presents the production scheduling model with these assumptions.

Since the single-machine configuration represents a building block for more complex configurations, researchers have dealt with it in some detail. On the other hand, the scheduling problems have been addressed by some researchers where job processing times are variable.

Kolahan and Liang (1998) considered a single machine sequencing problem with variable processing times and sequence-dependent setups. Allahverdi and Aydilek (2010) considered a two-machine flowshop scheduling problem to minimize maximum lateness where processing times were random variables with lower and upper bounds. Tseng et al. (2009) considered the single machine total tardiness problem with controllable processing times. Koulamas et al. (2010) presented a unified analysis for single-machine scheduling problems in which the actual job processing times are controlled by either a linear or a convex resource allocation function and also vary concurrently depending on either the job's position in the sequence and/or on the total processing time of the already processed jobs. Nowicki and Zdrzal-ka (1995) addressed the bicriterion problem of preemptive scheduling a number of jobs on parallel identical and parallel uniform machines, under the assumption that jobs have processing costs which are linear functions of variable processing times. Allahverdi (2006) addressed the two-machine flowshop scheduling problem where setup times were considered as separate from processing times and where the objective was to minimize total completion time and All setup and processing times on both machines were unknown variables between lower and upper bounds. Chenga et al. (1996) considered a due-date assignment and single-machine

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scheduling problem in which the jobs have compressible processing times. Ng et al. (2004) addressed the Single machine batch scheduling with jointly compressible setup and processing times.

The remainder of the paper is organized as follows. In section 2, the problem formulation and modeling is described. Solution method is described in section 3, and computational results are presented in section 4. Section 5 give short concluding remarks.

## 2. PROBLEM FORMULATION

In this section the problem is formulated. First, these are the main assumptions of the problem:

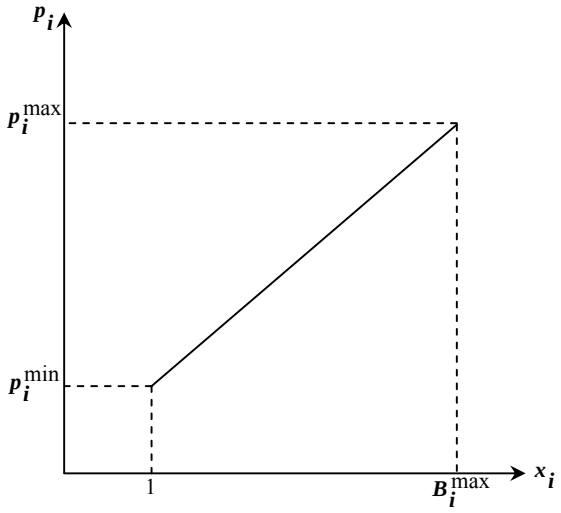


Figure 1. Variation of unit process time by batch size

- There are a number of job families to be processed over a single machine in the case of sequence dependent setup times.
- The process times are compressible, based on batch sizes according to Figure 1.
- Whole the processes should be finished before a known available time.
- Batch sizes of each job family should be less than an upper bound.
- The objective is to minimize the sum of weighted tardiness and earliness.

| Parameters | |
|---|---|
| $ct_i$ | unit tardiness cost of product *i* |
| $ce_i$ | unit earliness cost of product *i* |
| $st_i$ | setup time for producing a batch of product type *i* |
| $B_i^{max}$ | maximum batch size of product type *i* |
| $D_i$ | demand volume of product type *i* |
| $d_i$ | due date of delivering product type *i* |
| AT | total avaible time |
| **decision variables** | |
| $x_i$ | ordinary batch size of product type *i* |
| $w_i$ | size of the last batch of product type *i* |
| $y_i$ | number of batches of product type *i* |
| $E_i$ | earliness in delivering product type *i* |
| $T_i$ | tardiness in delivering product type *i* |
| $u_{i,j}$ | $\begin{cases} 1 & \text{if product type i is produced exactly after other } j\text{-1 product types} \\ 0 & \text{Otherwise} \end{cases}$ |
| $p_i$ | ordinary process time of product type *i* |

$$\min \quad Z = \sum_{i=1}^{n} \{ct_i \times T_i + ce_i \times E_i\} \tag{1}$$

s.t.

$$x_i \times (y_i - 1) + w_i = D_i \qquad \forall \ i \tag{2}$$

$$w_i \le x_i \le B_i^{max} \qquad \forall \ i \tag{3}$$

$$\sum_{i=1}^{n} \{st_i \times y_i + p_i \times x_i \times y_i + p_i^{'} \times w_i\} \le AT \tag{4}$$

$$p_i = p_i^{\max} - \frac{p_i^{\max} - p_i^{\min}}{B_i^{\max} - 1} \times (B_i^{\max} - x_i) \qquad \forall \ i \tag{5}$$

$$p_i^{'} = p_i^{\max} - \frac{p_i^{\max} - p_i^{\min}}{B_i^{\max} - 1} \times (B_i^{\max} - w_i) \qquad \forall \ i \tag{6}$$

$$\sum_{i=1}^{n} u_{i,j} = 1 \qquad \forall \ j \tag{7}$$

$$\sum_{j=1}^{n} u_{i,j} = 1 \qquad \forall \ i \tag{8}$$

$$F_i = S_i + st_i \times y_i + p_i \times x_i \times (y_i - 1) + p'_i \times w_i \qquad \forall \ i \tag{9}$$

$$S_i = \sum_{k \ne i}^{n} \sum_{j=2}^{n} \{F_k \times u_{k,j-1} \times u_{i,j}\} \qquad \forall \ i \tag{10}$$

$$T_i - E_i = F_i - d_i \qquad \forall \ i \tag{11}$$

$$T_i \times E_i = 0 \qquad \forall \ i \tag{12}$$

$$T_i, E_i, S_i, F_i, p_i, p_i^{'} \ge 0 \qquad \forall \ i \tag{13}$$

$$x_i, w_i, y_i \in Z_0^+ \tag{14}$$

$$u_{i,j} \in \{0,1\} \tag{15}$$

Objective (1) aims to minimize the total weighted tardiness plus earliness simultaneously. Constraint (2) indicates that the total demand of item *i* should be divided into $y_i$ number of batches such that ($y_i$-1) number of these batches are all of equal size xi and the last batch is of size $w_i$. Constraint (3) dictates the size of the last batch and ordinary batches before the last one. Constraint (4) indicates that the total productions should be done in a limited amount of available time. Constraints (5) and (6) indicate the relation between the unit process time and the batch size. In other words, these two constraints declare the compressibility of the process times by choosing the size of batches. Constraints (7) and (8) together determine the sequence of batches for different items. Constraints (9) and (10) together determine the finish time and the start time of processing different item types based in their sequences determined from (9) and (10). Constraints (11) and (12) together determine the amount of tardiness / earliness of each item type when the process of each item type could be contributed with being early finished, or tardy finished or on time delivered by regarding the corresponding due date. Also, constraint (12) confirms that we could not accept the cases that an item type encountered with both earliness and tardiness simultaneously. Finally, constraints (13), (14) and (15) state the type of decision variables.

## 3- BRANCH AND BOUND METHOD

In this section, the outline of the proposed branch and bound method is presented via the following pseudo codes:

```
1.  Let j=1.
2.  branch the current node
3.  call pruning rules
4.  If current node is pruned then
5.      if there are undecided nodes then
6.          backtrack to a neighbor node
7.          go to line 3
8.      else
9.          report the optimal solution
10.         stop the algorithm
11.     end
12. else
13.     if j = J then
14.         call Simplex algorithm
15.         if solution space is  null then
16.             go to line 5
17.         else
18.             update upper / lower bounds
19.             if the optimal solution of the Simplex is integral then
20.                 go to line 5
21.             else
22.                 call Separation algorithm
23.                 add new cutting planes to constraints
24.                 delete redundant constraints / cuts
25.                 go to line 14
26.             end
27.         end
28.     else
29.         go to line 2
30.     end
31. end
```

## 4. EXPERIMENT RESULTS

In this section, the results of solving the proposed model for five problems by branch and bound are presented. In each problem we put the $B_i^{max}$ equal to $D_i$ and solve the problem. Tables 1 up to 5 respectively present data for SMPSP with variable processing times with three, four, five, six and seven jobs.

**Table 1:** Data for single machine production scheduling problem with three jobs

| Jobs | $Ct_i$ | $Ce_i$ | $St_i$ | $P_i^{min}$ | $P_i^{max}$ | $B_i^{max} = D_i$ | $d_i$ | AT |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 5 | 20 | 30 | 50 | 2880 | 3360 |
| 2 | 1 | 1 | 4 | 15 | 20 | 44 | 2400 | 3360 |
| 3 | 1 | 1 | 5 | 17 | 23 | 33 | 1920 | 3360 |

**Table 2:** Data for single machine production scheduling problem with four jobs

| Jobs | $Ct_i$ | $Ce_i$ | $St_i$ | $P_i^{min}$ | $P_i^{max}$ | $B_i^{max} = D_i$ | $d_i$ | AT |
|---|---|---|---|---|---|---|---|---|

| Jobs | $Ct_i$ | $Ce_i$ | $St_i$ | $P_i^{min}$ | $P_i^{max}$ | $B_i^{max} = D_i$ | $d_i$ | AT |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 0 | 5 | 30 | 35 | 35 | 2250 | 3360 |
| 2 | 1.5 | 1 | 6 | 20 | 27 | 24 | 1420 | 3360 |
| 3 | 1 | 0.5 | 3 | 14 | 20 | 20 | 800 | 3360 |
| 4 | 2.5 | 1 | 4 | 9 | 12 | 30 | 950 | 3360 |

**Table 3:** Data for single machine production scheduling problem with five jobs

| Jobs | $Ct_i$ | $Ce_i$ | $St_i$ | $P_i^{min}$ | $P_i^{max}$ | $B_i^{max} = D_i$ | $d_i$ | AT |
|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 2.5 | 4 | 35 | 40 | 35 | 3000 | 4800 |
| 2 | 3.5 | 1.5 | 7 | 21 | 27 | 34 | 2420 | 4800 |
| 3 | 4 | 2 | 3 | 17 | 20 | 17 | 1006 | 4800 |
| 4 | 2 | 1 | 5 | 19 | 24 | 24 | 950 | 4800 |
| 5 | 4 | 2 | 4 | 22 | 27 | 12 | 700 | 4800 |

**Table 4:** Data for single machine production scheduling problem with six jobs

| Jobs | $Ct_i$ | $Ce_i$ | $St_i$ | $P_i^{min}$ | $P_i^{max}$ | $B_i^{max} = D_i$ | $d_i$ | AT |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0.5 | 2 | 27 | 32 | 30 | 2020 | 4800 |
| 2 | 3.5 | 1.5 | 3 | 13 | 18 | 24 | 1800 | 4800 |
| 3 | 2 | 1 | 4 | 17 | 22 | 26 | 1500 | 4800 |
| 4 | 4 | 2 | 6 | 23 | 28 | 18 | 2500 | 4800 |
| 5 | 3 | 1.5 | 5 | 19 | 25 | 29 | 1550 | 4800 |
| 6 | 2.5 | 1 | 4 | 12 | 15 | 15 | 1800 | 4800 |

**Table 5:** Data for single machine production scheduling problem with seven jobs

| Jobs | $Ct_i$ | $Ce_i$ | $st_i$ | $P_i^{min}$ | $P_i^{max}$ | $B_i^{max} = D_i$ | $d_i$ | AT |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0.5 | 2 | 12 | 15 | 41 | 1800 | 4800 |
| 2 | 1 | 0.5 | 1 | 7 | 13 | 32 | 2020 | 4800 |
| 3 | 1 | 0.5 | 3 | 9 | 14 | 43 | 1950 | 4800 |
| 4 | 1 | 0.5 | 2 | 13 | 19 | 29 | 1850 | 4800 |
| 5 | 1 | 0.5 | 4.5 | 10 | 14 | 35 | 1600 | 4800 |
| 6 | 1 | 0.5 | 3 | 8 | 13 | 34 | 1320 | 4800 |
| 7 | 1 | 0.5 | 1 | 11 | 17 | 27 | 1780 | 4800 |

Tables 1 to 10 show the results of solving SMPSP with three, four, five, six and seven jobs respectively. We solved the problems by branch and mound method.

**Table 6:** Solutions of the SMPSP with three jobs

| Jobs | $y_i$ | $x_i$ | $w_i$ | $E_i$ | $T_i$ | Position | $P_i$ | $S_i$ | $F_i$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 50 | 1 | 1 | 1630 | 0 | 1 | 20 | 9 | 1250 |
| 2 | 6 | 8 | 4 | 0 | 0 | 3 | 15.81 | 1902 | 2619.95 |
| 3 | 17 | 2 | 1 | 18 | 219.95 | 2 | 17.18 | 1250 | 1902 |

Concluding from table 6 the optimal sequence for this problem is 1 - 3 – 2. processing times for job 1, job 2 and job 3 are 20, 15.81 and 17.18 respectively. Optimal number of batch and batch size for job1, job 2and job 3 are (50,1), (6,8) and (17,2) respectively.

**Table 7:** Solutions of the SMPSP with four jobs

| Jobs | $y_i$ | $x_i$ | $w_i$ | $E_i$ | $T_i$ | Position | $P_i$ | $S_i$ | $F_i$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 6 | 6 | 5 | 0 | 134.2 | 4 | 30.74 | 1279.2 | 2384.2 |
| 2 | 5 | 5 | 4 | 140.793 | 0 | 3 | 21.22 | 741.2 | 1279.2 |
| 3 | 1 | 20 | 20 | 397 | 0 | 1 | 20 | 0 | 403 |
| 4 | 2 | 24 | 6 | 208 | 0 | 2 | 11.38 | 403 | 741 |

As shown in table 7 the optimal sequence is 3 – 4 – 2 – 1. Processing times, number of batches and size of batches for job1, job2, job3 and job4 are (30.74, 6, 6), (21.22, 5, 5), (20, 1, 20) and (11.38,2,24) respectively.

**Table 8:** Solutions of the SMPSP with five jobs

| Jobs | $y_i$ | $x_i$ | $w_i$ | $E_i$ | $T_i$ | Position | $P_i$ | $S_i$ | $F_i$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 4 | 4 | 0 | 61.43 | 5 | 37.43 | 1325.64 | 2061.43 |
| 2 | 3 | 4 | 4 | 94.36 | 0 | 4 | 23.56 | 1033 | 1325.64 |
| 3 | 5 | 3 | 2 | 328 | 0 | 2 | 18.5 | 413 | 672 |
| 4 | 1 | 17 | 17 | 1087 | 0 | 1 | 21.37 | 0 | 413 |
| 5 | 4 | 4 | 3 | 0 | 33 | 3 | 22 | 23.07 | 1033 |

As result of table 8 optimal sequence for this problem with 5 jobs is 4 – 3 – 5 – 2 – 1. Processing times, number of batches, and batch sizes for each job are shown in column of $P_i$, $y_i$ and $x_i$ respectively.

**Table 9:** Solutions of the SMPSP with six jobs

| Jobs | $y_i$ | $x_i$ | $w_i$ | $E_i$ | $T_i$ | Position | $P_i$ | $S_i$ | $F_i$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 30 | 0 | 0 | 803.66 | 5 | 32 | 1859.66 | 2823.66 |
| 2 | 1 | 24 | 24 | 1365 | 0 | 1 | 18 | 0 | 435 |

| Jobs | $y_i$ | $x_i$ | $w_i$ | $E_i$ | $T_i$ | Position | $P_i$ | $S_i$ | $F_i$ |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 7 | 4 | 2 | 0 | 359.66 | 4 | 17.6 | 1374.86 | 1859.66 |
| 4 | 1 | 18 | 18 | 0 | 833.66 | 6 | 28 | 2823.66 | 3333.66 |
| 5 | 2 | 29 | 0 | 175.14 | 0 | 3 | 25 | 639.86 | 1374.86 |
| 6 | 3 | 5 | 5 | 1160.14 | 0 | 2 | 12.86 | 435 | 639.86 |

Concluding from table 9 the optimal sequence is 2 – 6 – 5 – 3 – 1 – 4. Other solutions acquired from lingo i.e. processing times, number of batches and size of batch for each job type are shown in table 9.

**Table 10:** Solutions of the SMPSP with seven jobs

| Jobs | $y_i$ | $x_i$ | $w_i$ | $E_i$ | $T_i$ | Position | $P_i$ | $S_i$ | $F_i$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 9 | 5 | 1 | 0 | 917.96 | 7 | 12.3 | 2195.96 | 2717.96 |
| 2 | 16 | 2 | 2 | 0 | 175.96 | 6 | 7.19 | 1949.76 | 2195.96 |
| 3 | 9 | 5 | 3 | 0.24 | 0 | 5 | 9.48 | 1516 | 1949.76 |
| 4 | 2 | 22 | 7 | 1361 | 0 | 1 | 17.5 | 0 | 489 |
| 5 | 6 | 6 | 5 | 714 | 0 | 2 | 10.59 | 489 | 886 |
| 6 | 7 | 5 | 4 | 121 | 0 | 3 | 8.6 | 886 | 1199 |
| 7 | 14 | 2 | 1 | 264 | 0 | 4 | 11.23 | 1199 | 1516 |

## 5. CONCLUSIONS

This paper provided an efficient branch and bound method to solve a NP-Hard scheduling problem. The current research could be developed for the case of other scheduling environments such as job shop, flow shop and open shop. Also, the paper could be considered in multi-objective sense. Furthermore, all input parameters were deterministic and in real life cases they could be considered stochastic.

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# Integrating Taylor's expansion on the lexicographic approach for unconstrained optimization

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**Abstract**

In the present paper, the proposed methodology, transforms the original unconstrained optimization problem to an equivalent Lexicographic Multiobjective Optimization one, whereas the last objective function is equivalent to the original objective one. At the proposed methodology, which convergence is proved, each term of Taylor's expansion is treated as a separate entity. It is introduced a proper combination of the terms of Taylor's expansion of the initial unconstrained optimization problem, so as, well known iterative methods constitute a special case of the proposed methodology. Moreover, using the proposed methodology is like to use a method from a class of methods, like Broyden-Fletcher-Goldfarb-Shanno from the class of quadratic methods, by taking into account the advantages of methods from another class of methods, such as steepest descent method. Notice that, the proposed methodology uses the same optimization method from the beginning up to the end and it manages to converge to more optimum points than the optimization methods that it uses. Moreover, it can be used regardless of the chosen numerical optimization method. Last but not least, at the present paper the information included in infinite terms of Taylor's expansion is used without the need of computing all of them separately. Numerical applications to well known test functions, with promising results are presented.

*Keywords:* unconstrained optimization; lexicographic optimization; Taylor series; deterministic optimization methods

## 1 Introduction

Unconstrained optimization problems arise in virtually all areas of science and engineering, in many areas of the social sciences and generally are given by the following formula:

$$\begin{array}{l} Given \quad F : \mathbb{R}^n \to \mathbb{R}, \\ \min \{F(x) : x \in \mathbb{R}^n\} \end{array} \tag{1}$$

where the independent variables of the vector $x = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^n$ are referred to as *decision variables* and the function $F$ that wish to minimize is called objective function and returns a scalar. Many times the above problem is called decision problem and involves finding the best vector $x$ of all possible vectors in $\mathbb{R}^n$ or in a set $\Omega \subset \mathbb{R}^n$, [1, 8].

In order to manage the above problem iterative methods are usually applied. These methods start from an initial point, which is a first estimate of the optimum point and then produce a sequence of points leading to the optimal point, intermediate points are improving the initial estimate. The iterative process ends when properly stopping criteria are met, [1, 8, 9].

In this paper, the proposed methodology, called Taylor Lexicographic Sequential Optimization (TLSO), transforms the original unconstrained optimization problem given by Equation

(1) to an equivalent Lexicographic Multiobjective Optimization (LMO) one by using proper combinations of the Taylor's expansion terms, [2, 3, 4, 5, 10]. It is worth to notice that the last objective function of the proposed LMO problem is identical to the original objective function. The proposed methodology is based on the following two important points: firstly, optimize problems of smaller dimension and thus simpler problems than the original one, which arise from the transformation of the original optimization problem to an equivalent LMO one and secondly, take into account information which encloses the infinite terms of Taylor's expansion, without the necessity of calculating all of them.

In Section 2 is presented the proposed TLSO methodology and the proposed technique for using the truncation error of a Taylor's expansion, without having to calculate all the infinite terms. Moreover, it presented how deterministic optimization methods, which are belonging on different classes, can be combined under TLSO methodology. Some numerical applications to well-known test functions are given in Section 3. Conclusions and further research aspects are presented in Section 4.

# 2 The proposed idea

## 2.1 The proposed (TLSO) methodology and the proposed technique for the exact truncation error of a Taylor's expansion calculation

It is known that for a function $F$ that has $n$ continuous derivatives on the neighbourhood of a point $\alpha$, the Taylor's expansion of $F$ is:

$$\begin{aligned} F(x_1,\cdots,x_n) &= \sum_{k=0}^{m-1}\left\{\frac{1}{k!}\left[\sum_{j=1}^{n}(x_j-\alpha_j)\frac{\partial}{\partial x_j}\right]^k F(\alpha_1,\cdots,\alpha_n)\right\}_{x_1=\alpha_1,\ldots,x_n=\alpha_n} + \\ &+ \sum_{k=m}^{\infty}\left\{\frac{1}{k!}\left[\sum_{j=1}^{n}(x_j-\alpha_j)\frac{\partial}{\partial x_j}\right]^k F(\alpha_1,\cdots,\alpha_n)\right\} \end{aligned} \tag{2}$$

In most iterative optimization methods, the objective function is approached according to the relation (2), for a small $m$ and used the first sum of the relation (2). The infinite terms, given by the second sum of the relation (2), consist the truncation error. In the present paper, the terms of Taylor's expansion are introduced gradually and exhausted, according to the lexicographic concept [10], ending after gradually building, exactly the objective function of the original optimization problem. Moreover, the proposed technique, take into account information which encloses the infinite terms of Taylor's expansion, without the need of calculating all of them.

**Definition 2.1.** *For a function given by the relation (2) are defined* $(m+1)$*-tuple of sets as follows:*

$$t_0(x), t_1(x), t_2(x), \cdots, t_{m-1}(x), t_m(x)$$

*where*

$$t_k(x) = \frac{1}{k!}\left[\sum_{j=1}^{n}(x_j - \alpha_j)\frac{\partial}{\partial x_j}\right]^k F(\alpha_1, \cdots, \alpha_n), \quad k = 0, \ldots, m-1 \tag{3}$$

$$t_m(x) = F(x) - \sum_{k=0}^{m-1} t_k(x) \tag{4}$$

Obviously, by using the Definition (2.1) for the function $F(x)$, it is easy to calculate the truncation error $t_m(x)$ by relation (4), as the objective function and the terms $t_k(x)$, for $k = 1, ..., m-1$ are already known. Generally, it is possible to calculate the truncation error if the terms of the Taylor's expansion, calculated by low computational cost, subtracted from the original objective function.

**Definition 2.2.** *Define groups, $L_j$, $j = 1, ..., l$ where $l \leq m$, which consists of $n_j$ terms of Taylor's expansion, $t_k$ for $k = 0, 1, ..., m$, given by the Definition (2.1). The following relations must be hold:*

- $\sum_{j=1}^{l} n_j = m + 1$
- $\bigcap L_j = \emptyset$, $j = 1, ..., l$
- $\bigcup L_j \equiv (m+1)$*-tuple of sets given by the Definition (2.1)*

**Definition 2.3.** *The* ***Intermediate Objective function*** *$g_j$ is defined as the sum of the elements of a group $L_j$ where $j = 1, ..., l$.*

**Definition 2.4.** *The following LMO problem is defined*

$$\begin{aligned} \min_{x \in R^n} H(x) \quad &= \quad \{h_1(x), h_2(x), ..., h_l(x)\}, \\ &\qquad \textit{subject to } x \in \Omega \end{aligned} \tag{5}$$

*where*

$$h_k(x) = \sum_{j=1}^{k} g_j(x), \qquad k = 1, \ldots, l \tag{6}$$

*where the functions $g_j$ are given by the Definition (2.3).*

Implementing the principles of model described in [10] it is given the following proposition:

**Lemma 2.1.** *The LMO problem introduced in the Definition (2.4) is equivalent with the original optimization problem which is given by the relation (1).*

*Proof.* From the Definitions (2.2), (2.3) and (2.4) it is obviously that the last problem at the LMO problem, given by the relation (6), is identical with the original optimization one, given by the relation (1). □

An important observation is that the new proposed TLSO methodology can be used regardless of the chosen numerical optimization method for optimizing the problems given by the Definition (2.4). The proposed TLSO methodology is given by Algorithm 1.

The convergence theorem of the proposed methodology is given below.

**Algorithm 1** TLSO algorithm

**Require:** $\{F(x), x^0, \epsilon_j, l\}$ $\{$*The initial objective function* $F(x)$*, a point which will expand in Taylor* $x^0$*, the stopping criterion* $\epsilon_j$ *for* $j = 1, 2, ..., l$ *,* $l$ *the number of groups, maximum number of iterations* $MIT_j$ *for* $j = 1, 2, ..., l$ *for each subproblem*$\}$

**STEP 1** Create $(m + 1)$ - tuple of sets according to the Definition (2.1).

**STEP 2** Create $L_j$, for $j = 1, 2, ..., l$ groups according to the Definition (2.2).

**STEP 3** Set the LMO problem given by Equation (5) utilizing the Definition (2.3).

**STEP 4** Find minimizer $x^*$ of the sequential optimization problem given by Equation (5) using different stopping criteria $\epsilon_j$ and $MIT_j$ *for each* $j^{th}$ problem, and Zykina's lexicographic optimization model, [7, 10].

**return** $x^*$ $\{$*The solution of the last optimization problem found*$\}$.

**Theorem 2.1.** ***(Convergence Theorem)*** *The sequence* $x^i$ *generated by the Algorithm TLSO converges to the optimal value* $x^*$ *of the optimization problem given by equation*

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} F(x) \\ &\text{subject to } x \in \Omega \end{aligned}$$

*Proof.* According to the convergence theorem of lexicographic optimization is given in [10] the sequence $x^i$ generated by the Algorithm 1 converges to the optimal value $x^*$ of the $l$-th problem in the sequence of sequential optimization problems given by the relation (5). Due to the relation (5) and taking into account the Definitions (2.3), (2.4) it is hold the relation:

$$\lim_{k \to l} h_k = h_l = \sum_{j=1}^{l} g_j(x) = F(x),$$

Thus, the $l$-th problem in this sequence of sequential optimization problems is equivalent to optimization problem given by Equation (1). Hence, the theorem is proved. □

**Remark 2.1.** *For the intermediate LMO objective functions there are no need the convergence criteria to be as strict as for TLSO algorithm.*

## 2.2 The deterministic optimization methods under the TLSO methodology

As it is already mentioned, at the proposed TLSO methodology, each term of Taylor's expansion is treated as a separate entity and the information obtained from the calculation of derivatives is introduced gradually. Without loosing the generality, suppose that the required number of groups $l$ is equal $m$, the term $t_0$ belongs to the group $l_1$ and the group $l_j$, consists of the corresponding term of Taylor's expansion $t_j$, given by relation (3).

### 2.2.1 Steepest descent method under the TLSO methodology

Notice that if $L_1 = \{t_0, t_1\}$, the first objective function at the LMO formulation can be given by the following relation:

$$h_1(x) = \sum_{j=1}^{1} g_j(x) = g_1(x) = t_0(x) + t_1(x) = F(\alpha) + \sum_{j=1}^{n} (x_j - \alpha_j) \cdot \frac{\partial}{\partial x_j} F(\alpha_1, \ldots, \alpha_n). \quad (7)$$

Solving the problem (7) corresponds to using steepest descent method (also called Cauchy's method or gradient method) for the original optimization problem (1), as the general form for the steepest descent method is:

$$x^{k+1} = x^k - \lambda_k \nabla F(x^k), \quad x_0 \in \mathbb{R}^n$$

where the stepsize $\lambda_k > 0$ is chosen among some criteria, [1].

### 2.2.2 Newton type methods under the TLSO methodology

Notice that if $L_1 = \{t_0, t_1\}$ and $L_2 = \{t_2\}$, the second objective function at the LMO formulation will be given by the following relation:

$$\begin{aligned} h_2(x) &= \sum_{j=0}^{2} g_j(x) = g_1(x) + g_2(x) \\ &= t_0(x) + t_1(x) + t_2(x) = F(\alpha) + \sum_{j=1}^{n} (x_j - \alpha_j) \frac{\partial}{\partial x_j} F(\alpha_1, \ldots, \alpha_n) \\ &\quad + \sum_{j=1}^{n} (x_j - \alpha_j) \frac{\partial}{\partial x_j} F'(\alpha_1, \ldots, \alpha_n). \end{aligned} \tag{8}$$

At the corresponding conclusion leads the conditions $L_1 = \{t_0, t_1, t_2\}$, $L_2 = \{t_3\}$, where $t_i$ are given by the Definition (2.1), with the difference that the problem $h_1(x)$ would be given by the relation (8).

Solving the problem (8) corresponds to use Newton type methods (quasi - Newton methods, Conjugate Gradient methods) around a point $x^k$ for the original optimization problem (1), as the general form for the class of Quadratic methods, [1, 8], is:

$$m^k(x^k + d) = F(x^k) + \nabla F(x^k)^T d + \frac{1}{2} d^T \nabla^2 F(x^k) d$$

Focusing on Equation (5), which is depending on the choice of $m$, the corresponding intermediate $h_k$ functions, $k = 1, \ldots, m-1$, coincides with the $k$-th order of the Taylor terms. Therefore, regardless of the method used to minimize the optimization problem given by Equation (5), the minimization of $h_1$ corresponds to optimizing using steepest descent methodology, the minimization of $h_2$ corresponds to optimizing using second order methods (like Newton, quasi-Newton, conjugate gradient techniques etc.) and so on.

## 3 Numerical Results

The proposed methodology was applied to two well-known test functions and in order to compare the results from the new proposed algorithm and results from other methods, was used the well-known optimization method of Broyden - Fletcher-Goldfarb-Shanno (BFGS), [8]. Each optimization algorithm was tested in a dense grid of initial points in appropriately selected areas of the domain of the objective function.

The convergence criteria for all the algorithms and all the initial points were:

1. $\left|f(x^{k+1}) - f(x^k)\right| \leq 10^{-8}$,
2. $\left\|\nabla f(x^{k+1})\right\|_2 \leq 10^{-4}$,
3. MIT = 10000, where MIT is the maximum number of iterations.

In Tables (1 – 3) numerical results are presented and generally have the following structure:

1. the first column refers to the optimum point.
2. the second column describes the results depicted from the simple unconstrained methodology. Specifically, the proportion indicates the amount of initial points that converged to $x_i^*$ optima;
3. the average number of function evaluations (FE) required are shown on the next column.
4. accordingly, the next two columns represent the same information, after having used the same method with Taylor Lexicographic Sequential Optimization approach.

Moreover, for the following numerical results, we use Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm and apply the TLSO methodology, using $l = 3$ so $L_1 = \{t_0, t_1\}$, $L_2 = \{t_2\}$ and $L_3 = \{t_3\}$. The maximum iterations for optimization problem $h_1$ is equal to 1 and for optimization problem $h_2$ is equal to 2.

### 3.1 Six hump camel back function

The formula of the Six hump camel back function, [6], is given by:

$$F(x_1, x_2) = \left(4 - 2.1x_1^2 + \frac{x_1^4}{3}\right) x_1^2 + x_1x_2 + \left(-4 + 4x_2^2\right) x_2^2$$

and it has two global minima $x_1^* = (0.089842,\ -0.712656)$, $x_2^* = (-0.089842,\ 0.712656)$, and four local minima, $x_3^* = (-1.704, 0.796)$, $x_4^* = (1.704, -0.796)$, $x_5^* = (-1.607, -0.569)$, $x_6^* = (1.607, 0.569)$.

We apply the algorithms by using Taylor's expansion at point $(0, 1)$ and for boxes around $x_i^*$,for $i = 1, 2, \cdots, 6$, and results are presented at the tables (1) and (2).

| Six hump camel back function with initial values in the rectangle $x \in [-0.41, 0.589] \times [-1.21, -0.21]$ | | | | |
|---|---|---|---|---|
| Optimum Point | BFGS method | | TLSO BFGS | |
| | Area of attraction | Average FE | Area of attraction | Average FE [1] |
| $x_1^*$ | 88.96 % | 20.70 | 46.24 % | 28.44 |
| $x_2^*$ | 11.04 % | 23.05 | 38.04 % | 29.95 |
| $x_3^*$ | | | 6.12 % | 38.88 |
| $x_4^*$ | | | 6.76 % | 30.14 |
| $x_5^*$ | | | 2.00 % | 29.88 |
| $x_6^*$ | | | 0.84 % | 35.95 |

Table 1: TLSO for Six hump camel back function

| Six hump camel back function with initial values in the rectangle $x \in [-2.2, -1.2] \times [0.3, 1.3]$ | | | | |
|---|---|---|---|---|
| Optimum Point | BFGS method | | TLSO BFGS | |
| | Area of attraction | Average FE | Area of attraction | Average FE |
| $x_1^*$ | 3.76 % | 26.28 | 21.72 % | 42.11 |
| $x_2^*$ | 10.28 % | 27.65 | 27.52 % | 38.33 |
| $x_3^*$ | 84.68 % | 25.07 | 34.08 % | 35.13 |
| $x_4^*$ | 1.12 % | 26.89 | 10.20 % | 43.43 |
| $x_5^*$ | | | 4.76 % | 33.55 |
| $x_6^*$ | 0.16 % | 27.50 | 1.72 % | 35.05 |

Table 2: TLSO for Six hump camel back function

### 3.2 Branin function

The formula of the Branin function, [6], is given by:

$$F(x_1, x_2) = \left( x_2 - 5.1\frac{x_1^2}{4\pi^2} + 5\frac{x_1}{\pi} - 6 \right)^2 + 10\left(1 - \frac{1}{8\pi}\right)\cos(x_1) + 10$$

and has infinity global minima. We apply the algorithms by using Taylor's expansion at point $(0, 1)$ and for a box: $[-3.6, -2.6] \times [11.8, 12.8]$ and results are presented at the table (3).

| Branin function with initial values in the rectangle $x \in [-3.6, -2.6] \times [11.8, 12.8]$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| Optimum Point | BFGS method | | TLSO BFGS | | | | |
| | Area of attraction | Average FE | Area of attraction | Average FE | Point | Area of attraction | Average FE |
| $x_1^*$ | | | 5.72 % | 24.47 | $x_{14}^*$ | 0.12 % | 34.33 |
| $x_2^*$ | 100 % | 12.69 | 65.08 % | 16.51 | $x_{15}^*$ | 0.24 % | 47.00 |
| $x_3^*$ | | | 6.12 % | 23.59 | $x_{16}^*$ | 0.12 % | 53.33 |
| $x_4^*$ | | | 4.68 % | 26.21 | $x_{17}^*$ | 0.08 % | 35.50 |
| $x_5^*$ | | | 10.56 % | 23.85 | $x_{18}^*$ | 0.08 % | 43.50 |
| $x_6^*$ | | | 4.12 % | 26.98 | $x_{19}^*$ | 0.12 % | 44.33 |
| $x_7^*$ | | | 0.52 % | 58.08 | $x_{20}^*$ | 0.08 % | 56.50 |
| $x_8^*$ | | | 0.20 % | 70.00 | $x_{21}^*$ | 0.12 % | 39.33 |
| $x_9^*$ | | | 0.32 % | 36.63 | $x_{22}^*$ | 0.08 % | 46.00 |
| $x_{10}^*$ | | | 0.24 % | 30.83 | $x_{23}^*$ | 0.16 % | 50.75 |
| $x_{11}^*$ | | | 0.28 % | 31.14 | $x_{24}^*$ | 0.04 % | 51.00 |
| $x_{12}^*$ | | | 0.24 % | 40.17 | $x_{25}^*$ | 0.36 % | 11.00 |
| $x_{13}^*$ | | | 0.16 % | 34.25 | $x_{26}^*$ | 0.16 % | 11.00 |

Table 3: TLSO for Branin function

## 4 Conclusion and further research

At the present paper the proposed methodology transforms the original optimization problem given by Equation (1) to an equivalent LMO one, by defining the required objective functions through a combination of the terms of Taylor's expansion of the original objective function. Moreover, all the information that enclosing by the infinite terms of Taylor's expansion used, without calculating them.

By using TLSO methodology the user manages to pick out method at each iteration and as the numerical results indicate, if TLSO methodology applying in a grid of points then it converges to much more points than the corresponding optimization method that it uses. It is worth to notice that TLSO methodology gives the opportunity to take advantage of a class of optimization methods in the intermediate steps. An interesting point is to examine the effect of combinations of terms of Taylor's expansion of the original objective function, regardless the physical order of their appearance.

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# Minimal Cut Sets Generation Using Reverse Petri Nets

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**Abstract**

In this paper a new method for minicuts and minipaths generation of a coherent Fault Tree (FT) without repeated events is proposed. The approach is based on the special type of Petri Nets (PN) – Reverse Petri Net (RPN) which can be generated directly from the FT. Using RPN properties, it is shown that the problem of finding all minicuts becomes the following reachability problem: to find all dead markings reachable from marking $M = [0, 0, ..., 0, 1]^T$ of RPN. All minicuts are obtained as leaves of the reachability graph of RPN.

**KEYWORDS**

fault tree; minimal cut set; reverse petri net

## 1. INTRODUCTION

Coherent systems can be represented by FTs or by binary PNs (Ericson, 2005). Both approaches have been used widely for safety and reliability analyses where determining minimal cut sets (minicuts) generation is a crucial step. A new method for minicuts generation will be presented in the paper.

The fault tree analysis (FTA) was developed in the early 1960s just for the purpose of safety and reliability analyses of complex systems. PN was also developed in the early 1960s but for the study of the qualitative properties of systems exhibiting concurrency and synchronization characteristics. These two methods have been combined since late 1980s. Hura and Atwood (1988) provided a general approach to application of PN modeling for coherent FT analysis based on a state equation. FTA based on a state equation of PN is also given in (Huanqiu et al. 1998). In (McGraw, 1989) PN and FTA are combined on an example in military embedded software application. The author models a system using PN and then built a FT on the knowledge obtained from the PN modeling. In order to include failure probability or stochastic dependencies in the safety analysis, some authors convert a FT model into Stochastic PN (SPN) or Generalized Stochastic PN (GSPN) (Bobbio, et al. 2003), (Buchacker, 1999), (Malhotra & Trivedi, 1995). In (Kaiser et al. 2007) the authors use the extension of GSPN called Deterministic and Stochastic PN (DSPN) which allows all types of the dependencies and time modeling.

Determining minicuts using PN of a given fault tree can be found in literature. In (Liu & Chiou, 1997) a graphic method for minicuts and minipaths generation is presented. The top-down and bottom-up matrix methods for minicuts generation are proposed in (Yang & Liu, 1997). The authors also propose a marking transfer method based on a state equation and reorganization of the incidence matrix. Rochdi et al. (1999) present an algorithm for minicuts enumeration on a PN model of a coherent fault tree. They solve the complexity problem using PN modularization and reduction technique. In (Knezevic & Odoom, 2001) minicuts are generated using a matrix method and then Lambda-Tau methodology and the fuzzy set theory are used to obtain quantitative results and to include subjectivity in analysis. The minicuts identification algorithm based on simulation of PN is described in (Wang et al. 2002). PN model of fault propagation behaviors is built directly instead of developing a PN model of a given fault tree.

A new method for minimal cut sets generation based on Reverse Petri Nets will be presented here. Apart from the introduction (1) the paper is organized as follows. In Section 2 definitions and notations important for our method as well as widely used PN presentation of a logic gates are given. In section 3 we have developed the method for all minicuts generation using Reverse PN (RPN) of a given FT.

# 2. PRELIMINARIES

## 2.1 Fault Tree Analysis

The FTA is a systems analysis technique used to determine the root causes and probability of occurrence of a specified top event. FTA is based on a FT, a graphical model using logic gates and fault events to model the cause-effect relationships involved in causing the top event (Ericson, 2005). A coherent FT is a FT constructed using AND and OR logic operator only.

The cut set is a set of basic events which, when happen simultaneously, cause the top event to occur. The minimal cut set (minicut) is a cut set which has been reduced to the minimum number of events that cause the top event to occur. The minicuts cannot be further reduced and still guarantee occurrence of the top event. Once generated, the minicuts can be ranked according different criteria: number of events in minicuts, so called minicuts order; minicuts probability that drives the top event probability etc.

## 2.2. Petri Nets

PN are graphical and mathematical tools for modeling systems and their dynamics. The structure of PN is a directed, weighted, bipartite graph consisting of two kinds of nodes: transitions (t) which represent events or operations (transitions are represented by rectangles or bars), and places (p) which represent a condition and/or a result of events or buffer for resources necessary for execution of operations (places are represented by circles). Each place may contain one or several tokens, which allow modeling of dynamics of the system. Directed arcs join some places to some transitions, or some transitions to some places.

**Definition 1:** PN is a 4-tuple $PN = (P, T, F, M_0)$ where (Proth & Xie, 1996):

$P = \{p_1, p_2, ..., p_m\}$ is a finite set of places,

$T = \{t_1, t_2, ..., t_n\}$ is a finite set of transitions,

$F \subseteq (P \times T) \cup (T \times P)$ is a set of arcs (flow relation), $P \cap T = \varnothing$ and $P \cup T \neq \varnothing$,

$W : F \to \{1, 2, ...\}$ is the weight function attached to the arcs,

$M_0 : P \to \{0, 1, 2, ...\}$ is the initial marking.

Any distribution of tokens in the net is called marking. A marking is denoted by *M*, $M = [M(p_1), M(p_2), ..., M(p_m)]^T$, an *m*-vector where *m* is the number of places. The *i*th component of *M* is the number of tokens in place $p_i$ (Murata, 1989).

The dynamic of a PN is modeled by firing a transition which consists of: removing token from each input place and adding token to each output place of the transition. Formally, firing a transition *t* consists of transforming the initial marking $M_0$ into marking *M* defined as follows:

$$M(p) = \begin{cases} M_0(p) - W(p,t) & \text{if } p \in {}^0t \\ M_0(p) + W(t,p) & \text{if } p \in t^0 \\ M_0(p) & \text{otherwise} \end{cases} \quad (1)$$

where ${}^0t$ is a set of input places and $t^0$ is a set of output places of the transition $t$. A firing may occur only when transition is enabled e.g. if each of its input places contains $W(p,t)$ tokens. In this paper, the weights will always be 1. Marking *M* is said to be reachable from marking $M_0$.

**Definition 2:** $p$ is a source place if ${}^0p = \varnothing$; and a sink place if $p^0 = \varnothing$, where ${}^0p$ is a set of input transitions and $p^0$ is a set of output transitions of the place $p$.

**Definition 3:** (Proth & Xie, 1996) A marking *M* reachable from the initial marking $M_0$ is a deadlock if none of the transitions of the PN is enabled.

**Property 1:** Let *M* be a marking with tokens only in sink places. Marking *M* is a deadlock.

**Proof:** If only sink places contain tokens all other places in PN are empty. That means that there is no enabled transition, e.g. transition whose input places contain a token. According to definition 3, marking *M* is a deadlock. ♦

**Illustration 1:** Figure 1 shows logic gate AND (a) and its equivalent binary PN (b).

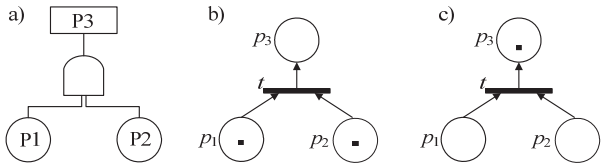


Figure 1. Logic gate AND and its equivalent binary PN

Basic events P1 and P2 are represented by source places $p_1$ and $p_2$, and top events P3 is represented by sink place $p_3$. The existence of a token in the place models the occurrence of a corresponding event. A logic operator AND is modeled by transition *t* which is enabled only if both of its input places $p_1$ and $p_2$ contain tokens, e.g. if the initial marking is $M_0 = [1,1,0]^T$. After transition *t* fires, the resulting marking will be $M = [0,0,1]^T$ (Figure 1c), which models the occurrence of the top event P3. This marking is a deadlock.

**Illustration 2:** Figure 2 shows a logic gate OR (a) and its equivalent binary PN (b).

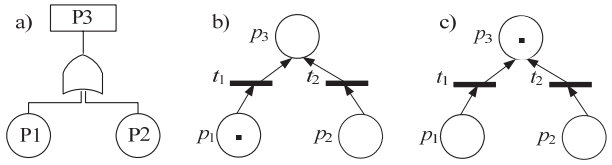


Figure 2. Logic gate OR and its equivalent binary PN

Logic operator OR is modeled by transition $t_1$ which is enabled only if its input place $p_1$ contains tokens and transition $t_2$ which is enabled only if its input place $p_2$ contains tokens, e.g. if the initial marking are $M_{01} = [1,0,0]^T$, $M_{02} = [0,1,0]^T$ or $M_{03} = [1,1,0]^T$. The initial marking of PN in figure 2 is $M_{01} = [1,0,0]^T$. After transition $t_1$ fires, the resulting marking will be $M = [0,0,1]^T$ which models the occurrence of the top event P3 (Figure2c). This marking is a deadlock.

## 2.3. Reachability problem

*Let* $M_0$ be the initial marking of PN and $R(M_0)$ the set of all possible markings reachable from $M_0$. The reachability problem for PN is the problem of finding all $M \in R(M_0)$. The transformation from one marking to another in a state space is given by a state equation:

$$M = M_0 + A \cdot V_\sigma , \tag{2}$$

where *A* is the incidence matrix and $V_\sigma$ is the firing count vector (Proth & Xie, 1996).

Incidence matrix $A = [a_{ij}], i = 1,...,m; j = 1,...,n$ is a matrix with a row for each transition and a column for each place. Element $a_{ij}$ is defined as follows:

$$a_{ij} = \begin{cases} 1 & \text{if } t_j \in {}^0p_i \\ -1 & \text{if } t_j \in p_i^0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where ${}^0p_i$ is a set of input transitions and $p_i^0$ is a set of output transitions of the place $p_i$.

The firing count vector is a $n \times 1$ column vector of nonnegative integers, where $n$ is a number of transitions in PN. The $j$th entry of $V_\sigma$ denotes the number of times that transition $t_j$ must fire to transform $M_0$ to $M$ (Murata, 1989).

**Illustration 3:** Transformation from the initial marking is $M_0 = [1,0,0]^T$ to marking $M = [0,0,1]^T$ by firing transition $t_1$ (Illustration 2), can be expressed using a state equation:

$$\begin{matrix} M & & M_0 & & A & & V_\sigma \end{matrix}$$
$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

## 2.4. Reverse PN

**Definition 2** (Priese & Wimmel, 2008)**:** A Reverse Petri Net (RPN) of a given PN, with net structure $N = (P,T,F)$, is a PN with the net structure: $RN = (P,T,F^R)$, where $F^R$ is a set of reverse directed arcs from $F$: $F^R = \{(u,v) \mid (v,u) \in F\}$.

**Remark 1:** Since arcs are reversed, the incidence matrix of RPN can be expressed as $-A = [-a_{ij}]$, $i = 1,...,m; j = 1,...,n$, where $A = [a_{ij}], i = 1,...,m; j = 1,...,n$ is the incidence matrix of the given PN.

**Property 2:** Let $M_0$ and $M$ be markings of PN and its RPN, and let $M$ be reachable from $M_0$ of PN. Then, marking $M_0$ is reachable from marking $M$ of RPN.

**Proof:** Transformation from $M_0$ to $M$ of PN is given by state equation (2), which is equivalent to the following equations:

$$M_0 = M - A \cdot V_\sigma \Rightarrow M_0 = M + (-A) \cdot V_\sigma \quad (4)$$

The equation (4) is the state equation which describes transformation from $M$ to $M_0$ of PN with a matrix incidence $-A$, which is RPN. ♦

**Remark 2:** The sequences of transitions firing which transform $M_0 \to M$ and $M \to M_0$ are reverse. However, the firing count vector $V_\sigma$ is the same in both equations because it denotes only the number of transitions firing but not the sequence of firing.

All markings reachable from the initial marking $M_0$ can be represented by a reachability graph. The reachability graph is a graph with a node for each marking and an arc for each firing transition. At each level of the reachability graph, each marking leads to as many markings as the enabled transition number. Marking $M_0$ is a root node and the deadlock markings are leaves of the reachability graph.

# 3. MINICUTS GENERATION USING RPN

An equivalent Petri Net (EPN) model of a given coherent FT can be obtained by replacing the AND and OR logic operators with their equivalent PN structure.

**Illustration4.** Figure 3 gives an example of a coherent FT and its EPN. The given coherent FT has 7 minicuts: P8, P5P7, P6P7, P3P4P5, P1P2P7, P3P4P6 and P1P2P3P4. Basic events of FT: P1, P2, P3, P4, P5, P6, P7 and P8 are modeled by places of PN: $p_1$, $p_2$, $p_3$, $p_4$, $p_5$, $p_6$, $p_7$ and $p_8$, respectively. Intermediate events: G1, G2, G3, G4 and G5 are modeled by places: $p_9$, $p_{10}$, $p_{11}$, $p_{12}$ and $p_{13}$, respectively. The top event T is modeled by place $p_{14}$.

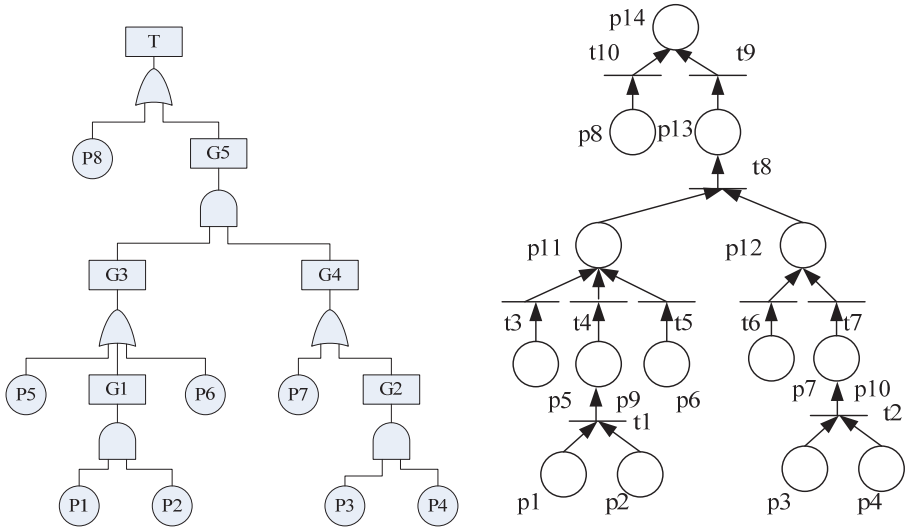


Figure 3. A coherent FT Figure and its EPN

**Property 3:** Let $P=\{p_1, p_2, ..., p_m\}$ be a set of places of an EPN, where place $p_m$ models top events. The main properties of the EPN are:

1. EPN is a PN with a source place for each basic event and with one sink place for the top event T of a FT.
2. A cut set of a FT can be represented by marking of EPN with tokens in source places that model its basic events.
3. Any marking $M=[M(p_1),...,M(p_{m-1}),1]^{\mathrm{T}}$, where $M(p_m)=1$, represents the occurrence of the top event T, e.g. system failure.
4. Each initial marking $M_0$ with tokens only in source places, from which marking $M=[M(p_1),...,M(p_{m-1}),1]^{\mathrm{T}}$ is reachable, represents a cut set.
5. If initial marking $M_0$ models a minicut then, according to the definition of minicuts, the final (dead) marking of EPN will be $M=[0,0,...,0,1]^{\mathrm{T}}$.

**Illustration5:** Minicut P2P6S2 on the FT in Figure 3 can be represented by initial marking $M_0=[0,0,1,1,1,0,0,0,0,0,0,0,0,0]^{\mathrm{T}}$ of its EPN. Source places: $p_3$, $p_4$ and $p_5$ contain tokens and all other places are empty. Transformations from initial marking $M_0=[0,0,1,1,1,0,0,0,0,0,0,0,0,0]^{\mathrm{T}}$ to $M=[0,0,0,0,0,0,0,0,0,0,0,0,0,1]^{\mathrm{T}}$, can be expressed by a state equation as follows:

$$
\underset{M}{\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix}}
=
\underset{M_0}{\begin{bmatrix}0\\0\\1\\1\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}}
+
\underset{A}{\begin{bmatrix}
-1&0&0&0&0&0&0&0&0&0\\
-1&0&0&0&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0&0&0\\
0&0&-1&0&0&0&0&0&0&0\\
1&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&-1&0&0&0&0&0\\
0&0&0&0&0&-1&0&0&0&0\\
0&1&0&0&0&0&-1&0&0&0\\
0&0&1&1&1&0&0&-1&0&0\\
0&0&0&0&0&1&1&-1&0&0\\
0&0&0&0&0&0&0&1&-1&0\\
0&0&0&0&0&0&0&0&0&-1\\
0&0&0&0&0&0&0&0&1&1
\end{bmatrix}}
\cdot
\underset{V_\sigma}{\begin{bmatrix}0\\1\\1\\0\\0\\0\\1\\1\\1\\0\end{bmatrix}}
$$

If any other place in $M=[M(p_1),...,M(p_{m-1}),1]^T$ contains a token, for instance $M(p_i)=1, i\neq m$, a cut set, modeled by corresponding initial marking, is not a minicut and can be reduced by removing a basic failure modeled by place $p_i$.

**Remark 3:** The problem of finding all minicuts becomes the following reverse reachability problem: find all initial markings $M_0$ of EPN such that $M=[0,0,...,0,1]^T \in R(M_0)$ (from which marking $M=[0,0,...,0,1]^T$ is reachable). Reachability problem can be solved using RPN (Makajić Nikolić et al. 2004).

**Property 4:** Let RPN be a reverse net of EPN. According to Definition 2 and Properties: 1, 2 and 3, main RPN properties are:

1. RPN is a PN with a sink place for each basic event and with one source place for the top event T of a given FT.
2. A cut set of a FT can be represented by marking of RPN with tokens only in sink places which models its basic events. Such marking is dead marking.
3. Any marking $M=[M(p_1),...,M(p_{m-1}),1]^T$, where $M(p_m)=1$, represents the occurrence of the top event T, e.g. system failure.
4. Each final (dead) marking $M_0$ with tokens only in source places, reachable from $M=[M(p_1),...,M(p_{m-1}),1]^T$, represents a cut set.
5. Each final (dead) marking $M_0$ reachable from $M=[0,0,...,0,1]^T$, represents a minicut.

**Remark 4:** The problem of finding all minicuts becomes the following reachability problem: find all dead markings $M_0$ of RPN such that $M_0 \in R(M)$, $M=[0,0,...,0,1]^T$ (all markings reachable from $M=[0,0,...,0,1]^T$).

The reachability problem from $M=[0,0,...,0,1]^T$ to markings which model minicuts can be solved using the reachability graph. Marking $M=[0,0,...,0,1]^T$ is a root node and the deadlock markings which model minicuts are leaves of the reachability graph.

**Illustration 6:** Figure 4 shows RPN of the EPN in Figure 3 with an initial marking $M=[0,0,0,0,0,0,0,0,0,0,0,0,0,1]^T$ which models occurrence of the top event.

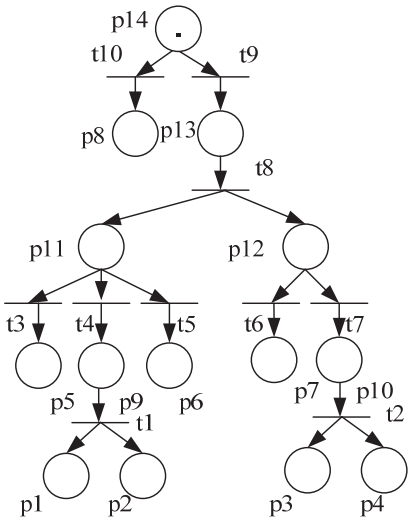


Figure 4. An RPN of the EPN in Figure 8

The reachability graph of the RPN is shown in Figure 5. The root of the graph is marking $M = [0,0,0,0,0,0,0,0,0,0,0,0,0,1]^{T}$.

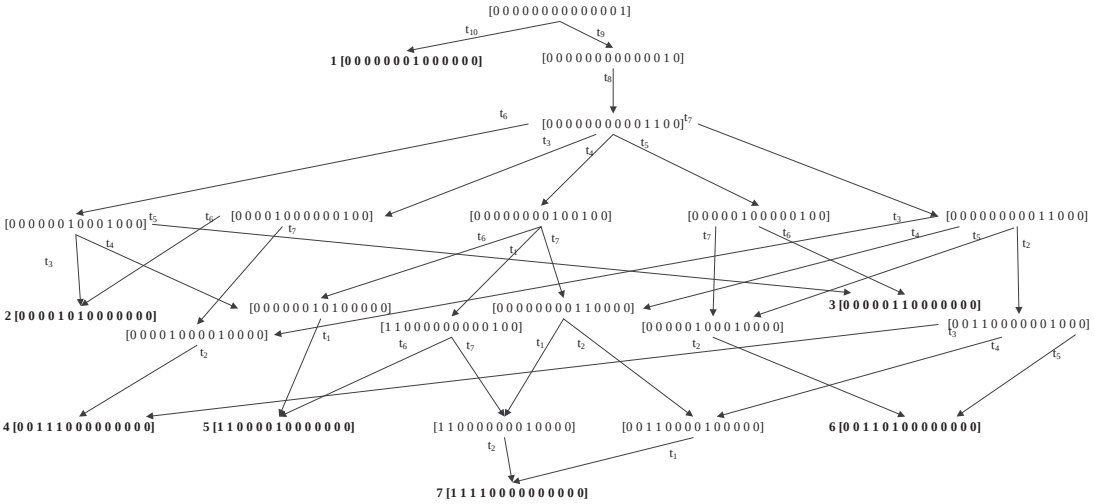


Figure 5. A reachability graph of the RPN in Figure 4

The reachability graph on Figure 5 has 7 leaves which are minicuts of the coherent FT from illustration 4.

Table 1. Minicuts of the coherent FT in Figure 7

| Leaves of reachability graph | | RPN paces contain token | FT minicuts |
|---|---|---|---|
| 1. | [00000001000000] | p8 | P8 |
| 2. | [00001010000000] | p5, p7 | P5P7 |
| 3. | [00000110000000] | p6, p7 | P6P7 |
| 4. | [00111000000000] | p3, p4, p5 | P3S4P5 |
| 5. | [11000010000000] | p1, p2, p7 | P1P2P7 |
| 6. | [00110100000000] | p3, p4, p6 | P3P4P6 |
| 7. | [11110000000000] | p1, p2, p3, p4 | P1P2P3P4 |

It is easily understandable that a reachability graph may have very large numbers of nodes. One of the ways to limit the size of the tree is the use of the branching criteria which allows a choice of nodes that should be developed. (Jensen, 1997).

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## CONCLUSION

In this paper a new approach for minicuts of a coherent fault tree generation is presented. It is described how a coherent fault tree without repeated events can be matched with an equivalent Petri Net and then transformed to a Reverse Petri Net. Then, using the properties of Reverse Petri Net and its reachability graph, starting from initial marking $M = [0,0,...,0,1]^{T}$, minicuts are formulated. Leaves of the reachability graph represent minicuts of a given ft. The efficiency of approach is highly depended on the structure of the given coherent Fault tree.

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# Modelling Advertising Scheduling to Improve Media Planning

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**Abstract**

This paper presents one flexible mathematical model for determining optimal scheduling of advertising messages (commercials) that are broadcasted on television. The 0-1 linear programming model maximizes target rating points (TRP), while keeping constraints on daily TRP, number of commercials available budget and desired reach. This model encompasses specific characteristics of media and media buying process in Serbia.

**KEYWORDS**

Advertising, Media, Scheduling, Mathematical model

## 1. INTRODUCTION

In the contemporary conditions of business and development of media market, planning and buying of media space becomes complex activity. Belch E.G. and Belch A. M. (2004), Fill C. (2005) and Pickton D. and Broderick A. (2001) discussed numerous factors that have influenced the change of media planning and the change of perception of its significance at the global level. The most important are: appearance and development of new types of media, fragmentation of media, advertising cost increase, saturation of the market by promotional messages, improvement of the existing and appearance of new media research methodologies, changes in the buyer's behaviour – evolution of needs and the way of media consumption etc. Not until recently was in the marketing practice in Serbia special significance and deserved attention given to this part of realization of the process of the marketing communication and advertising. The fact that in Serbia there are no specialized studies for the job title of the 'media planner', as well as the fact that in the domestic literature about marketing communication and advertising insignificant attention has been dedicated to it, contributes to this attitude.

The transition from centrally planned to market economy that started at the end of the past century in Serbia, had an influence on the media market also, that is on the business of marketing agencies, that are responsible for buying media space and time. Concerning the above mentioned, the MIP model for media planning, suggested by the authors in this study, took into consideration every specific aspect of media planning characteristic for a business environment in a country in transition:

- Media environment of Serbia – number, type and characteristics of available media;
- Characteristics of the process of media planning – from the client's request, the way the negotiations with media house are realized and media plan making, to its implementation and evaluation;
- Available strategies for media planning concerning time schedule, characteristics of the target group and competition activity.

Taking into consideration real needs and problems that the media planners face in practice, this model refers to media planning process for national and local television in Serbia.

Since the 1960, linear and MI programming appeared as a suitable tool for modelling and optimizing, whereby we dispose advertisement to various types of media. Miller, Starr and Day (1962) established the criteria to apply linear programming principles for media selection via answering the questions when (time) and where (space) advertisement should appear, depending on budget constraints. Lee (1962) developed a LP model in order to optimize the number of advertisements of a given time length to ensure a required level of attention. In order to improve and extend previous models, Little and Lodish (1969) developed MEDIAC, which includes market segments, segment-specific sales potentials, and probability of each media type rating.

1962Afterwards, Aaker (1975) suggests media planning model ADMOD which is focused on specific segment of population chosen from various segments. Starsch (1975) developed the model to select markets, in which advertising appears in order to harmonize variations among the sales potentials, which are specific for each market, and the circulation of selected media, which serves those markets. Television planning model (VIDEAC) which provides standard data, such as: program availability, cost and rating (by segments) directly to advertising agencies is presented in (Rust, 1985). Makajic-Nikolic, Kostic-Stankovic and Janicic (1997) developed the mathematical model of goal programming for media planning on the segmented market.

Kelton and Schneider Stone (1998) formulated binary integer models have for solving network television-scheduling problems under three different competitive environments: myopic, Nash competitive and cooperative and then solved them for the three major USA TV networks. Mihiotis and Tsakiris (2004) developed integer programming model and algoritham for satisfactory solution for placements of commercial with the goal of the highest rating and subject to constrained advertisement budgets. Cetin&Tolun Esen (2006) proposed a media and budget allocation model, based on the weapon-target assignment problem and then solved a hypothetical numerical example using MS Excel as a tool of spreadsheet modeling. The approach presented by Perez-Gladisha et al. (2010), focuses on audience impact fuzzy predictions and their articulation in two-objective decision model: to achieve the highest audience impact and to reduce advertising costs. Impact is measured depending on the logarithm of ad repetitions.

The article is organized as follows. Section 2 deals with the description and the formulation of the TV ads scheduling problem. In section 3 model formulation is presented. Results of the real numerical example solved using Large-Scale LP Solver are presented in section 4. Concluding remarks are made in Section 5.

## 2. PROBLEM DESCRIPTION

We next describe and formulate the problem of schedule of broadcast commercials (TV ads) optimization. The commercials have been made for a known buyer – advertiser and will be broadcast on several TV stations. Target group, the type of the campaign, duration, budget and TV stations on which the ad will be broadcasted are defined in advance for the given TV commercial (ad). Basic assumptions of the problem are:

- The programme schedules on the TV stations are relatively constant and the number of terms for commercial broadcasting is the same every working day and every Saturday and Sunday. This assumption allows use of the historical data for media plan optimization.
- The data on the target group rating (TRP) of some programs and commercial breaks are available. TRP represents the percentage of the target audience that watch TV show at the moment of observation (Katz H, 1995).
- Television stations in Serbia introduced different coefficients for price correction of a broadcasted commercial, depending on the position in the advertising block, period of the day and the duration of commercial.
- Advertiser and the Agency define the type of the advertising campaign according to the dynamics and intensity: Burst, Continous or Puls. The daily number of the broadcasted ads and the daily TRP depend on the chosen type.
- An agreement is made about the wanted ratio of the commercials that will be broadcasted in *prime time* and the commercials that will be broadcasted in *non-prime time*.
- There could be more than one version of the commercial, regarding the spot length. In the suggested model, the problem is narrowed down to the two most frequent cases: commercial in duration of 30 sec and commercial shorter than 30 sec. Even though the price lists of the domestic TV stations show different coefficients for more than one option of shorter commercial (for example, 10, 15, 20 and 25 sec.), in practice there is usually only one shorter version of the same commercial.
- Reach is expressed as a percentage of population which is at least once exposed to a certain programme during the observed time. So, as a measure of accumulation of auditorium, reach is the number of different individuals (or households) that are exposed to the given media (Surmanek, 1996). In Europe reach is also called ’opportunity to see’ (OTS). In the electronic media reach is most often calculated for the period of four

weeks. Precise data on the accomplished reach can be gathered only after the broadcast of commercials. However, experienced media planners achieve higher reach by limitation of the number of broadcasts of commercials in the same types of programs (episodes of the same serie, daily news in the same term).

- The budget for the campaign and one rating point price (cost per point - CPP) defined by the advertiser and TV station, but these data are not available for the research and modelling. Therefore, we introduced for the modelling the NET which is calculated as the "total budget/CPP". The NET was obtained from the Agency and used for indirect budget limitation.
- When allocating commercials on different programs and weeks, media planner himself defines: - how many 30 sec long commercials and how many shorter versions of the same commercial will be broadcasted in the observed period; in what week the broadcasting of the shorter version will start; whether both versions can be broadcasted in the same advertising block. In practice, at the beginning of the campaign the whole version of 30 sec is usually broadcasted, and after one or two weeks, when it is evaluated that the great part of auditorium has seen the whole commercial, broadcasting of the shorter version starts.

## 3. MODEL FORMULATION

The notation used to define sets, parameters and decision variables is as follows.

Sets:
$s$ – number of weeks in the planning period, $t$=1,...,$s$,
$J_t$ –set of days in a week, $t$=1,...,$s$,
$J_t = J_1 \cup J_2$, $J_1$=(1,2,3,4,5) – work days, $J_2$=(6,7) – Saturday and Sunday, $j \in J$
$N$ – set of all programs broadcasted in the planned period,
$N_j$ – set of programs broadcasted on every day $j$, $i \in N_j$, $\bigcup_{\substack{t=1 \\ j \in J_t}}^{s} N_j = N$ .

$D$ – set of time periods during the day, $d \in D, D = \{1,2,3...\}$ . It is now possible to group the programs that are broadcasted during one day as follows:
$N_{jd}$ – set of programs broadcasted on the $j$ day in the period $d$,.
$N_{jD_p}$ - set of programs broadcasted in *prime time* on the day $j$ and
$N_{jD_{np}}$ - set of programs broadcasted in *non-prime time* on the day $j$.
$N_{jD_p} \cup N_{jD_{np}} = N_j$

$L$ – set of positions of commercials in a block, $l \in L, L = \{1,2,3...\}$ , where 1 marks first or ultimate position, 2 – second or penultimate, etc.
$N_{J_1}$ - set of programs broadcasted every day in the same term, $i \in N_{J_1}$
$N_{J_2}$ - set of programs broadcasted every work day in the same term, $i \in N_{J_2}$
$N_{J_3}$ - set of programs broadcasted every Saturday in the same term, $i \in N_{J_3}$
$N_{J_4}$ - set of programs broadcasted every Sunday in the same term, $i \in N_{J_4}$ , $N_{J_1} \cup N_{J2} \cup N_{J_3} \cup N_{J_4} \subset N$

Parameters:
$p$ - number of commercials in *prime time*
$trp_{tji}$ - TRP of the show $i$ on the day $j$ of the week $t$,
$NET$ – total cost of media buying upon which an agency and a client have agreed (budget/cpp)
$k_d$ - price coefficient for the time of the day, $d \in D$ .
$k_l$ - price coefficient for the position of the commercial in the block, $l \in L$ .
$k_k$ - price coefficient for the shorter duration of time of the commercial.
$ddr_j, gdr_j$ - bottom and upper bounds of the TRP on the day $j$, $j \in J_t, t = 1,...s$ .

$dbs_j, gbs_j$ - bottom and upper bounds of the total number of commercials on the day *j*.

$a_1$, $a_2$, $a_3$, $a_4$ – total number of broadcasting of commercials in the planned period in the programs (terms) from the sets: $N_{J_1}$, $N_{J_2}$, $N_{J_3}$ and $N_{J_4}$, respectively.

*b* – minimal requested participation of 30sec long commercials in the total number of commercials

Variables

$$x_{tjil} = \begin{cases} 1 & \text{if 30 sec long commercial is broadcasted on the } j\text{-day of the k- week in the } i\text{-show on the } l\text{-position in the block} \\ 0 & \text{otherwise} \end{cases}$$

$$y_{tjil} = \begin{cases} 1 & \text{if the commercial shorter than 30 sec is broadcasted on the } j\text{-day of the k- week in the } i\text{-show on the } l\text{-position in the block} \\ 0 & \text{otherwise} \end{cases}$$

Using this notation, the 0-1 linear programming model of broadcast commercials scheduling can be stated as:

$$\max f(x,y) = \sum_{t=1}^{s} \sum_{j \in J_t} \sum_{i \in N_j} trp_{tji} \sum_{l \in L} (x_{tjil} + y_{tjil}) \tag{1}$$

s.t.

$$\sum_{t=1}^{s} \sum_{j \in J_t} \sum_{d \in D} \sum_{i \in N_{jd}} trp_{tji} k_d \sum_{l \in L} k_l x_{tjil} + k_k \sum_{t=1}^{s} \sum_{j \in J_t} \sum_{d \in D} \sum_{i \in N_{jd}} trp_{tji} k_d \sum_{l \in L} k_l y_{tjil} \le NET \tag{2}$$

$$\sum_{i \in N_{jD_p}} \sum_{l \in L} (x_{tjil} + y_{tjil}) - p \cdot \sum_{i \in N_j} \sum_{l \in L} (x_{tjil} + y_{tjil}) \ge 0, \ \ j \in J_t, \ t = 1,...,s \tag{3}$$

$$ddr_j \le \sum_{i \in N_j} trp_{tji} \sum_{l \in L} (x_{tjil} + y_{tjil}) \le gdr_j, \ \ j \in J_t, \ t = 1,...,s \tag{4}$$

$$dbs_j \le \sum_{i \in N_j} \sum_{l \in L} (x_{tjil} + y_{tjil}) \le gbs_j, \ \ j \in J_t, \ t = 1,...,s \tag{5}$$

$$\begin{aligned} &\sum_{t=1}^{s} \sum_{j \in J_t} \sum_{l \in L} (x_{tjil} + y_{tjil}) \le a_1, \ i \in N_{J_1} \\ &\sum_{t=1}^{s} \sum_{j \in J_t} \sum_{l \in L} (x_{tjil} + y_{tjil}) \le a_2, \ i \in N_{J_2} \\ &\sum_{t=1}^{s} \sum_{j \in J_t} \sum_{l \in L} (x_{tjil} + y_{tjil}) \le a_3, \ i \in N_{J_3} \\ &\sum_{t=1}^{s} \sum_{j \in J_t} \sum_{l \in L} (x_{tjil} + y_{tjil}) \le a_4, \ i \in N_{J_4} \end{aligned} \tag{6}$$

$$(1-b) \sum_{t=1}^{s} \sum_{j \in J_t} \sum_{i \in N_j} \sum_{l \in L} x_{tjil} - b \sum_{t=1}^{s} \sum_{j \in J_t} \sum_{i \in N_j} \sum_{l \in L} y_{tjil} \ge 0 \tag{7}$$

$$\sum_{j \in J_t} \sum_{i \in N_j} \sum_{l \in L} y_{tjil} = 0, \ t = 1 \tag{8}$$

$$\sum_{l \in L} (x_{tjil} + y_{tjil}) \le 1, \ i \in N_j, j \in J_t, t = 1,...,s \tag{9}$$

$$x_{tjil} = \{0,1\}, \ i \in N_j, j \in J_t, t = 1,...,s, l \in L$$

$$y_{tjil} = \{0,1\}, \ i \in N_j, j \in J_t, t = 1,...,s, l \in L$$

Objective function (1) maximizes the total achieved TRP. Budget (NET) constraint is presented with (2). Constraint (3) ensures the ratio between the number of commercials in *prime time* and *non-prime time* on the day *j*. Since: $N_{jD_p} \cup N_{jD_{np}} = N_j$ and $p+np=1$, parameter *np*, which represent the number of commercials in *non-prime time*, is omitted. Balances of the TRP and the number of commercials per days are met by (4) and

(5), respectively. Constraints (6) prevent unnecessary repetition of broadcast and, indirectly, provide wonted reach. Constraint (7) refers to the ratio between the numbers of longer and shorter commercials during observed period. Constraint for non-broadcasting of shorter commercials during first week is represented by (8). Constraint (9) models the decision of media planer not to broadcast longer and shorter version of the commercial in the same advertising block.

## 4. OPTIMIZATION RESULTS

Given mathematical model is applied for obtaining optimal broadcasting plan within the campaign, which was done for a company that produces alcoholic drinks. Real data were used, which covered a period of 28 days and three most viewed TV stations in Serbia. Specificity of campaign which is promoting the alcoholic drinks is that advertisements can not be broadcast before 6 p.m., so total number of advertising blocks, where advertisements could be broadcast in that period, was 1001. Two advertisements were used in the campaign; one of 30 seconds (longer advertisement) and one of 7seconds (shorter advertisement).

Three groups of data were required to create this mathematical model:

- Customer's requirements.
- List price of television stations. Factors that influence on price are: day in week, time of the day, type of programme, length of the advertisement, etc.
- TV programme schedule. Here were used TV programme schedules and ratings of programmes from previous months; because it is way the media planner works.

The final model had 2002 variables and 1188 constraints, and for his optimization was used Premium Solver, i.e. Large-Scale Linear Programming Solver Engine. The time of executing on computer with 2.0GHc and 512MB RAM was 03h48m03s. In order to validate the result, the advertising plan made by media planner over the same data was used. The results of the media planner and optimization are given in the Table 1. Scenario No.1 completely satisfies customer's requirements that deviation in daily GRP is little, whereas with Scenario No.2 it is slightly deviated from this requirement.

Table 1 Results of media planer and optimization

| | Media planner | Optimization (Scenario No.1) | Optimization (Scenario No.2) |
|---|---|---|---|
| NET | 6259 | 6259 | 6259 |
| Total GRP | 908,74 | 1249,07 (137%) | 1353,02 (149%) |
| Total number of advertisements | 264 | 310 | 352 |
| Daily number of advertisements | 2 - 16 | 7-11 | 7 – 12 |
| Daily GRP | 4,33 - 57,23 | 32,32 - 39,92 (30-40) | 20,55 - 49,87 (20-50) |

On the basis of the table, we can conclude that solution obtained by optimizing is better, by all performances than solutions obtained by media planner. With the same budget, i.e. NET, achieved Total GRP by Scenario No.1 is 137% of Total GRP of the media planner, i.e. 149% by Scenario No.2. Total number of advertisements is also greater, and it is achieved better equalization of daily number of advertisement and daily GRP.

Since the *reach* was modelled indirectly, indicator which would enable direct comparison of this performance does not exist. However, it was performed detailed analyses of programmes which had been chosen by media planner on the basis on broadcasting plan and programmes, obtained by optimization. It was determined that in the plan of media planer number of advertisements, within the same type programmes, is greater than it should be. Media planer's schedule contains 72 advertisements which, in praxis, do not increase *reach,* which can be considered as a irrational used clients budget. Since given mathematical model contains this type constraints, it can be concluded, indirectly, that expected *reach*, on the basis of the optimization, is greater.

## 5. CONCLUSION

In this paper the authors presented one flexible mathematical model for optimal scheduling of commercials on TV stations, considering the specific characteristics of media environment in a transition country. Since the process of media planning and buying has become one of the key factors of advertising campaign efficiency, scheduling of commercials cannot be left to the intuitive and subjective estimation of media planners. The model suggested in this paper has taken into consideration all the constraints such as budget, prime time and non-prime time ads ratio, balance of the daily TRP, daily number of broadcasted commercials and other, in order to maximize total TRP achieved during one advertising campaign. Reach has been modelled indirectly and it can be evaluated only in the post-bye analysis.

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# Two new variants of Christofides heuristic for the Static TSP and a computational study of a nearest neighbor approach for the Dynamic TSP

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**Abstract**

The Traveling Salesman Problem (TSP) is a standard test-bed for algorithmic ideas. From this point of view we evaluate the computational behavior of two new variants of Christofides heuristic for the Symmetric TSP. Moreover, we compare these numerical results with those arose from known heuristics with same worst-case time complexity. As a further step, we consider the Symmetric Dynamic TSP in which the travel cost between specific pairs of nodes may change through time. In this case, we apply an algorithmic approach based on Nearest Neighbor heuristic in order to avoid a link which has affected negatively cause of a surcharge. Effective implementations were developed and tested on TSPLib 95 benchmark instances.

**KEYWORDS**

Traveling Salesman Problem, Heuristics, Static and Dynamic TSP, Computational Study

## 1. INTRODUCTION

## 1.1 The Traveling Salesman Problem

The Traveling Salesman Problem (TSP) is one of the most well studied combinatorial optimization problems. Given a complete graph $G = (N, E)$ on nodes with edge costs $c \in R^E$, the objective is to find a Hamiltonian cycle or tour in $K_n$ of minimum cost. This problem is known to be NP-hard [Garey & Johnson (1979)] even in the case where the corresponding costs satisfy the triangle inequality, i.e. when $c_{ij} + c_{jk} \geq c_{ik}, \forall i, j, k \in N$ [Laweler et al. (1985)]. There exist plenty of algorithms used to find optimal tours, but none of them is efficient when large instances are handled since they all grow exponentially [Flood (1956)]. This fact leaves the researchers with two major strategies; either look for heuristics that merely find quickly near-optimal solutions, or attempt to develop optimization algorithms that work well on 'real world' rather than worst-case instances. In practice, the TSP is usually solved by using heuristics that are able to find a locally optimal solution.

In addition to the choice of the solution method, one would like to have some guarantee on the quality of the solutions found. Such a guarantee exists if a lower bound for the length of the shortest possible tour is known. In general, lower bounds are obtained by solving relaxations of the original problem. The optimal solution of the relaxed problem gives a valid lower bound for the optimal solution of the original problem. Using different relaxations, different lower bounds can be obtained. Hence, the main goal is to find appropriate relaxation problems for the TSP that can be solved efficiently and which give lower bounds as tight as possible. If the optimal solution of the relaxed problem is a valid tour, then the optimal solution of the

original TSP has been found. In this paper we are going to use lower bounds created by the Held-Karp algorithmic approach [Held & Karp (1970)].

## 1.2 Applications and algorithmic approaches

TSP has many real world applications. For example, it could be used from logistics and data clustering applications to genome sequencing and optimization of drilling in VLSI design. There have been proposed several heuristics (Table 1) with or without a worst-case ratio as far as their results [Papadimitriou & Steiglitz (1977)] during the past decades but Christofides heuristic with worst-case time complexity $O(n^3)$ [Christofides (1976)] and worst-case ratio *3/2* [Vazirani (2003)] remains the best. Other solution approaches are neural networks, tabu search, simulated annealing and genetic learning.

Table 1. Worst-case time complexity and worst-case ratio of the basic TSP heuristics

| Heuristic | Worst-case time complexity | ~above Held-Karp lower bound | Worst-case ratio |
|---|---|---|---|
| Nearest Neighbor | $O(n^2)$ | *25%* | - |
| Greedy | $O(n^2 \log_2 n)$ | *15%-20%* | - |
| Double MST | $O(n^2 \log_2 n)$ | *30%* | *2* |
| Christofides | $O(n^3)$ | *10%* | *1.5* |

Furthermore, apart from tour construction algorithms, there are also some tour improvement algorithms [Lin & Kernighan (1973)]. In tour improvement algorithms, a given initial solution is improved, if possible, by transposing two or more points in the initial tour. With regard to improvement algorithms, two strategies are possible. We may either consider all possible exchanges or choose the one that result in the greatest saving and continue this process until no further reduction is possible.

## 1.3 Proposed methods

It is well-known that TSP is a difficult problem that can approximately be solved by a bunch of heuristics when it is static. But in real world applications we need good results in reasonable time in order to decide which tour should be followed. To this matter, we suggest two heuristics based totally on Christofides heuristic. Moreover, there are many applications in which the transitions costs between specific pairs of nodes may change dynamically through time, i.e. real time traffic routing and router packet scheduling. From this point of view, we are going to examine the computational behavior and the yielding results of an iterative step by step application of the Nearest Neighbor heuristic for the Dynamic TSP. We selected this heuristic due to its ability to adapt to dynamic changes and its efficient computational behavior.

Following this section, in section 2, we describe the data set used in our experiments. In section 3, we describe briefly the proposed solution methods for the Static and the Dynamic TSP by presenting the pseudocodes and a destruction scheme. In section 4, we present the numerical results of our computational study and finally, our conclusions are presented in section 5.

# 2. DATA SET DESCRIPTION

## 2.1 The TSPLib 95

In order to test the computational behavior of the proposed algorithms we have used benchmark instances from TSPLib 95 [Reinelt (1991)]. TSPLib 95 is a well-known suite containing many hard to solve

optimization problems for the TSP and related problems from various sources and of various types. In our paper, we took under consideration the Symmetric Traveling Salesman Problem in which the distance from node $i$ to node $j$ is the same as from node $j$ to node $i$ ($c_{ij} = c_{ji}, i, j = 1,2,...,n, i \neq j$). These instances are all very different from each other, especially in the structure of the cost matrix. The entries of the cost matrix were computed using various geographical distances. These formats, in our occasion, were the Euclidean, the pseudo-Euclidean, the Geographical and the full Matrix. Table 2 demonstrates four benchmark instances with all the different types of entries of the cost matrix used in this survey.

Table 2. Four TSPLib 95 benchmark instances with all the different types of entries of the cost matrix

| Name | #cities | Type | Optimal weight |
|---|---|---|---|
| att48 | 48 | ATT | 10268 |
| brazil58 | 58 | MATRIX | 25395 |
| ch130 | 130 | EUC_2D | 6110 |
| gr666 | 666 | GEO | 294358 |

# 3. ALGORITHMS DESCRIPTION

## 3.1 Algorithms for the Static TSP

In this section we are going to describe the proposed solution methods for the Static TSP. More specifically, we are going to introduce two variations of Christofides heuristic with much better computational behavior and results, as we will see, comparable with those of the Nearest Neighbor and the Greedy heuristic ones. In the pseudocode that follows, we describe the main steps of Christofides heuristic.

**Christofides heuristic – $O(n^3)$**
**Require:** $G = (N, E)$

**STEP 0.** Compute a minimum spanning tree $T$ for $G$ (every tree has an even number of nodes that are odd-degree).
**STEP 1.** Let $N^* \subset N$ be the set of nodes of odd degree in $T$.
**STEP 2.** Compute a minimum weight perfect matching $M$ on the complete graph $G^* = (N^*, E^*)$.
**STEP 3.** Add the edges from $M$ to $T$, and find an Euler tour $\Gamma$, of the resulting (multi-)graph.
**STEP 4.** For each node that occurs more than once in $\Gamma$, remove all but one of its occurrences.
**STEP 5. return** $\Gamma$

### 3.1.1 Proposed methods for computing minimum weight perfect matching

The problem of computing a minimum weight perfect matching is $P$ ($O(n^3)$) but many times doesn't suit well in real time applications. From this point of view we compute this by using the Nearest Neighbor and Greedy heuristic in a very simple way as we can see in Figure 1. More specifically, after the STEP 1 of Christofides heuristic, for the subset of nodes with odd degree, we apply the above heuristics in order to create a tour for these nodes. Then, we choose the cheaper perfect matching from the two available options. Below are presented the pseudocodes.

**ApproximatePerfectMatching1 – $O(n^2)$**
**Require:** $G = (N, E)$, $N^*$

**STEP 0.** Create the complete graph $G^* = (N^*, E^*)$ from the subset $N^*$ created in STEP 1 of Christofides heuristic.
**STEP 1.** Find a tour $C$ over $G^*$ by using the Nearest Neighbor heuristic.
**STEP 2.** Start from node *C (i = 1)*, compute the total weight of the perfect matching $M_1$ by adding the distance from node *C (i)* to *C (i + 2)* and then increase $i$ by 2 in order to compute the second

pairing. Repeat $|N| / 2$ times.

**STEP 3.** Start from node $C(i = 2)$, repeat the above procedure $|N| - 1$ times in order to compute the perfect matching $M_2$ and then add to $M_2$ the $C(|N|)$ to $C(1)$ distance.

**STEP 4.** Choose the cheapest perfect matching from $M_1$ and $M_2$ and then continue with the STEP 3 of Christofides heuristic.

**<u>ApproximatePerfectMatching2</u> – $O(n^2 log_2 n)$**

**Require:** $G = (N, E)$, $N*$

**STEP 0.** Create the complete graph $G* = (N*, E*)$ from the subset $N*$ created in STEP 1 of Christofides heuristic.

**STEP 1.** Find a tour $C$ over $G* = (N*, E*)$ by using the Greedy heuristic.

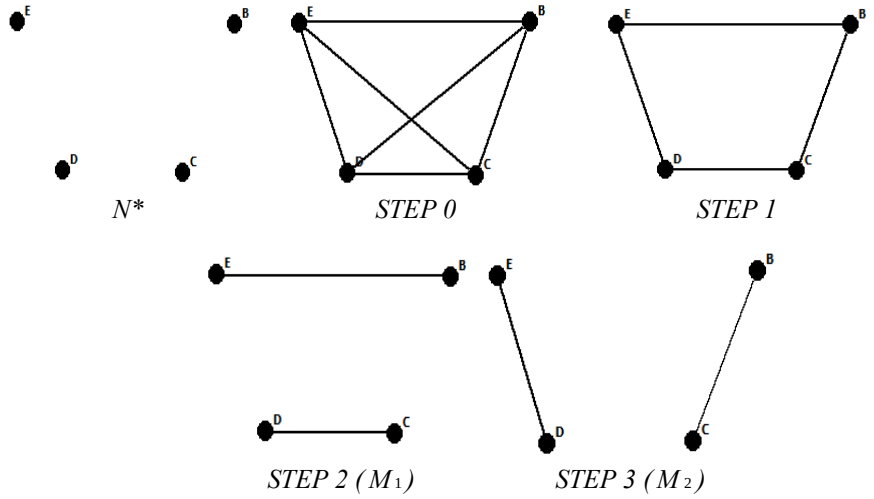
**STEP 2.** Start from node $C(i = 1)$, compute the total weight of the perfect matching $M_1$ by adding the distance from node $C(i)$ to $C(i + 2)$ and then increase $i$ by $2$ in order to compute the second pairing. Repeat $|N| / 2$ times.

**STEP 3.** Start from node $C(i = 2)$, repeat the above procedure in order to compute the perfect matching $M_2$ and then add to $M_2$ the $C(|N|)$ to $C(1)$ distance.

**STEP 4.** Choose the cheapest perfect matching from $M_1$ and $M_2$ and continue with the STEP 3 of Christofides heuristic.

Moreover, as we know from the basic principles of the asymptotic analysis, the proposed methods, i.e. Christofides1 (Christofides heuristic using ApproximatePerfectMatching1) and Christofides2 (Christofides heuristic using ApproximatePerfectMatching2) have worst-case time complexity $O(n^2)$ and $O(n^2 log_2 n)$ respectively. As a result, when applied to Christofides heuristic, they reduce its worst-case time complexity to the same values but unfortunately with no guarantee for the quality of the created tour.

Figure 1. Visualization of the proposed methods for finding perfect matching




## 3.2 Algorithm for the Dynamic TSP

**<u>DynamicTSPNearestNeighbor</u>**

**Require:** $G = (N, E)$

**STEP 0.** Select arbitrarily a starting node $u$.

**STEP 1.** While you haven't visited all nodes

Compute the subset $N* \subset N$ of nodes whose selection won't create a cycle.

Let $v \in N*$ be the node nearest to $u$.

Apply the destruction scheme on $G = (N, E)$.

If a destruction has affected $N^*$
Let $v \in N^*$ be the node nearest to $u$.
$u \leftarrow v$
else
$u \leftarrow v$
End_If
End_While
**STEP 2.** Return the created tour.

### 3.2.1 Destructions scheme

We examined five different types of destructions with different probabilities of occurrence and surcharge due to the fact that not all real phenomena happen and disrupt networks the same way, i.e. rain is more possible and causes fewer surcharges than a hurricane. For our experiments we consider five types of destruction (Table 3) with decreasing probability of occurrence and increasing surcharge:

Table 3. The five different types of destruction with their surcharges

| Phenomenon | Probability of occurrence | Surcharge |
|---|---|---|
| A | 33.33% | 50% |
| B | 26.66% | 100% |
| C | 20.00% | 150% |
| D | 13.33% | 200% |
| E | 6.66% | 250% |

Furthermore, we took into account two basic factors, first that there are periods of no destruction at all and second that not all nodes on a network are affected when destruction does occur, i.e. there are periods without snow and when it does it certainly doesn't snow everywhere. (These two factors where modeled with the parameters of the destruction probability and the percentage number of nodes affected from destruction).

# 4. COMPUTATIONAL STUDY

## 4.1 Computing environment

All heuristics were implemented in the MathWorks MATLAB environment. MATLAB is a widely used tool in scientific computing community. The main reasons for this choice were the inherent capability of MATLAB for matrix operations and the support of sparse matrices. The reported CPU times were computed with the built-in function `cputime`. The given times are net times and do not include times for the input. The characteristics of the hardware and software can be a crucial factor for the performance of an algorithm. Our computational experiments ran in the computing environment described in Table 4.

Table 4. Description of the computing environment

| | |
|---|---|
| CPU | Intel(R) Core2Duo™, 2.26 GHz (2 processors) |
| RAM size | 3096 MB |
| L2 Cache size | 3 MB |
| L2 Cache size | 128 KB |
| Operating System | Linux O/S – Ubuntu, Kernel: 2.6.31 - generic |
| MATLAB version | 7.8.0.347 (R2009a) |

## 4.2 Computational results for the Static TSP

Analyzing the numerical results, it is worth mentioning that our proposed methods in order to solve the Static TSP returned tours with very good results. More specifically, the Christofides1 (Christofides using the ApproximatePerfectMatching1) with worst-case time complexity $O(n^2)$ returned tours with less weight for the 65 of the 87 TSPLib 95 instances (~74.70%) and Christofides2 (Christofides using the ApproximatePerfectMatching2) with worst-case time complexity $O(n^2 log_2 n)$ returned better tours in 36 out of our 87 instances (~41.40%). Table 5 demonstrates the yielding results for 20 out of our 87 TSPLib 95 instances whose selection based on their returned weights which belong to a comparable scale. Figure 2 shows the total weight of all TSPLib 95 instances used in this survey using various heuristics and Figure 3 the total weight of all TSPLib 95 instances in contrast with the Double Minimum Spanning Tree heuristic (worst-case ratio: 2) and the optimal weight results.

Table 5. Numerical results for our proposed methods

| Name | n | Nearest Neighbor weight | Nearest Neighbor time (secs) | Christofides1 weight | Christofides1 time (secs) | Greedy weight | Greedy time (secs) | Christofides2 weight | Christofides2 time (secs) | Optimal weight |
|---|---|---|---|---|---|---|---|---|---|---|
| att532 | 532 | 35516 | 0.18 | 35647 | 0.95 | 34002 | 1.29 | 32626 | 1.36 | 27686 |
| brazil58 | 58 | 30774 | 0.01 | 27950 | 0.03 | 30458 | 0.05 | 27623 | 0.05 | 25395 |
| d493 | 493 | 41665 | 0.15 | 42063 | 0.97 | 41361 | 1.37 | 41077 | 1.37 | 35002 |
| d657 | 657 | 61627 | 0.21 | 57944 | 1.19 | 56612 | 1.33 | 58682 | 1.90 | 48912 |
| d1291 | 1291 | 60214 | 0.40 | 62735 | 4.32 | 61253 | 7.84 | 63691 | 6.02 | 50801 |
| gr96 | 96 | 75065 | 0.03 | 68364 | 0.06 | 66197 | 0.07 | 63597 | 0.08 | 55209 |
| gr202 | 202 | 47080 | 0.06 | 47683 | 0.18 | 47130 | 0.23 | 45670 | 0.26 | 40160 |
| kroA100 | 100 | 27807 | 0.03 | 27127 | 0.07 | 24287 | 0.07 | 25387 | 0.08 | 21282 |
| kroA150 | 150 | 33633 | 0.04 | 31440 | 0.13 | 31892 | 0.16 | 30834 | 0.17 | 26524 |
| kroA200 | 200 | 35859 | 0.04 | 35794 | 0.18 | 34554 | 0.15 | 35948 | 0.25 | 29368 |
| kroB200 | 200 | 36980 | 0.05 | 36123 | 0.19 | 35975 | 0.20 | 35975 | 0.24 | 29437 |
| lin318 | 318 | 54019 | 0.10 | 52288 | 0.39 | 49898 | 0.33 | 51062 | 0.44 | 42029 |
| nrw1379 | 1379 | 68964 | 0.52 | 68545 | 7.24 | 66106 | 7.18 | 67600 | 9.04 | 56638 |
| P654 | 654 | 43457 | 0.19 | 41224 | 1.11 | 40278 | 0.99 | 39911 | 1.45 | 34643 |
| pcb442 | 442 | 61979 | 0.13 | 56851 | 0.71 | 61076 | 0.66 | 56343 | 0.85 | 50778 |
| pcb1173 | 1173 | 71978 | 0.47 | 67946 | 4.82 | 70060 | 6.45 | 66203 | 5.31 | 56892 |
| pr124 | 124 | 69297 | 0.03 | 65379 | 0.07 | 64998 | 0.08 | 70714 | 0.10 | 59030 |
| pr299 | 299 | 59890 | 0.08 | 58546 | 0.40 | 63333 | 0.53 | 59649 | 0.50 | 48191 |
| u159 | 159 | 54675 | 0.03 | 49248 | 0.13 | 49589 | 0.11 | 47108 | 0.16 | 42080 |
| u574 | 574 | 50549 | 0.20 | 44357 | 1.21 | 45043 | 1.15 | 46214 | 1.42 | 36905 |
| **Total** | **-** | **1021028** | **2.95** | **977254** | **24.35** | **974102** | **30.24** | **965914** | **31.05** | **816962** |

Figure 2. Computational results of our proposed methods

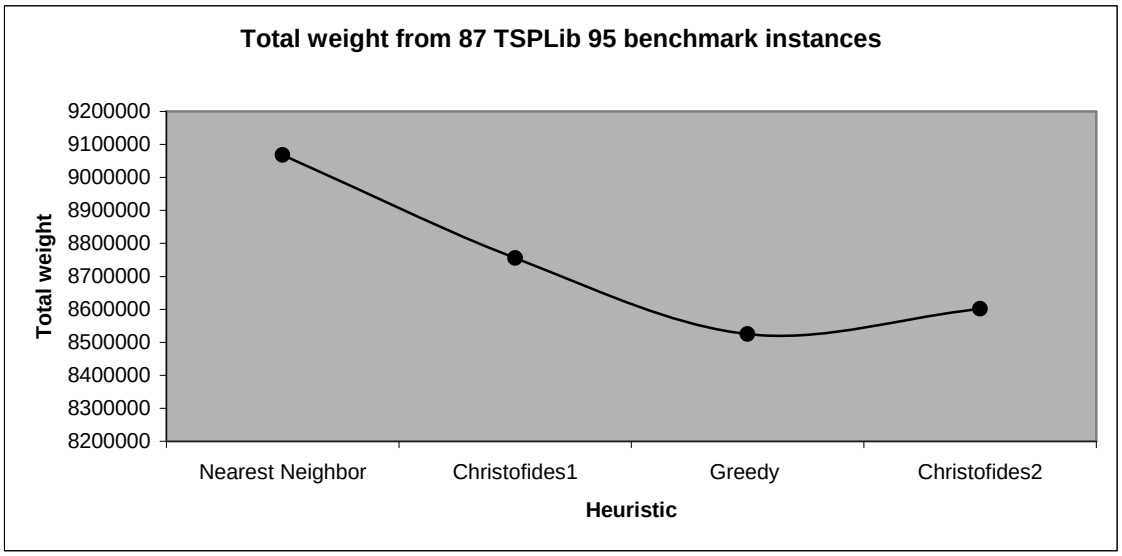
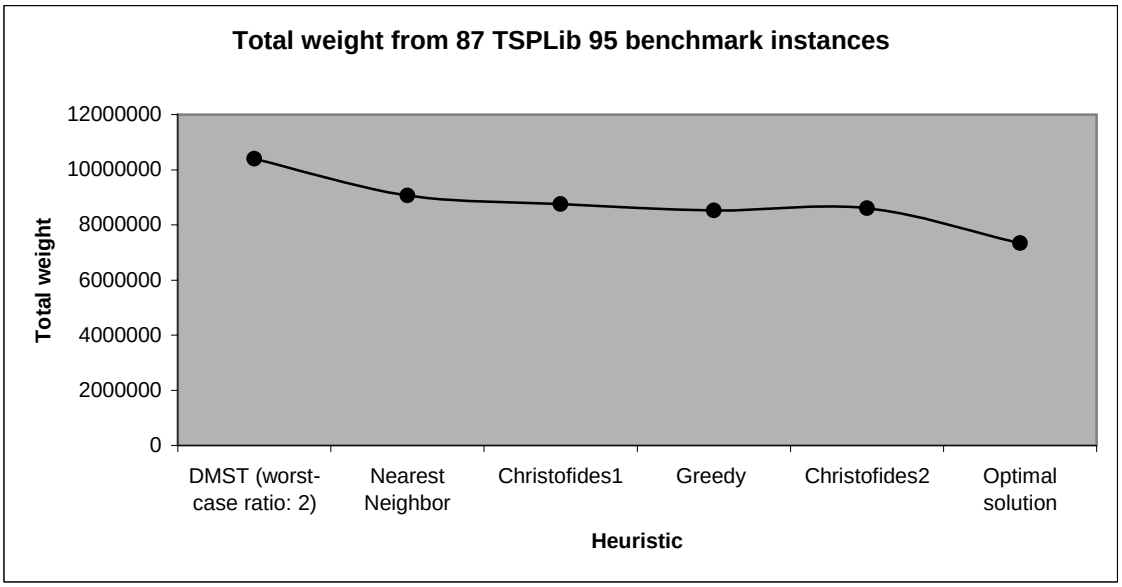


Figure 3. Computational results of our proposed methods in contrast with the DMST and optimal ones




## 4.3 Computational results for the Dynamic TSP

In Table 6 we present our results each of which correspond to 10 iterations on each test set and we provide 20 out of the 87 sets which were totally tested, in order to have a coherent outcome (returned weight) scale from approximately 30000 up to 80000. Column *n* corresponds to the number of cities on the set, column *Average weight without avoidance* corresponds to the weight found without changing course despite of destruction and columns *Average weight* and *Average time with avoidance* correspond to the average weight and time in cpu seconds for the 10 iterations. Column *Average extra weight with avoidance* is the mean out of 10 iterations when we allow the destruction to be taken accounted for and column *Best weight with avoidance* corresponds to the best route found with making changes due to destruction. Column *#of avoided destructions / #of total destructions* show the times we choose a different node instead of the originally chosen out of the total choices we made to complete the rout. Columns *Weight without destructions* and *Time without destructions* correspond to the weight and time when no destruction took place. Finally, column *Optimal weight* corresponds to the best known solution. Two visualized examples of the performance of the algorithms are presented in Figure 4 and 5 by demonstrating the yielding results corresponding to pcb442 TSPLib 95 instance.

Table 6. Numerical results (possibility of destruction: 50%, percentage of affected nodes in each destruction: 10%)

| Name | n | Average weight without avoidance | Average weight with avoidance | Average time (sec) with avoidance | Average extra weight with avoidance | Best weight with avoidance | #of avoided destructions/ # of total destructions | Weight without destructions | Time (sec) without destructions | Optimal weight |
|---|---|---|---|---|---|---|---|---|---|---|
| att532 | 532 | 37171.05 | 36002.40 | 0.28 | 260.10 | 34350 | 22/29 | 35516 | 0.19 | 27686 |
| brazil58 | 58 | 32383.45 | 32779.45 | 0.03 | 632.75 | 30286 | 2/3 | 30774 | 0.03 | 25395 |
| d493 | 493 | 43387.70 | 43859.80 | 0.26 | 302.10 | 41428 | 27/31 | 41665 | 0.18 | 35002 |
| d657 | 657 | 65609.65 | 62933.45 | 0.36 | 296.85 | 59272 | 14/24 | 61627 | 0.26 | 48912 |
| d1291 | 1291 | 63691.70 | 64207.50 | 0.71 | 570.20 | 60674 | 36/62 | 60214 | 0.41 | 50801 |
| gr96 | 96 | 78241.85 | 75161.45 | 0.04 | 884.65 | 72353 | 1/5 | 75065 | 0.02 | 55209 |
| gr202 | 202 | 49331.40 | 50833.80 | 0.10 | 719.80 | 47346 | 2/3 | 47080 | 0.05 | 40160 |
| kroA100 | 100 | 29565.20 | 27305.50 | 0.04 | 174.50 | 24554 | 6/10 | 27807 | 0.02 | 21282 |
| kroA150 | 150 | 36020.75 | 33652.20 | 0.06 | 205.30 | 31942 | 7/7 | 33633 | 0.02 | 26524 |
| kroA200 | 200 | 37974.35 | 39159.00 | 0.08 | 514.80 | 36739.50 | 5/6 | 35859 | 0.03 | 29368 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| kroB200 | 200 | 39487.85 | 38774.15 | 0.09 | 513.95 | 37221 | 5/6 | 36980 | 0.04 | 29437 |
| lin318 | 318 | 57293.15 | 53704.05 | 0.14 | 411.85 | 50084 | 12/16 | 54019 | 0.11 | 42029 |
| nrw1379 | 1379 | 72880.85 | 72607.60 | 0.92 | 705.40 | 70526.50 | 45/52 | 68964 | 0.54 | 56638 |
| P654 | 654 | 45458.50 | 43773.50 | 0.32 | 104.50 | 39562 | 13/14 | 43457 | 0.20 | 34643 |
| pcb442 | 442 | 65371.45 | 63055.30 | 0.20 | 441.40 | 60320 | 20/24 | 61979 | 0.13 | 50778 |
| pcb1173 | 1173 | 75662.90 | 75140.20 | 0.76 | 391.40 | 73456 | 43/58 | 71978 | 0.48 | 56892 |
| pr124 | 124 | 72884.95 | 73548.80 | 0.05 | 745.10 | 70967 | 5/6 | 69297 | 0.04 | 59030 |
| pr299 | 299 | 63698.05 | 62850.15 | 0.13 | 245.80 | 57712.50 | 10/14 | 59890 | 0.08 | 48191 |
| u159 | 159 | 58078.20 | 54885.00 | 0.06 | 720.70 | 47614 | 7/9 | 54675 | 0.04 | 42080 |
| u574 | 574 | 52961.35 | 49350.90 | 0.30 | 702.50 | 46788 | 13/17 | 50459 | 0.23 | 36905 |
| **Total** | **-** | **~1077154** | **~1053584** | **4.93** | **~9544** | **~993196** | **295/396** | **1020938** | **3.10** | **816962** |

Figure 4. Numerical results for the pcb442 instance when the probability of destruction was 70%

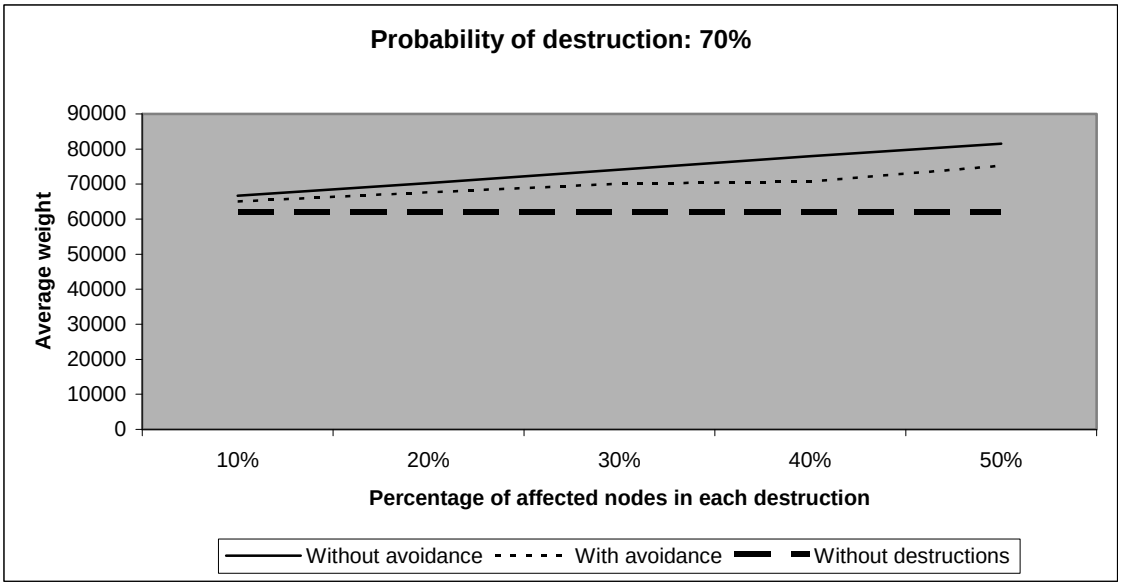
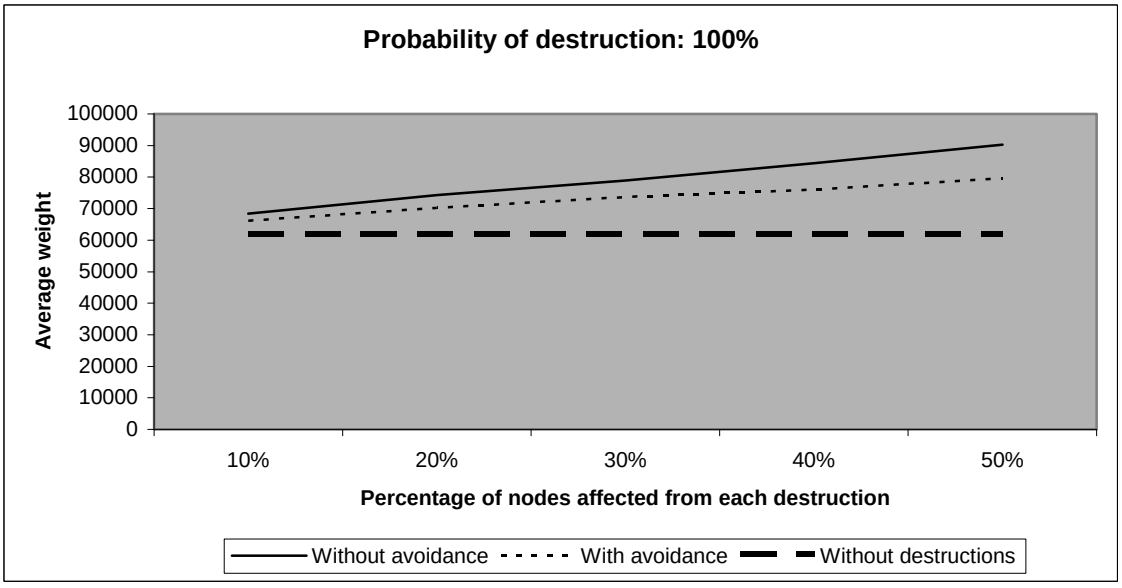



Figure 5. Numerical results for the pcb442 instance when the probability of destruction was 100%



## 5. CONCLUSION

In this paper we tried to evaluate the computational behavior of several proposed heuristics for the Static and the Dynamic Symmetric Traveling Salesman Problem while in parallel we suggested some algorithmic approaches for solving much different instances not only effectively but also in real time. As our experiments have shown, Christofides1 and Christofides2 heuristics behave very well in general. More specifically, Christofides1 heuristic returned less weighted tours than Nearest Neighbor for the 74.70% of our instances. However, this heuristic seemed to behave in the most of the cases in his worst-case time complexity. On the other hand, Christofides2 heuristic returned less weighted tours than Greedy for the 41.40% of our instances in reasonable time. In our opinion, these results indicate that our proposed heuristics may easily apply in real applications in which the time responses play a crucial role. It is worth mentioning here that a random selection between the perfect matching $M_1$ and $M_2$ during the STEP 4 of our proposed methods (ApproximatePerfectMatching1 and ApproximatePerfectMatching2) doesn't work effectively as we clearly understood through experimentation. Ending, DynamicTSPNearestNeighbor heuristic seemed to behave very well in general as our hundreds of experiments have shown. Moreover, the yielding results of this heuristic were far better for all of our experiments and the time responses were closely related to those of Nearest Neighbor without the property of avoiding a surcharged link cause of a destruction.

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# A parallel implementation of the revised simplex algorithm using OpenMP: some preliminary results

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**Abstract**

Linear Programming (LP) is a significant area in the field of operations research. The simplex method is the most widely used method for solving LP problems. The aim of this paper is to present a parallel implementation of the revised simplex algorithm. Our parallel implementation focuses on the reduction of the time taken to perform the basis inverse, due to the fact that the total computational effort of an iteration of simplex type algorithms is dominated by this computation. This inverse does not have to be computed from scratch at any iteration. In this paper, we compute the basis inverse with two well-known updating schemes: (i) The Product Form of the Inverse (PFI) and (ii) A Modification of the Product Form of the Inverse (MPFI); and incorporate them with revised simplex algorithm. Apart from the parallel implementation, this paper presents a computational study that shows the speedup among the serial and the parallel implementations in large-scale LP problems. The test set used in the computational study is the Netlib set of linear problems. The parallelism is achieved using OpenMP in a shared memory multiprocessor architecture.

**KEYWORDS**

Linear Programming, Revised Simplex Method, Basis Inverse, Parallel Computing, OpenMP.

## 1. INTRODUCTION

Linear Programming (LP) is the process of minimizing or maximizing a linear objective function $z = \sum_{i=1}^{n} c_i x_i$ to a number of linear equality and inequality constraints. Several methods are available for solving LP problems, among which the simplex algorithm is the most widely used. We assume that the problem is in its general form. Formulating the linear problem, we can describe it as shown below:

$$\begin{aligned} \min \quad & c^T x \\ \textit{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad (1)$$

where $A \in R^{mxn}$, $(c, x) \in R^n$, $b \in R^m$, and T denotes transposition. We assume that A has full rank. The simplex method searches for an optimal solution by moving from one feasible solution to another, along the edges of the feasible set. The dual problem associated with the linear problem in equation (1) is shown below:

$$\begin{aligned} \min \quad & b^T w \\ \textit{subject to} \quad & A^T w + s = c \\ & s \geq 0 \end{aligned} \qquad (2)$$

where $w \in R^m$ and $s \in R^n$. As in the solution of any large scale mathematical system, the computational time for large LP problems is a major concern. Parallel programming is a good practice for solving computationally intensive problems in operations research. The application of parallel processing for linear

programming has been introduced in the early 1970s. However, only since the beginning of the 1980s attempts have been made to develop parallel implementations.

One of the earliest parallel tableau simplex methods on a small-scale distributed memory Multiple-Instruction Multiple-Data (MIMD) machines is the one introduced by Finkel (1987). Stunkel (1988) implemented both the tableau and the revised simplex method on a 16-processor Intel hypercube computer, achieving a speedup of between 8 and 12 for small problems from the Netlib set (Gay, 1985). Helgason, Kennington and Zaki (1988) proposed an algorithm to implement the revised simplex using sparse matrix methods on shared memory MIMD computer. Furthermore, Shu and Wu (1993) and Shu (1995) parallelized the explicit inverse and the LU decomposition of the basis simplex algorithms. Hall and McKinnon (1996; 1998) have implemented two parallel schemes for the revised simplex method. The first of Hall and McKinnon's parallel revised simplex implementations was ASYNPLEX (1996). In this implementation one processor is devoted to the basis inversion and the remaining processors perform simplex iterations. ASYNPLEX was implemented on a Cray T3D, achieving a speedup of between 2.5 and 4.8 for four modest Netlib problems. The second of Hall and McKinnon's parallel revised simplex implementations was PARSMI (1998). PARSMI was tested on modest problems from the Netlib set, resulting in a speedup of between 1.7 and 1.9. Hall (2005) implemented a variant of PARSMI on a 8-processor shared memory Sun Fire E15k, leading in a speedup of between 1.8 and 3.

Simplex algorithms for general LP problems on Single Instruction Multiple Data (SIMD) have been reported by Agrawal et al. (1989). Luo and Reijns (1992) presented an implementation of the revised simplex method, achieving a speedup of more than 12, when solving modest Netlib problems on 16 transputers. Eckstein et al (1995) implemented a parallelization of standard and revised simplex method in a CM2 machine. Lentini et al (1995) worked on the standard simplex method with the tableau stored as a sparse matrix, resulting in a speedup of between 0.5 and 2.7, when solving medium sized Netlib problems on four transputers. Thomadakis and Liu (1996) worked on the standard simplex method on MasPar MP-1 and MP-2 machines, achieving a speedup of up to three, when solving large randomly-generated problems. Badr et al (2006) implemented a dense standard simplex method on eight computers, leading in a speedup of five when solving small random dense LP problems.

The use of GPUs for general purpose computations is a quite recent topic, which was applied to linear programming. Greeff (2004) implemented the revised simplex method on a GPU using OpenGL and Cg and was able to achieve a speedup of up to 11.4 over an identical CPU implementation. Jung and O'Leary (2008) and Owens et al (2008) also presented an implementation using Cg and OpenGL. Spampinato and Elster (2009) proposed a GPU implementation of the revised simplex method, based on the CUDA architecture and achieved a speedup of up to 2.5. Recently, Bieling, Peschlow and Martini (2010) also presented an implementation of the revised simplex algorithm and achieved a speedup of up to 10.

Finally, computational results for parallelizing the network simplex method are reported in (Chang et al, 1988; Barr and Hickman, 1994; Peters, 1999).

This paper presents a parallelization of the revised simplex algorithm on a shared memory multiprocessor architecture. The focus of this parallelization is on the basis inverse. The structure of the paper is as follows. In Section 2, the revised simplex algorithm is described and presented. In Section 3, two methods that have been widely used for basis inversion are analyzed. Section 4 presents the parallel revised simplex algorithm and section 5 gives the computational results. Finally, the conclusions of this paper are outlined in section 6.

## 2. REVISED SIMPLEX ALGORITHM

The linear problem in equation (1) can be written as:

$$\begin{aligned} \min \quad & c_B^T x_B + c_N^T x_N \\ \textit{subject to} \quad & A_B x_B + A_N x_N = b \\ & x_B, x_N \geq 0 \end{aligned} \qquad (3)$$

In the above equation, B is a mxm non-singular sub-matrix of A, called basic matrix or basis. The columns of A which belong to subset B are called basic and those which belong to N are called non basic. The solution of the linear problem $x_B = B^{-1}b, x_N = 0$ is called a basic solution. A solution $x = (x_B, x_N)$ is

feasible if x > 0. Otherwise the solution is infeasible. The solution of the linear problem in equation (2) is computed by the relation $s = c - A^T w$, where $w = (c_B)^T B^{-1}$ are the simplex multipliers and s are the dual slack variables. The basis B is dual feasible if $s \geq 0$.

In each iteration, simplex algorithm interchanges a column of matrix B with a column of matrix N and constructs a new basis $\overline{B}$. Any iteration of simplex type algorithms is relatively expensive. The total work of an iteration of simplex type algorithms is dominated by the determination of the basis inverse. This inverse however, does not have to be computed from scratch during each iteration. Simplex type algorithms maintain a factorization of basis and update this factorization in each iteration. There are several schemes for updating basis inverse. The most well-known schemes are (i) the Product Form of the Inverse (PFI) and (ii) a Modification of the Product Form of the Inverse, developed by Benhamadou (2002). These methods, in order to compute the new basis, use only information about the entering and leaving variables along with the current basis. A formal description of the revised simplex algorithm (Dantzig, Orden and Wolfe, 1953) is given below.

Table 1. Revised Simplex Algorithm.

**Step 0.** (*Initialization*).

Start with a feasible partition (B, N). Compute $B^{-1}$ and vectors $x_B$, w and $s_N$.

**Step 1.** (*Test of optimality*).

if $s_N \geq 0$ then STOP. The linear problem is optimal.

else

Choose the index l of the entering variable using a pivoting rule.

Variable $x_l$ enters the basis.

**Step 2.** (*Minimum ratio test*).

Compute the pivot column $h_l = B^{-1} A_{.l}$.

if $h_l \leq 0$ then STOP. The linear problem is unbounded.

else

Choose the leaving variable $x_{B[r]} = x_k$ using the following equation:

$$x_{B[r]} = \frac{x_{B[r]}}{h_{il}} = \min\left\{\frac{x_{B[i]}}{h_{il}} : h_{il} < 0\right\} \qquad (4)$$

**Step 3.** (*Pivoting*).

Swap indices k and l. Update the new basis inverse $\overline{B}^{-1}$, using PFI or MPFI.

Go to Step 1.

# 3. METHODS USED FOR BASIS INVERSION

The revised simplex algorithm differs from the original method. The former uses the same recursion relations to transform only the inverse of the basis in each iteration. It has been implemented to reduce the computation time of the basis inversion and is particularly effective for sparse linear problems. In this section, we will review two methods that have been widely used for basis inversion: (i) the Product Form of the Inverse and (ii) a Modification of the Product Form of the Inverse.

## 3.1 Product Form of the Inverse

The PFI scheme, in order to compute the new basis, uses information only about the entering and leaving variables along with the current basis. The new basis inverse can be updated at any iteration using the equation below:

$$\overline{B}^{-1} = (BE)^{-1} = E^{-1} B^{-1} \qquad (5)$$

where $E^{-1}$ is the inverse of the eta-matrix and can be computed by:

$$E^{-1} = I - \frac{1}{h_{rl}}(h_l - e_l)e_l^T = \begin{bmatrix} 1 & & -h_{1l}/h_{rl} & & \\ & \ddots & \vdots & & \\ & & 1/h_{rl} & & \\ & & \vdots & \ddots & \\ & & -h_{ml}/h_{rl} & & 1 \end{bmatrix}$$

If the current basis inverse is computed using regular multiplication, then the complexity of the PFI is $\Theta(m^3)$.

## 3.2 A Modification of Product Form of the Inverse

MPFI updating scheme has been presented by Benhamadou (2002). The key idea is that the current basis inverse $\overline{B}^{-1}$ can be computed from the previous inverse $B^{-1}$ using a simple outer product of two vectors and one matrix addition. Namely:

$$(\overline{B})^{-1} = (\overline{B})^{-1}_{r.} + v \otimes (B)^{-1}_{r.} \qquad (6)$$

The updating scheme of the inverse is shown in Fig. 1.

Figure 1. A modification of the PFI scheme.

$$B^{-1}: \left| \begin{array}{ccccc} b_{r1} & \ldots & b_{rr} & \ldots & b_{rm} \end{array} \right|$$

$$\overline{B}^{-1} = \begin{vmatrix} b_{11} & \ldots & b_{1m} \\ \ldots & \ldots & \ldots \\ 0 & 0 & 0 \\ \ldots & \ldots & \ldots \\ b_{ml} & \ldots & b_{mm} \end{vmatrix} + \begin{vmatrix} -\frac{h_{1l}}{h_{rl}} \\ \ldots \\ -\frac{1}{h_{rl}} \\ \ldots \\ -\frac{h_{m1}}{h_{rl}} \end{vmatrix}$$

The outer product requires $m^2$ multiplications and the addition of two matrices requires $m^2$ additions. The total cost of the above method is $2m^2$ operations (multiplications and additions). Hence, the complexity is $\Theta(m^2)$.

# 4. PARALLEL REVISED SIMPLEX ALGORITHM

The parallelization of all the individual steps of the revised simplex method is limited and very hard to achieve. However, it is also essential for any algorithm to perform basis inverse in parallel with simplex iterations, otherwise basis inverse will become the dominant step and limit the possible speedup. Our parallel implementation focuses on the reduction of the time taken to perform the basis inverse. The basis inversion is done with the Product Form of the Inverse and a Modification of the Product Form of the Inverse, as described in the previous section.

Both methods take as input the previous basis inverse ($B^{-1}$), the pivot column ($h_l$), the index of the leaving variable (k) and the number of the constraints (n).

The most time-consuming step of PFI scheme is the matrix multiplication of relation (5). Our parallel algorithm uses the block matrix multiplication algorithm for this step. This algorithm suggests a recursive divide-and-conquer solution, as described in (Hake, 1993; Horowitz and Zorat, 1983). This method has significant potential for parallel implementations, especially on shared memory implementations.

Let us assume that we have *p* processors. We use the following steps to compute the new basis inverse $\overline{B}^{-1}$ with the PFI scheme:

Table 2. Parallel Product Form of the Inverse.

**Step 0.**
Compute the column vector:

$$v = \left[ -\frac{h_{1l}}{h_{rl}} \quad \cdots \quad \frac{1}{h_{rl}} \quad \cdots \quad -\frac{h_{ml}}{h_{rl}} \right]^T$$

Each processor computes in parallel n/p elements of v.
**Step 1.**
Replace the $r^{th}$ column of an identity matrix with the column vector v. Each processor assigns in parallel n/p elements to the identity matrix. This matrix is the inverse of the Eta-matrix.
**Step 2.**
Compute the new basis inverse using relation (5) with block matrix multiplication. Each processor will compute n/p rows of the new basis.

We use the following steps to compute the new basis inverse $\overline{B}^{-1}$ with the MPFI scheme:

Table 3. Parallel Modification of Product Form of the Inverse.

**Step 0.**
Compute the column vector:

$$v = \left[ -\frac{h_{1l}}{h_{rl}} \quad \cdots \quad \frac{1}{h_{rl}} \quad \cdots \quad -\frac{h_{ml}}{h_{rl}} \right]^T$$

Each processor computes in parallel n/p elements of v.
**Step 1.** (The following steps are computed in parallel)
**Step 1.1**. Compute the outer product $v \otimes B^{-1}{}_r$ with block matrix multiplication.
**Step 1.2**. Copy matrix $B^{-1}$ to matrix $\overline{B}^{-1}$. Set the $r^{th}$ row of $\overline{B}^{-1}$ equal to zero. Each processor computes in parallel n/p rows of $\overline{B}^{-1}$.
**Step 2.**
Compute the new basis inverse using relation (6). Each processor computes in parallel n/p rows of the new basis.

# 5. COMPUTATIONAL EXPERIMENTS

The three most usual approaches to analyzing algorithms are i) worst-case analysis, ii) average-case analysis and iii) experimental analysis. Computational studies have been proven useful tools in order to examine the practical efficiency of an algorithm or even compare algorithms by using the same problem sets. The computational comparison has been performed on a quad-processor Intel Xeon 3.2 GHz with 2 Gbyte of main memory running under Ubuntu 10.10 64-bit and performed on GCC 4.5.2. The algorithms have been implemented using C++ and OpenMP.

## 5.1 Problem Instances

Below there are some useful information about the data set, which was used in the computational study. The first column of the table includes the name of benchmarks, the second the number of constraints, the third the number of variables, the fourth the non-zero elements of matrix A and the fifth the sparsity of matrix A.

Table 4. Statistics of the benchmarks.

| Problem | Constraints | Variables | Non-Zeros A | Sparcity A |
|---|---|---|---|---|
| agg | 488 | 163 | 2410 | 3.03% |
| agg2 | 516 | 302 | 4284 | 2.75% |
| agg3 | 516 | 302 | 4300 | 2.76% |
| bandm | 305 | 472 | 2494 | 1.73% |

| | | | | |
|---|---|---|---|---|
| brandy | 220 | 249 | 2148 | 3.92% |
| e226 | 223 | 282 | 2578 | 4.10% |
| fffff800 | 524 | 854 | 6227 | 1.39% |
| israel | 174 | 142 | 2269 | 9.18% |
| lotfi | 153 | 308 | 1078 | 2.29% |
| sc105 | 105 | 103 | 280 | 2.59% |
| sc205 | 205 | 203 | 551 | 1.32% |
| scfxm1 | 330 | 457 | 2589 | 1.72% |
| scfxm2 | 660 | 914 | 5183 | 0.86% |
| scfxm3 | 990 | 1371 | 7777 | 0.57% |
| scrs8 | 490 | 1169 | 3182 | 0.56% |
| share1b | 117 | 225 | 1151 | 4.37% |
| share2b | 96 | 79 | 694 | 9.15% |
| ship04l | 402 | 2118 | 6332 | 0.74% |
| ship04s | 402 | 1458 | 4352 | 0.74% |
| ship08l | 778 | 4283 | 12802 | 0.38% |
| ship08s | 778 | 2387 | 7114 | 0.38% |
| ship12l | 1151 | 5427 | 16170 | 0.26% |
| ship12s | 1151 | 2763 | 8178 | 0.26% |
| stocfor1 | 117 | 111 | 447 | 3.44% |
| klein2 | 477 | 54 | 4585 | 17.80% |
| klein3 | 994 | 88 | 12107 | 13.84% |

## 5.2 Results

Table 5, presents the results from the execution of the serial and parallel implementations of the above mentioned updating schemes. For each implementation, the table shows the basis inverse and the total time. All times are displayed in seconds.

Table 5. Basis inverse and total time of the serial and parallel implementations.

| PROBLEM | Serial implementations | | | | Parallel implementations | | | |
|---|---|---|---|---|---|---|---|---|
| | PFI | | MPFI | | PFI | | MPFI | |
| | Time of basis inverse | Total time | Time of basis inverse | Total time | Time of basis inverse | Total time | Time of basis inverse | Total time |
| agg | 2.83 | 4.58 | 2.28 | 4.05 | 1.52 | 3.32 | 1.13 | 2.95 |
| agg2 | 3.54 | 5.65 | 2.78 | 4.97 | 1.81 | 3.96 | 1.52 | 3.69 |
| agg3 | 3.28 | 5.61 | 2.62 | 5.03 | 1.85 | 4.05 | 1.48 | 3.78 |
| bandm | 1.01 | 1.62 | 0.74 | 1.41 | 0.84 | 1.52 | 0.61 | 1.29 |
| brandy | 1.30 | 2.76 | 1.06 | 2.55 | 1.04 | 2.48 | 0.76 | 2.22 |
| e226 | 1.66 | 3.34 | 1.38 | 3.09 | 1.22 | 2.82 | 0.85 | 2.50 |
| fffff800 | 6.10 | 12.77 | 4.86 | 11.64 | 5.18 | 11.58 | 4.31 | 10.89 |
| israel | 0.82 | 1.73 | 0.63 | 1.65 | 0.53 | 1.48 | 0.45 | 1.31 |
| lotfi | 0.33 | 0.85 | 0.29 | 0.80 | 0.25 | 0.78 | 0.21 | 0.68 |
| sc105 | 0.08 | 0.09 | 0.03 | 0.07 | 0.01 | 0.07 | 0.02 | 0.06 |
| sc205 | 0.55 | 0.91 | 0.51 | 0.85 | 0.40 | 0.74 | 0.20 | 0.56 |
| scfxm1 | 3.72 | 7.11 | 2.96 | 6.38 | 3.17 | 6.31 | 2.49 | 5.80 |
| scfxm2 | 30.76 | 62.34 | 24.26 | 56.40 | 26.34 | 58.56 | 21.22 | 52.54 |
| scfxm3 | 109.06 | 244.48 | 83.97 | 219.22 | 91.67 | 224.76 | 72.93 | 209.23 |
| scrs8 | 11.17 | 22.20 | 8.69 | 19.81 | 9.56 | 20.20 | 7.34 | 17.90 |
| share1b | 0.15 | 0.29 | 0.09 | 0.26 | 0.11 | 0.25 | 0.08 | 0.23 |
| share2b | 0.05 | 0.11 | 0.04 | 0.10 | 0.01 | 0.10 | 0.03 | 0.09 |
| ship04l | 5.20 | 14.53 | 4.18 | 13.65 | 4.43 | 13.65 | 3.56 | 12.90 |
| ship04s | 1.76 | 4.55 | 1.52 | 4.31 | 1.54 | 4.22 | 1.21 | 4.10 |
| ship08l | 33.78 | 94.70 | 26.30 | 86.45 | 30.02 | 91.90 | 22.10 | 82.50 |
| ship08s | 7.43 | 19.07 | 5.99 | 17.49 | 6.50 | 18.09 | 5.01 | 16.95 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| ship12l | 120.21 | 335.98 | 89.94 | 305.95 | 100.05 | 317.80 | 75.64 | 292.00 |
| ship12s | 17.20 | 43.52 | 13.45 | 39.16 | 12.99 | 42.84 | 10.77 | 36.60 |
| stocfor1 | 0.04 | 0.06 | 0.03 | 0.05 | 0.02 | 0.06 | 0.02 | 0.04 |
| klein2 | 17.36 | 33.01 | 13.53 | 28.80 | 9.85 | 25.30 | 7.55 | 22.74 |
| klein3 | 192.50 | 413.33 | 148.69 | 368.50 | 106.92 | 326.70 | 75.30 | 295.01 |

Table 6, presents the speedup obtained by the parallel implementations regarding the basis inverse and the total time, for both PFI and MPFI schemes.

Table 6. Speedup obtained by the parallel implementations.

| **PROBLEM** | **Speedup** | | | |
|---|---|---|---|---|
| | **PFI** | | **MPFI** | |
| | **Basis inverse** | **Total** | **Basis inverse** | **Total** |
| agg | 1.86 | 1.38 | 2.02 | 1.37 |
| agg2 | 1.96 | 1.43 | 1.83 | 1.35 |
| agg3 | 1.77 | 1.39 | 1.77 | 1.33 |
| bandm | 1.20 | 1.07 | 1.21 | 1.09 |
| brandy | 1.25 | 1.11 | 1.39 | 1.15 |
| e226 | 1.36 | 1.18 | 1.62 | 1.24 |
| fffff800 | 1.18 | 1.10 | 1.13 | 1.07 |
| israel | 1.55 | 1.17 | 1.40 | 1.26 |
| lotfi | 1.32 | 1.09 | 1.38 | 1.18 |
| sc105 | 8.00 | 1.29 | 1.50 | 1.17 |
| sc205 | 1.38 | 1.23 | 2.55 | 1.52 |
| scfxm1 | 1.17 | 1.13 | 1.19 | 1.10 |
| scfxm2 | 1.17 | 1.06 | 1.14 | 1.07 |
| scfxm3 | 1.19 | 1.09 | 1.15 | 1.05 |
| scrs8 | 1.17 | 1.10 | 1.18 | 1.11 |
| share1b | 1.36 | 1.16 | 1.13 | 1.13 |
| share2b | 5.00 | 1.10 | 1.33 | 1.11 |
| ship04l | 1.17 | 1.06 | 1.17 | 1.06 |
| ship04s | 1.14 | 1.08 | 1.26 | 1.05 |
| ship08l | 1.13 | 1.03 | 1.19 | 1.05 |
| ship08s | 1.14 | 1.05 | 1.20 | 1.03 |
| ship12l | 1.20 | 1.06 | 1.19 | 1.05 |
| ship12s | 1.32 | 1.02 | 1.25 | 1.07 |
| stocfor1 | 2.00 | 1.00 | 1.50 | 1.25 |
| klein2 | 1.76 | 1.30 | 1.79 | 1.27 |
| klein3 | 1.80 | 1.27 | 1.97 | 1.25 |
| Average | 1.79 | 1.15 | 1.44 | 1.17 |

From the above results, we observe: (i) the MPFI scheme is in most problems faster than PFI both in serial and in parallel implementation, (ii) using PFI scheme, the speedup gained from the parallelization is of average 1.79 for the time of basis inverse and 1.15 for total time, and (iii) using MPFI scheme, the speedup is of average 1.44 for the time of basis inverse and 1.17 for total time.

## 6. CONCLUSIONS

A parallel algorithm for the revised simplex method has been described in this paper. Some preliminary computational results on Netlib problems have reported a speedup of average 1.79 and 1.44 regarding the basis inverse procedure, using PFI and MPFI updating schemes respectively. These results could be further improved by performance optimization. In future work, we plan to implement our parallel algorithm combining the Message Passing Interface (MPI) and OpenMP programming models to exploit parallelism beyond a single level. Furthermore, we intend to port our algorithm to a GPU implementation based on the CUDA architecture.

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# Optimization of Working Shifts in Call Center via Discrete Differential Evolution

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**Abstract**

The Call Centers have realized the need to improve the performance of their services to better serve its customers. This need motivated the development of this paper, in which, a procedure was developed to solve the working shifts problem in Call Centers. The problem concerns the determination of the weekly working shifts which consists of: given a set with kinds of working shifts, find the amount of each type in order to meet the demand of telephone operators for each hour of the week. The main objective is to optimize the scheduling of telephone operators in Call Centers, in order to reduce wage costs. It is a complex optimization problem due to the large number of restrictions and requirements. The working shifts were optimized by applying Discrete Differential Evolution (DDE). To improve the DDE have been included some additional and alternative searches procedures with adequate randomization. These searches were essential to obtain significant improvements in DDE. The DDE model results were compared with those of a mathematical model of Integer Linear Programming. The DDE found adequate results, and are equal to optimal or close to optimal.

**Keywords**: Discrete Differential Evolution, working shifts, optimization.

## 1. Introduction

Several companies have realized the need to improve the performance of its services to better serve its customers. The need for improvement of Call Centers is due mainly to current legislation. Due to demand variation of calls in Call Centers arise various problems, including determining the required number of telephone operators in each shift to meet the demands at each moment. This need motivated the development of this paper, in which, a procedure was developed to solve the working shift scheduling problem in Call Centers.

According Hojati (2011) are generally considered three steps in performing employee scheduling for a service organization. The first step is demand forecast for each working hour of each day during the next planning period. The second step is to transform the demand forecasts into employee necessities for each working hour of the planning period in order to provide a given customer service level. The third step is to find the employee requirements for each working hour of the planning period by determining the work days, and a shift on each work day, for each employee. According Ernst *et al*. (2004) personnel scheduling, or rostering, is the process of constructing work timetables for its staff so that an organization can satisfy the demand for its goods or services.

The step discussed in this paper is an intermediate step of employee scheduling problem; the problem concerns the determination of the weekly working shifts which consists of: given a set of shifts, find the amount of each type in order to meet the demand of telephone operators for each hour of the planning period. The types of shifts must meet labor and operating restrictions the company. The main objective of this paper is to optimize the weekly working shifts of the Call Centers telephone operators for the period of one week, seeking the best combination of types of working shifts, in order to reduce wage costs.

In literature, there are several papers that deal with scheduling and shifts assignment, given the importance in seeking better solutions to the staff scheduling problem. Many models and different approaches have been proposed and used to solve scheduling problems due to differences between the organizations.

Mathematical programming or network modeling may provide an optimal solution. However, there may be problems with excessive solution times, and computer memory requirements [Ashley, 1995]. With the

appearing of metaheuristics such as Genetic Algorithms, Tabu Search and Simulated Annealing opened a new horizon in resolving this problem. Although such methods do not guarantee obtaining the optimal solution, they are provided with mechanisms to escape from minimum (maximum) local. Thompson (1997) developed a method called SSAH (Specialized Heuristic Shift Assignment) for assigning telephone operators satisfying the demand of calls and seeking to meet the preferences and priorities of telephone operators. Yoshimura & Nakano (1998) propose an effective mutation operator and successfully apply to the telephone operators scheduling problem. Barboza (2000) proposed a solution for the elaboration and assignment of working hours in a Call Center. She used the technique of simulation to determine the number of telephone operators needed to meet historical demand. It was developed an Integer Linear Programming (ILP) model to resolve the telephone operators scheduling problem. This model was solved with the software package LINGO. Finally, it carried out the assignment of working hours to telephone operators, maximizing their satisfaction regarding their work schedules. Saltzman (2005) developed a methodology for scheduling that combines Linear Programming, Tabu Search and Simulation. In one case, applying the hybrid approach reduced total daily costs of the Call Center's existing staffing plan by nearly 18%. Bhulai *et al*. (2008) describe simple methods for working shifts, existing methods single skill in the Call Center. They introduce a new method for shift scheduling in multiskill Call Centers. The method consists of two steps. The scheduling problem is based on a linear programming model.

In this paper, we propose a working shift scheduling model with predetermined intervals of rest. In this model we used the Discrete Differential Evolution metaheuristic to optimize working shifts of Call Center telephone operators for the period of one week.

The remainder of this paper is organized as follows: The approach to optimize the working shifts of the Call Centers telephone operators is presented in Section 2. The topic on the Discrete Differential Evolution metaheuristic is given in Section 3. The experiments and results are given in Section 4. Finally, we conclude our paper in Section 5.

## 2. Working Shifts of Telephone Operators of a Call Center

For the purposes of this study, it was used data from a particular Call Center. In this Call Center, the rostering of the telephone operators are made manually with the support of Excel spreadsheets. Each month is done estimating the calls number using previous month's data to estimate the required of telephone operators number. Then, the rostering of the telephone operators are prepared. These reasons have motivated the development of this paper to optimize working shifts of Call Center telephone operators.

In this study, the working shift is defined as the period of daily working of a telephone operator.

The telephone operators are contracted to work six hours and twenty minutes per day, with two pauses of ten minutes each and an interval of twenty minutes to rest and feeding. The first pause of ten minutes, it must satisfy the following restriction: beginning between one hour and two hours after beginning the working shift. The second pause of ten minutes, it must satisfy the following restriction: beginning between one hour and ten minutes and two hours and ten minutes before the ending of the working shift. The interval should meet the following restrictions: beginning between one hour and two hours after the first pause, and ending between one hour and two hours before the beginning of the second pause. The planning periods (days) were fragmented from ten to ten minutes. A shift is therefore composed of 38 fragments of ten minutes each. The beginning of a shift belongs to the set {00:00, 00:30,..., 23:00, 23:30}. Due to problems with transport after midnight and before 6 o'clock in the morning were imposed constraints to the beginning and ending of shift, in this period. But it was allowed to violate these limits at twenty minutes to the ending of shift.

## 2.1 Problem Formulation

The mathematical formulation of the problem was adapted from the original formulation of Integer Programming (IP) of the shift scheduling problem developed by Dantzig (1954):

$$\text{Minimize} \quad \sum_{n \in N} c_n x_n \tag{1}$$

$$\text{Subject to:} \quad \sum_{n \in N} (a_{np} - b_{np}) x_n \geq r_p , \qquad \forall p \in P \tag{2}$$

$$x_n \geq 0 \quad \text{and integer} \qquad \forall n \in N \tag{3}$$

where

- $n \rightarrow$ index for the shifts
- $N \rightarrow$ set of shifts
- $c_n \rightarrow$ cost of having an telephone operator to work on the shift $n$
- $p \rightarrow$ index to the fragments of the planning period
- $P \rightarrow$ set of fragments of the planning period
- $x_n \rightarrow$ number of telephone operators assigned to the shift $n$
- $a_{np} \rightarrow$ 1, $p$ if the fragment is a fragment of the shift $n$; 0, otherwise
- $b_{np} \rightarrow$ 1, $p$ if the fragment is a pause fragment or interval the shift $n$; 0, otherwise
- $r_p \rightarrow$ number of telephone operators needed to meet demand in the fragment $p$

The objective of the model is: to minimize the cost of shift scheduling (1), subject to the restrictions: the number of telephone operators required is present in each fragment of the planning period, considering the breaks and interval (2) and there are no partial assignments of telephone operators to shifts (3).

## 3. Discrete Differential Evolution for Working Shifts

## 3.1 Differential Evolution

Several techniques are used to optimize the shift schedule as mentioned in the previous section. In this study it was used algorithm Differential Evolution (DE) (Storn & Price, 1995 and 1997) is considered an Evolutionary Algorithm, because it employs operators such as crossover, mutation and selection.

The classic ED (Price & Storn, 1995 and 1997) is a global stochastic optimization algorithm based on population, which can be used to minimize non-linear functions and non-differentiable continuous space, with real-valued parameters. Thus, like other evolutionary algorithms, ED begins with a population of individuals, followed by the mutation operators, crossover and selection, respectively, until a termination criterion is satisfied.

An *i*-th individual in the population of *NP* individuals in generation *g* has *n* chromosomes, i.e.,

$$X_i(g) = [x_{i,1}(g), x_{i,2}(g), \ldots, x_{i,n}(g)], \quad i = 1, 2, \ldots, NP \tag{4}$$

where $x_{i,j}(g)$ are the chromosomes (components) of the *i*-th individual.

In mutation operation, for each individual $X_i$ (target individual) of the population, are chosen three other individuals named, $X_{r1}(g)$, $X_{r2}(g)$ e $X_{r3}(g)$, with random indexes *r1*, *r2* e *r3* $\in \{1, 2, \ldots, NP\}$ mutually different, i.e., $i \neq r1 \neq r2 \neq r3$. An mutant individual, $V_i(g+1)$, is generated by the combination of these three individuals, according to:

$$V_i(g+1) = X_{r1}(g) + F[(X_{r2}(g) - X_{r3}(g)] \tag{5}$$

where *F* is a scaling factor.

In crossover operation, an individual named attempt individual, $U_i(g+1)$, is determined according to:

$$U_i(g+1) = \begin{cases} v_{i,j}(g+1), & \text{if } rand_{i,j} \leq CR \text{ or } j = I_{rand} \\ x_{i,j}(g), & \text{if } rand_{i,j} > CR \text{ or } j \neq I_{rand} \end{cases} \tag{6}$$

where *CR* is the crossover rate, $rand_{i,j}$ is a random number chosen from the interval [0, 1 [and $I_{rand}$ is an integer random chosen from the set {1, 2, ..., *n*}.

The selection is made according to the following formulation:

$$X_i(g+1) = \begin{cases} U_i(g+1), & \text{if } f(U_i(g+1)) \leq f(X_i(g)) \\ X_i(g), & \text{otherwise} \end{cases} \tag{7}$$

The problem discussed in this paper has characteristic discrete, so it will be used Discrete Differential Evolution algorithm.

## 3.2 Discrete Differential Evolution

The advantages of DE are: simple structure, the speed in finding the solutions and robustness. However, the main obstacle to applying DE in combinatorial problems is due to its continuous nature. Regarding the resolution of combinatorial optimization problems with discrete characteristics, new approaches to discrete DE has been proposed in the literature. Another difficulty would be that in combinatorial optimization problems, differential vectors do not have a clear correspondence with the directions in the search space.

The ability of DE to run with good performance in continuous search spaces is widely documented. The arithmetic reproduction operator used by DE is simple, but the manner in which the operator is defined, makes it virtually impossible to apply the standard DE to other spaces (Pampará *et al*., 2006).

The mutation operator of the original DE leads to the conclusion, which it can only keep closure in the field of real numbers. Another problem is that in the original DE cannot keep the closure, when it is applied in discrete domain. Thus it can not be used in directly discrete optimization problems [Deng *et al*., 2009].
Some papers involving applications of DDE in scheduling problems can be cited, such as: Flowshop Scheduling [Onwubolu e Davendra, 2006], Permutation Flowshop Scheduling [Pan *et al*., 2008], Bi-Criteria No-Wait Flowshop Scheduling Problems [Pan *et al*., 2009], Production Scheduling [Jun *et al*., 2008], Task Scheduling in Heterogeneous Computing Systems [Kang & He, 2009], Resource Planning and Scheduling of Payload for Satellite [Li, 2009], Multi-Objective Flowshop Scheduling [Qian *et al*., 2009] and Blocking Flowshop Scheduling [Wang *et al*., 2010].

## 3.3 Encoding of individuals

A solution of the problem is composed of several types of feasible shifts of the planning period (one week is the planning period adopted here). Thus, each individual (*i*) represents a weekly schedule, each chromosome (*c*) represents a shift, and their genes (*g*) represent the characterization of the shift. Each chromosome has 5 genes. The genes have the following characterization: the first indicates whether the shift is part of the daily schedule (binary encoding); the second represents the initial moment of the shift; the third represents the start of the first break; the fourth represents the beginning of the interval; and the fifth represents the start of the second break. Thus, the first gene is a binary value, the second is a value belonging to the set {1, 2, ..., 432} which represents one of the 432 fragments of the planning period (one week) and the third, fourth and fifth genes are values belonging to the set {1, 2, ..., 38} which represents the 38 fragments of a shift. The generic representation of a chromosome, can be given:

$$C_{ic} = [j_{ic}, it_{ic}, ip1_{ic}, ii_{ic}, ip2_{ic}], \quad i = 1,2,\dots,NI; \quad c = 1,2,\dots NC \tag{8}$$

where

- $C_{ic}$ → *c*-th chromosome of *i*-th individual
- $j_{ic}$ → shift of *c*-th chromosome of *i*-th individual
- $it_{ic}$ → beginning of shift of *c*-th chromosome of *i*-th individual
- $ip1_{ic}$ → beginning of first pause of *c*-th chromosome of *i*-th individual
- $ii_{ic}$ → beginning of interval if *c*-th chromosome of *i*-th individual
- $ip2_{ic}$ → beginning of second pause of *c*-th chromosome of *i*-th individual
- $NI$ → number of individuals
- $NC$ → number of chromosomes

The generic representation of an individual, can be given:

$$I_i = [C_{i1}, C_{i2}, \dots, C_{iNC}], \quad i = 1, \dots, NI \tag{9}$$

where $I_i$ is the *i*-th individual.

A frequently form used (adopted in this paper) to initialize the population (at generation g = 0) is to generate random values for the components of each individual within the limits of variation of the problem.

## 3.4 Computing of Fitness

It is intended to minimize the number of weekly working shifts. Thus, the fitness function adopted is the following:

$$F_i = \sum_{c=1}^{NC} j_{ic}, \quad \forall i \in \{1, 2, \dots, NI\} \tag{10}$$

where $F_i$ is the fitness of the i-th individual.

## 3.5 Mutation

A mutant population of individuals is generated from a population named target population. Three individuals $X_{r1}(g)$, $X_{r2}(g)$ e $X_{r3}(g)$ of the target population are random chosen. For the target individual, $Y_i(g)$, one mutant individual, $V_i(g+1)=[v_{i1}, v_{i2}, ..., v_{iNC}]$, is generated according to:

$$V_i(g+1) = X_{r1}(g) \oplus F \otimes [X_{r2}(g) \odot X_{r3}(g)] \tag{11}$$

where $v_{ic}=[jv_{i1}, itv_{ic}, ip1_{ic}, iiv_{ic}, ip2_{ic}]$.

The symbols $\oplus$, $\otimes$ and $\odot$ replace the traditional addition symbols, subtraction and multiplication, respectively.

The components of the chromosomes have two kinds of variables (integer and binary). First, we define the operations to the integer variables as follows: $\odot$ is the difference between the genes of the chromosome c of the individual $X_{r2}$ and the genes of the chromosome $c$ of the individual $X_{r3}$; $\otimes$ is multiplying the constant $F$ by result of the operation $\odot$, rounding to the integer below; $\oplus$ is adding $X_{r1}$ to the result of the operation $\otimes$.

For the binary operation, referring to the first gene of the chromosome, we have:

$$jv_{ic}(g+1) = j_{r1c}(g) \oplus F \otimes [j_{r2c}(g) \odot j_{r3c}(g)] \tag{12}$$

The operation $\otimes$ keeps the result of the operation $\odot$. Thus, the equation (13) becomes:

$$jv_{ic}(g+1) = j_{r1c}(g) \oplus [j_{r2c}(g) \odot j_{r3c}(g)] \tag{13}$$

The other operations are defined as follows: $\odot$ is the AND binary operator, i.e. returns 1 if both operands are 1, and 0 otherwise; $\oplus$ is the OR binary operator, i.e. returns 1 when at least one of the operands is 1, and 0 otherwise .

## 3.6 Crossover

An individual named trial individual, $U_i(g+1)$, is produced of the genes of the mutant individual, $V_i(g+1)$, and of the genes of the target individual, $Y_i(g)$. The chromosomes of the trial individual are formed according:

$$u_{ic}(g+1) = \begin{cases} v_{ic}(g+1), & if\ R \le CR\ \ or\ \ c = I_{rand} \\ y_{ic}(g), & otherwise \end{cases} \tag{14}$$

where $R \in [0,1[$ and $I_{rand} \in \{1,2, \dots, NC\}$ are random numbers generated.

The selection operation used in this paper is the same as the original version, according to equation (7).

## 3.7 Parameters

Some research shows that fixed values of $F$ and $CR$ can lead to premature convergence and poor performance (Noman & Iba, 2008). To resolve this problem, Zhang *et al*. (2009) describe an adaptive adjustment strategy, according to:

$$F = F_0 \cdot e^{-a_1 \cdot \left(\frac{G}{G_{max}}\right)} \tag{15}$$

$$CR = CR_0 \cdot e^{-a_2 \cdot \left(\frac{G}{G_{max}}\right)} \tag{16}$$

where $G_{max}$ and $G$ are respectively current evolutionary generation and maximum; $F_0$ and $CR_0$ are, respectively, the initial value of scale factor and crossover rate, $a_1$ and $a_2$ are positive constants.

For integer variables, the strategy adopted is similar, changing only the base of the exponential, shown below. Empirically, the base 2 has brought better results than the base $e$.

$$F = F_0 \cdot 2^{-a_1\left(\frac{G}{G_{max}}\right)} \quad (17)$$

$$CR = CR_0 \cdot 2^{-a_2\left(\frac{G}{G_{max}}\right)} \quad (18)$$

## 3.8 Local Search

A proper increase in the randomization is fundamental for obtaining significant improvements in the DE functioning (Neri & Tirronen, 2010). The randomization was used in the DDE so that local search moves were inserted. For this, a extra mutation was introduced applied to genes $ip1v_{ic}$, $iiv_{ic}$ e $ip2v_{ic}$. This mutation is performed randomly, i.e. new genes are determined in the same manner as were generated in the initial population. The factor of this extra mutation called ME, is given:

$$ME = ME_0 \cdot 2^{-a_3\left(\frac{G}{G_{max}}\right)} \quad (19)$$

where $ME_0$, $a_3$ are a constant defined empirically.

The randomization to modify the binary genes has been introduced after the crossover. If the binary gene is 1 applies to rate named $shift_{out}$ that is the probability of gene is replaced by 0 and if the binary gene is 0 applies to rate denominated $shift_{in}$ that is the probability of gene is replaced by 1. The rates are set according to:

$$shift_{out} = shift_{out_0} \cdot 2^{-a_4\left(\frac{G}{G_{max}}\right)} \quad (20)$$

$$shift_{in} = shift_{in_0} \cdot 2^{-a_5\left(\frac{G}{G_{max}}\right)} \quad (21)$$

where $shift_{out_0}$, $shift_{in_0}$, $a_4$ and $a_5$ are constants defined empirically.

## 4. Experiments and Results

The number of calls in the Call Center varies throughout the day, although it shows similar patterns such as peaks in time for lunch or dinner. This affects the demand for attendants in every moment.

We used data of a functioning month of the Call Center. Demand patterns were identified and grouped on weekdays, Saturdays and Sundays. The demands these groups showed similarities, so we opted to use their arithmetic averages obtained from a simulator in accordance with Table 1.

Table 1: data arithmetic averages of a functioning month of the Call Center.

| SUNDAYS | | | SATURDAY | | | WEEKDAY | | |
|---|---|---|---|---|---|---|---|---|
| Time | | Demand | Time | | Demand | Time | | Demand |
| Beginning | End | | Beginning | End | | Beginning | End | |
| 00:00 | 01:00 | 3 | 00:00 | 01:00 | 3 | 00:00 | 01:00 | 2 |
| 01:00 | 02:00 | 2 | 01:00 | 02:00 | 3 | 01:00 | 02:00 | 2 |
| 02:00 | 03:00 | 1 | 02:00 | 03:00 | 1 | 02:00 | 03:00 | 1 |
| 03:00 | 04:00 | 1 | 03:00 | 04:00 | 3 | 03:00 | 04:00 | 1 |
| 04:00 | 05:00 | 1 | 04:00 | 05:00 | 1 | 04:00 | 05:00 | 1 |
| 05:00 | 06:00 | 2 | 05:00 | 06:00 | 1 | 05:00 | 06:00 | 2 |
| 06:00 | 07:00 | 2 | 06:00 | 07:00 | 3 | 06:00 | 07:00 | 3 |
| 07:00 | 08:00 | 3 | 07:00 | 08:00 | 4 | 07:00 | 08:00 | 5 |
| 08:00 | 09:00 | 5 | 08:00 | 09:00 | 10 | 08:00 | 09:00 | 16 |
| 09:00 | 10:00 | 7 | 09:00 | 10:00 | 15 | 09:00 | 10:00 | 25 |
| 10:00 | 11:00 | 8 | 10:00 | 11:00 | 18 | 10:00 | 11:00 | 28 |
| 11:00 | 12:00 | 9 | 11:00 | 12:00 | 17 | 11:00 | 12:00 | 27 |
| 12:00 | 13:00 | 8 | 12:00 | 13:00 | 18 | 12:00 | 13:00 | 22 |
| 13:00 | 14:00 | 8 | 13:00 | 14:00 | 13 | 13:00 | 14:00 | 24 |
| 14:00 | 15:00 | 6 | 14:00 | 15:00 | 12 | 14:00 | 15:00 | 23 |
| 15:00 | 16:00 | 5 | 15:00 | 16:00 | 11 | 15:00 | 16:00 | 22 |
| 16:00 | 17:00 | 5 | 16:00 | 17:00 | 10 | 16:00 | 17:00 | 23 |
| 17:00 | 18:00 | 5 | 17:00 | 18:00 | 11 | 17:00 | 18:00 | 20 |
| 18:00 | 19:00 | 5 | 18:00 | 19:00 | 10 | 18:00 | 19:00 | 13 |
| 19:00 | 20:00 | 5 | 19:00 | 20:00 | 8 | 19:00 | 20:00 | 11 |
| 20:00 | 21:00 | 4 | 20:00 | 21:00 | 7 | 20:00 | 21:00 | 8 |
| 21:00 | 22:00 | 4 | 21:00 | 22:00 | 6 | 21:00 | 22:00 | 6 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 22:00 | 23:00 | 4 | 22:00 | 23:00 | 5 | 22:00 | 23:00 | 5 |
| 23:00 | 00:00 | 2 | 23:00 | 00:00 | 3 | 23:00 | 00:00 | 3 |

For validating the proposed model with application of the DDE is made a comparison their results with the results of a Integer Linear Programming (ILP) model.

For implementation the ILP model was developed a "Generator of Working Shifts" which aims to generate the kinds of working shifts considering time restrictions. For the resolution the model we used the service of optimization resources online NEOS Server (Network Enabled Optimization System). This service is a portal for optimization software and services via Internet and was developed by the Division of Computer Science and Mathematics at Argonne National Laboratory. Among the optimization applications offered by NEOS have been chosen for the resolution the MOSEK.

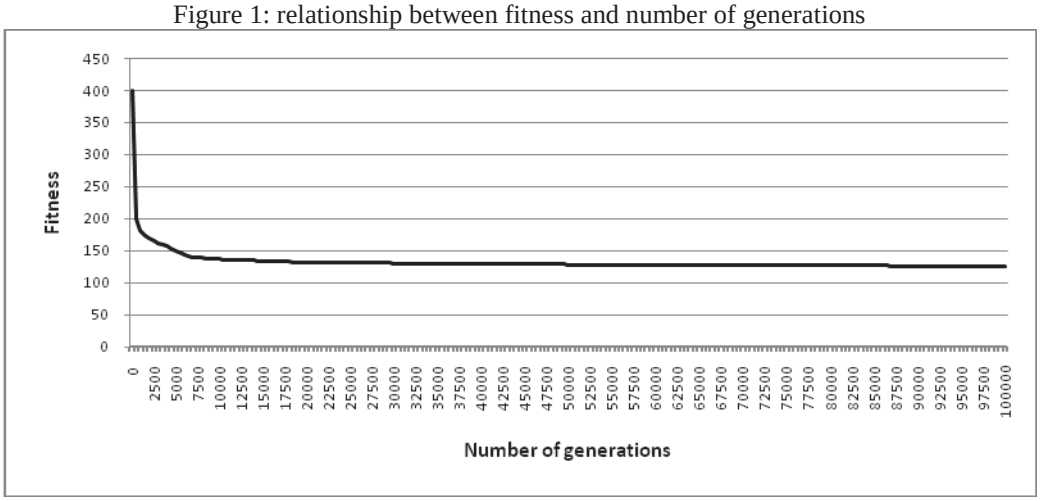
The parameters that showed better results in implementing the DDE were:

- 5 individuals;
- $CR_0 = F_0 = shift_{out_0} = 0{,}1$;
- $ME_0 = shift_{in_0} = 0{,}01$;
- $a_1 = a_2 = a_3 = a_4 = a_5 = 3$.

We adopted (empirically) 400 chromosomes for each individual in the population. For all Binaries genes of the initial population was adopted the value 1. As stopping criterion was adopted 100,000 generations.

The results obtained in solving the problem by ILP model were as follows: 23 working shifts to Sundays, 61 to weekdays and 42 to Saturdays (total of 126 shifts). For the problem were carried out 10 executions of DDE. It was found the value 126 for the objective function in 4 executions. In other 4 executions was found the value 127 and in the others 4 executions was found the value 128. In figure 1 shows the graphic of the relationship between fitness and number of generations of the execution of DDE that obtained the best result. The result equal to 126 for the objective function was found at generation 86,522.

Figure 1: relationship between fitness and number of generations




The DDE was programmed in Object Pascal in Delphi 7 and run on a Pentium IV, 2.8GHz and 1GB of RAM. The average time of the 10 executions was equal to 159.6 seconds.

## 5. Conclusions

In this paper, it was described a procedure to resolve the weekly working shifts problem in particular Call Center. For the determination of working shifts has been proposed a model with application of DDE and the obtained results were compared with of an ILP model. The DDE found optimal result in some executions and found results close to optimal in other executions.

It is concluded that the application of DDE proposed for the working shifts problem of Call Center telephone operators was appropriate as generated satisfactory results in reasonable computational time, since it was found in some tests, the optimal result and in other tests, results close to optimal.

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# Hybrid Genetic Algorithm for solving of Maximum Betweenness Problem

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**Abstract**

In this paper hybrid-genetic algorithm (GA) on Maximum Betweennes Problem (MBP) is applied.Integer represetation preserved feasibility of individuals is used, while allowing application of standard genetic operators. Fast interchange local search procedure additionally improves performances of proposed GA. Experimental results on instances up to 50 elements and 1000 triples from the literature show the effectiveness of presented approach. On smaller instances all known optimal values, except one, are reached. Proposed approach gives better or equal results on all instances compared to previous method.

**KEYWORDS**

Evolutionary approach. Genetic algorithms. Betweenness.

## 1. INTRODUCTION

Let $A$ be a finite set and let $C$ be a collection of triples $(a,b,c)$ of distinct elements from $A$. Let $f:A \rightarrow \mathbf{R}^+$. Let Obj($f$) be a number of betweenness constraints $f(a) < f(b) < f(c)$ or $f(a) > f(b) > f(c)$ satisfied by function $f$.

Now, the maximum betweenness problem (MBP) can be formulated as finding Max(Obj($f$)) over all functions $f:A \rightarrow \{1,...,|A|\}$ which are "1-1". Explicitly, to find

$$\max_f \text{Obj}(f) \quad f \in \{h \mid h: A \rightarrow \{1,\text{K},|A|\}, h \text{ is "1-1"}\}$$

Let us demonstrate this on one little illustrative example.

**Example 1.** Let $A = \{1,2,3,4,5\}$ and let collection $C$ have 5 triples. Let them be (1,3,2), (4,1,5), (5,3,4), (2,1,4), (3,5,1). Then the optimal solution, obtained by total enumeration, is the function $f$ given with $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 5 & 3 \end{pmatrix}$. Optimal value is Obj($f$) = 4, where function $f$ satisfies betweenness constraints in triples 1, 2, 4 and 5.

Problem arose in late seventies when Opatrny (Opatrny 1979) showed that finding $n$ totally ordered elements that satisfies $m$ such betweenness constraints is NP-complete. Furthermore, the problem is MAX SNP complete, and for every $a > 47/48$ finding a total order that satisfies at least $a$ of the $m$ constraints is NP-hard (even if all the constraints are satisfiable) (Chor-Sudan 1998). Complete previous works is out of this paper scope and can be found in (Savić 2009). It is easy to see that lower bound of the objective function must be equal or greater than 1/3 of constraints, because even if we randomly order the elements at least third of betweenness constraints will be satisfied. Most considerations of this problem were purely theoretical (Goerdt 2009, Guttmann-Maucher 2006.). Mixed integer linear programming model could be found in (Savić et al. 2010) and will be used for finding optimal solutions with CPLEX.

## 2. MATHEMATICAL FORMULATION

Here will be presented the MILP formulation of MBP as given in (Savić et al. 2010).

Let $n$ be a number of elements in finite set $A$. Without loss of the generality we can assume $A = \{1,...,n\}$. Let $n_t$ be a number of triples in collection $C$. Let us denote the $i$-th triple from the collection $C$ with $(a_i, b_i, c_i)$, $i = 1, \ldots, n_t$. Also, let $\alpha \in (0,1]$.

Let us define vector of variables $(x,y,z)$.

$$x_j = \frac{f(j)-1}{n}, \quad j = 1,\mathrm{K}\,,n$$

and for every $i = 1,...,n_t$ let define

$$y_i = \begin{cases} 1 & f(a_i) < f(b_i) < f(c_i) \\ 0 & \text{otherwise} \end{cases}, \quad z_i = \begin{cases} 1 & f(a_i) > f(b_i) > f(c_i) \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, the formulation is as follows:

$$\max \sum_{i=1}^{n_t} y_i + z_i$$

subject to

$$x_{a_i} - x_{b_i} + y_i \le 1 - \frac{\alpha}{n}, \quad i = 1,\mathrm{K}\,,n_t$$

$$x_{b_i} - x_{c_i} + y_i \le 1 - \frac{\alpha}{n}, \quad i = 1,\mathrm{K}\,,n_t$$

$$-x_{a_i} + x_{b_i} + z_i \le 1 - \frac{\alpha}{n}, \quad i = 1,\mathrm{K}\,,n_t$$

$$-x_{b_i} + x_{c_i} + z_i \le 1 - \frac{\alpha}{n}, \quad i = 1,\mathrm{K}\,,n_t$$

$$x_j \in [0,1], \quad j = 1,\mathrm{K}\,,n\,, \; y_i, z_i \in [0,1], \quad i = 1,\mathrm{K}\,,n_t$$

## 3. PROPOSED GA METHOD

In the recent times metaheuristics have started to play a major role in solving problems, which decade ago couldn't even be thought to solve, because of the computational time and resources. Now, with metaheuristics researchers can topple even the hardest NP-hard problems. One of more well-known metaheuristics are genetic algorithms (GA). As their name suggests, they have roots in biology and real genetics. GA are emulating nature in capacity that they are stochastic methods for finding solutions for various problems similar to nature's finding answers to environmental problems. As in nature, GA works with individuals which are constituting population. And also, as in nature, GA works to find individuals better suited to cope with problem. These better individuals are allowed to transfer their good qualities to the next generations of individuals. This is done through genetic operators of crossover and mutation. Deciding which individuals will pass their good qualities in GA is done by evaluating fitness function. The betterment of individuals is iteratively repeated until optimum of fitness function is achieved or some other stopping criterion. The field of applications of GA's in recent years has grown fast and can't be covered in this paper's scope. Some descriptions can be found in (Mitchell 1999). Extensive computational experience on various optimization problems shows that GA often produces high quality solutions in a reasonable time. Some of recent applications are:

- hub location (Kratica et al. 2007),
- metric dimension of graphs (Kratica et al. 2009),
- traveling purchase (Goldbarg 2009),
- multiprocessor open shop (Marie 2009),
- order acceptance (Rom-Slotnick 2009)

- airline crew-pairing and rostering (Souai-Teghem 2009).

Encoding of individuals in this paper is integer. Genetic code in this case has $n$ elements. Since $i$-th element of the gene should be between 0 and $n$-$i$-1 it is obtained with division by mod($n$-$i$). If gene has a value $q$ on position $m$ than $m$-th element of permutation will be obtained as ($q$+1)-th element from the line of unused elements. Obtaining $n$-th element of permutation is unique because there is only one element left unused.

For example, let $n$=4, then gene 1-2-0 generates permutation $\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 4 & 1 & 3\end{pmatrix}$. Number 1 in gene notes that the first element of permutation should be the second element from 1-2-3-4 which is 2. Number 2 in gene notes that the second element of permutation should be the third element from 1-3-4 which is 4. Number 0 in gene means that the third element in permutation should be the first element from 1-3 which is 1. Last element is unique, that is the only element which is left over and that is 3. Similarly, gene 3-0-1 generates permutation $\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 1 & 3 & 2\end{pmatrix}$.

Advantage of this method is that all individuals are feasible. Crossover and mutation operators will again give feasible individuals because it will again be permutation. Disadvantage is that with crossover operator in the next generation from two numbers in parent permutations it is not sure that both numbers will figure in descendant permutation.

For example let us have those two genes from the previous example. They are 1-2-0 and 3-0-1. In this paper one point crossover operator is used which means that the first numbers in genes will not be exchanged. As a result of crossover operator from parent genes 1-2-0 and 3-0-1 descendant genes are 1-0-1 and 3-2-0. This implies that from permutations $\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 4 & 1 & 3\end{pmatrix}$ and $\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 1 & 3 & 2\end{pmatrix}$, two descendant permutations are obtained, which are $\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix}$ and $\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 1 & 2\end{pmatrix}$. As can be seen, on the second position descendant permutations have numbers 1 and 3 though their parents have numbers 4 and 1.

The proposed GA was in every step combined with local search procedure to improve value of the best solution found in that step. Local search implemented in this GA is as follows. In permutation p, which is incumbent record, positions $i$ and $j$ are chosen. For permutation $p$ existed list of certain number of triples which satisfied betweenness constraints. From this list we first remove those where $p(i)$ was included. Afterward from list are removed those triples which included $p(j)$. Now, in permutation $p$ positions $i$ and $j$ change places and new permutation $p_1$ is obtained. Now in earlier revised list is expanded from collection C with all those triples that include $i$ and permutation $p_1$ satisfies betweenness constraint. Afterward, list of triples is expanded by all those which include $j$ but do not include $i$, and for which permutation $p_1$ satisfies betweenness constraint. If the new list is greater than the old one, it means that local search has found better permutation and $p_1$ is now new incumbent record, otherwise some other pair ($i$, $j$) is chosen.

Local search is executed until first improvement when new incumbent record is obtained. This procedure repeats until there are no new improvements. Initialization of GA is random. This guaranties most heterogeneous and diversified genetic pool of population in the first generation.

The very important parts of proposed GA are: crossover and mutation, selection and fitness function. In the text that follows their function and utilization will be shortly explained.

Fitness function $f_{ind}$ is responsible for decision which individual is fit to continue in evolution and which is not. Its values are computed by scaling objective values $obj_{ind}$ of all individuals into the interval [0,1], so that the best suitable individual $ind_{\max}$ gets the value 1 and the worst $ind_{\min}$ gets 0.

After obtaining values of fitness function, two cases arise which must be considered. First, number of individuals with same objective function but different genetic code must be limited by some constant $N_{rv}$. This is done so that other genetic material with different potential can stay in genetic pool. To obtain this, fitness of all individuals with the same value of objective function but different genetic material will be set to 0 except the first $N_{rv}$ of them. If for example $N_{\text{pop}}$ = 150 than adequate maximum of individuals with same fitness, is $N_{rv}$ = 40. In order to avoid premature convergence and to provide the maximal diversity of genetic

material duplicated individuals are removing from population. Setting their fitness value to zero for all individuals, except the first one, they lose the opportunity to appear in the next generation.

In population, as in nature, we differentiate elite and non-elite individuals. Their treatment with GA is accordingly also different. All elite individuals, their number being $N_{elite}$, are automatically passed to the next generation. All those non-elite individuals, their number being $N_{nnel} = N_{pop}-N_{elite}$, are subject to genetic operators. This differentiation reduces computational time, because evaluation of objective function in elite individuals do not change from generation to generation and need to be calculated only once, in the first generation.

Selection of non-elite individuals is done through tournaments. Purpose of tournament is to allow its winner to pass to the next generation. Winner of the tournament is individual with the highest value of objective function. To ensure that population do not shrink in size, number of tournaments is equal to the number of non-elite individuals $N_{nnel}$, so that exactly $N_{nnel}$ of parents can be chosen for crossover. It is allowed that same individual from the current generation can participate in more than one tournament.

Implementation of selection is improved with tournament selection operator, Fine-grained tournament selection - FGTS (Filipović 2003). Here, the tournament size is a real parameter $F_{tour}$, which represents preferable average tournament size. FGTS performs best with value of $F_{tour}$ set on 5.4, as can be seen in (Filipović 2003). The same value is used in this work because numerical and statistical data presented here are exceptional. For detailed information about FGTS see (Filipović 2003).

Adequate ratio between number of elite and number of non-elite individuals is necessary to achieve the most satisfactory results of GA. For example, for $N_{pop} = 150$ adequate proportion is $N_{elite} = 100$ and $N_{nnel} = 50$.

After selection of non-elite individuals, these individuals can be processed by crossover operator. They are now randomly paired in $\lfloor N_{nnel} / 2 \rfloor$ pairs. Intention of GA algorithm through its crossover operator is that two selected individuals exchange genes and produce offspring which will have improved qualities and especially fitness function, than parent individuals. In every crossover two offspring are produced which will replace their parents in the next generation. In this paper standard one point crossover operator is used. Probability of realization of crossover operator is 85%. This means, that approximately 85% pairs of individuals will exchange genes.

For purposes of this paper in genetic algorithm simple mutation operator is used. Direct application of this operator changes randomly selected gene with some mutation rate. The operator is improved with modification for dealing with problems of frozen genes. Frozen genes are those genes that are on certain position in all individuals in population. Existence of frozen genes reduce search space and increase the possibility of premature convergence. For example, if there is $q$ frozen genes in population, then search space will be $q!$ times smaller. Modification of simple mutation operator in GA is that mutation rate is increased only on frozen genes. For proposed GA in this paper increased mutation rate for frozen genes is 2.5 greater than mutation rate on unfrozen genes.

Finally, for improved performance of proposed GA caching technique is implemented. The main idea is to avoid evaluation of objective function for individuals with the same genetic code. The values of objective function that are already computed, are stored by the least recently used (LRU) caching technique into the hash-queue data structure. Now, if new individual has same genetic code as some old one, value of his objective function is not computed, but found in cache memory. This results in significant time savings. Limitation for already calculated values of objective function in this implementation is 5000. If cache memory is full then we remove least recently used cash memory block. Detailed information about caching GA can be found in (Kratica 1999).

## 4. EXPERIMENTAL RESULTS

All computations were executed on 2.0 GHz Double DuoCore MacPro computer with 3 Gb RAM under Windows. Genetic algorithm was coded in C programming language. For experimental testing instances from literature (Savić 2009) were used. Names of instances reflects dimension of problem, for example mbp-11-100 indicates that set $A$ has 11 elements and that collection $C$ has 100 triples of elements from set $A$. These instances include different number of elements in set $A$ ($N$ = 10, 11, 12, 15, 20, 30, 50) and different number of triples in $C$ (ranging from 20 to 1000). In order to check GA, all instances were solved by CPLEX solver applied on mathematical formulation given in Section 2, with parameter $\alpha$=0.01.

The finishing criterion of GA is the maximal number of generations $N_{gen}$ = 1000. The algorithm also stops if the best individual or the best objective value remains unchanged through $N_{rep}$ = 500 successive generations. Since the results of GA are nondeterministic, the GA was run 20 times on each of the instances.

Table 1 summarizes the GA results on these instances. In the first column are given names of the instances. In the second column are given total times for GA to find solution. Average times needed to detect the best GA values are given in *t* column. The solution quality in all 20 executions is evaluated as a percentage gap named *agap*, with respect to the optimal solution $sol_{opt}$}, with standard deviation σ of the average gap. A percentage gap *agap* is defined as $agap = \frac{1}{20}\sum_{i=1}^{20} gap_i$ , where $gap_i = 100\frac{GA_i - GA_{best}}{GA_{best}}$ and $GA_i$ represents the GA solution obtained in the *i*-th run, while o is the standard deviation of $gap_i$, $i$ = 1,2,...,20, obtained by formula $\sigma = \sqrt{\frac{1}{20}\sum_{i=1}^{20}(gap_i - agap)^2}$ . The next two columns are related to the caching: *eval* represents the average number of evaluations, while *cache* displays savings (in percent), achieved by using caching technique. The best GA values $GA_{sol}$ are given in the next column. In the last column mark *opt* is given if optimal solution is reached and there is no difference between that solution and optimal solution obtained by CPLEX.

Table 1. GA results on MBP

| Instance name | $t_{tot}$ | *T* (sec) | *agap* | σ | *eval* | *Cache* (%) | $GA_{sol}$ | *opt* |
|---|---|---|---|---|---|---|---|---|
| mbp-10-20 | 0.088 | 0.003 | 1.250 | 2.565 | 10260.9 | 61.7 | 16 | opt |
| mbp-10-50 | 0.090 | 0.004 | 0.000 | 0.000 | 10442.5 | 60.5 | 29 | opt |
| mbp-10-100 | 0.116 | 0.017 | 2.500 | 1.100 | 11969.4 | 60.2 | 50 | opt |
| mbp-11-20 | 0.089 | 0.002 | 2.500 | 3.495 | 12022.4 | 54.1 | 14 | opt |
| mbp-11-50 | 0.095 | 0.006 | 2.273 | 1.346 | 11302.5 | 58.5 | 33 | opt |
| mbp-11-100 | 0.115 | 0.018 | 2.636 | 2.669 | 12031.8 | 59.9 | 55 | opt |
| mbp-12-20 | 0.090 | 0.005 | 1.765 | 2.766 | 11007.1 | 59.0 | 17 | opt |
| mbp-12-50 | 0.104 | 0.013 | 3.030 | 1.390 | 10978.8 | 62.6 | 33 | opt |
| mbp-12-100 | 0.119 | 0.019 | 2.946 | 2.970 | 11358.7 | 62.2 | 56 | |
| mbp-15-30 | 0.101 | 0.011 | 8.400 | 4.285 | 13389.0 | 53.0 | 25 | |
| mbp-15-70 | 0.110 | 0.009 | 4.022 | 2.368 | 12953.4 | 54.0 | 46 | |
| mbp-15-200 | 0.149 | 0.021 | 2.048 | 2.458 | 13884.8 | 52.9 | 105 | |
| mbp-20-40 | 0.171 | 0.043 | 11.351 | 6.360 | 21927.5 | 36.5 | 37 | opt |
| mbp-20-100 | 0.268 | 0.104 | 4.697 | 2.597 | 27135.8 | 37.8 | 66 | |
| mbp-20-200 | 0.225 | 0.051 | 1.842 | 0.940 | 20707.1 | 36.9 | 114 | |
| mbp-30-60 | 0.341 | 0.136 | 8.113 | 4.544 | 28692.8 | 31.9 | 53 | |
| mbp-30-150 | 0.598 | 0.251 | 11.261 | 3.970 | 34560.8 | 33.2 | 111 | |
| mbp-30-300 | 1.002 | 0.449 | 6.313 | 3.176 | 37397.3 | 33.8 | 179 | |
| mbp-50-100 | 1.163 | 0.603 | 5.523 | 4.249 | 47340.6 | 19.6 | 86 | |
| mbp-50-200 | 3.837 | 2.371 | 4.801 | 4.019 | 62088.7 | 20.3 | 151 | |
| mbp-50-400 | 7.800 | 5.019 | 6.415 | 4.413 | 63049.0 | 20.6 | 265 | |
| mbp-50-1000 | 19.854 | 12.269 | 3.355 | 2.630 | 62085.2 | 21.1 | 532 | |

As can be seen in Tables 1, total running time of GA on all instances is really small. Average execution on the largest instance is smaller than 12.5 seconds. As can be seen GA has reached almost all optimal solutions for smaller instances (*N* = 10, 11, 12). For greater *N* optimal solutions could not been found with CPLEX so GA results could not be verified for optimality. For some instances optimal solution was obtained from (Savić 2009).

In Table 2 are given comparative results of CPLEX, best results of GA without local search given in (Savić 2009) and finally best results of GA with local search. With best results are respectively given their running times. For experimental results was used CPLEX 10.1 on the same computer as was GA. Where running time of CPLEX exceeded 7200 seconds solving, process was stopped. In cases where running time was smaller than 7200 seconds, it was assumed that optimal solution was reached.

Table 2. Comparative results and running times of CPLEX, GA without LS and GA with LS on MBP

| Instance name | CPLEX | *T* (sec) | $GA_{sol}$ without LS | *T* (sec) | $GA_{sol}$ with LS | *T* (sec) |
|---|---|---|---|---|---|---|
| mbp-10-20 | 16 | 0.437 | 16 | 0.194 | 16 | 0.088 |
| mbp-10-50 | 29 | 7203.843 | 29 | 0.195 | 29 | 0.090 |
| mbp-10-100 | 42 | 7201.594 | 50 | 0.652 | 50 | 0.116 |
| mbp-11-20 | 14 | 2.125 | 14 | 0.200 | 14 | 0.089 |
| mbp-11-50 | 33 | 7203.625 | 33 | 0.214 | 33 | 0.095 |
| mbp-11-100 | 55 | 7201.656 | 55 | 0.243 | 55 | 0.115 |
| mbp-12-20 | 17 | 1.156 | 17 | 0.197 | 17 | 0.090 |
| mbp-12-50 | 32 | 7203.781 | 33 | 0.228 | 33 | 0.104 |
| mbp-12-100 | 54 | 7202.157 | 56 | 0.246 | 56 | 0.119 |
| mbp-15-30 | 26 | 3.172 | 25 | 0.217 | 25 | 0.101 |
| mbp-15-70 | 45 | 7202.797 | 46 | 0.231 | 46 | 0.110 |
| mbp-15-200 | 98 | 7201.375 | 105 | 0.289 | 105 | 0.149 |
| mbp-20-40 | 37 | 1.625 | 36 | 0.340 | 37 | 0.171 |
| mbp-20-100 | 63 | 7201.812 | 65 | 0.398 | 66 | 0.268 |
| mbp-20-200 | 111 | 7201.109 | 113 | 0.400 | 114 | 0.225 |
| mbp-30-60 | 54 | 7201.844 | 51 | 0.538 | 53 | 0.341 |
| mbp-30-150 | 105 | 7200.781 | 102 | 0.627 | 111 | 0.598 |
| mbp-30-300 | 165 | 7200.547 | 173 | 0.749 | 179 | 1.002 |
| mbp-50-100 | 84 | 7200.922 | 84 | 1.147 | 86 | 1.163 |
| mbp-50-200 | 154 | 7200.391 | 140 | 1.385 | 151 | 3.837 |
| mbp-50-400 | 225 | 7200.343 | 240 | 1.535 | 265 | 7.800 |
| mbp-50-1000 | 420 | 7200.219 | 504 | 2.169 | 532 | 19.854 |

As can be seen in Table 2 GA with local search has obtained all optimal solutions except one (instance name mbp-15-30). Also in all other instances except one (instance name mbp-50-200) GA with LS found better results than both GA without LS and CPLEX. Running times are also better than those of GA without LS except 3 largest instances though it can be due to different testing computers. Also as can be seen in all greater instances (N =30, 50) CPLEX did not produce better results though running times was 2 hours compared to GA with LS which was in the worst case smaller than 20 seconds. From this it can be concluded that hybridization with local search was justified.

## 5. CONCLUSION

In this paper the GA metaheuristic combined with local search for solving MBP is presented. The integer representation of the necessary permutations was used. Fine-grained tournament selection refinement was applied in selection process. One-point crossover and simple mutation with frozen genes were used. Computational performance of GA was improved by caching. Adequate and very efficient local search adjusted for the MBP is implemented. For almost all instances, except one, GA calculates solutions that match optimal ones obtained through literature or by CPLEX solver. GA results were obtained in a very small running time.

Based on the results, hybridization of GA with local search was justified and it has the potential to be useful metaheuristic for solving other similar problems. Parallelization of the GA and hybridization with other heuristics and/or exact methods are also promising directions of future work.

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# An Efficient Hybrid Algorithm for Locating Long-Term Care Facilities

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**Abstract**

This paper considers the problem of determining locations for long-term care facilities with the objective of balancing the numbers of patients assigned to the facilities. We propose an efficient hybrid approach (HA) based on the combination of an evolutionary algorithm (EA) and a variant of variable neighborhood search method (VNS) for solving this problem. For evaluation of the suggested hybrid algorithm, computational experiments are performed on test data sets with up to 80 potential facility locations. Presented computational results show that the proposed HA quickly reaches all optimal solutions from the literature, and also produces solutions on problem instances that were unsolved up to now.[1]

## 1 Introduction

The use of Operational Research (OR) in health-care has developed significantly in the past decade, from the aspect of the number of OR applications in this area but also of the number and scope of topics covered. Numerous OR problems in health-care systems are related to the design and efficiency of emergency medical services. Facility location models have been widely applied in real OR problems which include the sitting and response of emergency services, such as medical service, police, fire stations. There are many research articles on facility location problems dealing with the design and optimization of health-care systems and health-care service delivery. The most common objectives in these problems are to minimize the maximal waiting time of a customer, minimize the number of facilities or emergency vehicles, determine the locations of emergency service facilities required to satisfy the demand for the service [5], [7], [18]. In some emergency service delivery problems, one or more service level objective is optimized [14]. Various models capture the stochastic nature of emergency services related to demand or travel time [1], [19], [13]. A detailed review of facility location modeling applications in emergency response services may be found in papers [17], [12] and [10].

In this paper we consider one facility location problem which arises from designing and optimizing long-term health-care systems, named the Long-Term Care Facility Location Problem (LTCFLP). The set of potential facility locations $J$ is given and no facility is previously established. The set of potential facility sites coincides with the set of patient groups, which means that a facility may be located at one of the locations of the patient groups. The elements of the set $J$ will be referred as "nodes". The objective is to locate certain number of long-term facility sites, in order to minimize the maximum load of established facilities. We assume that all potential facilities have the same capacity, that is, the numbers of sickbeds in the facilities

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are the same. We specify locations and demand quantities (the numbers of patients) for each patient group. The problem involves single allocation scheme, which means that each patient group is assigned to exactly one, previously established facility. All patients in each patient group are to be served by a facility located nearest to the location of the patient group. The maximum number $K$ of facilities to be established is pre-determined, and inequality $K \leq |J|$ holds. No capacity restrictions on established facilities are assumed. Fixed costs for locating facility sites are not considered in this model.

The Long-Term Care Facility Location Problem was introduced by Kim et al. in [9]. The authors presented a mathematical model for the LTCFLP and develop a heuristic algorithm, by adopting and modifying the add, drop and interchange methods and by applying them sequentially. The solution generated by heuristic is used as an upper bound in the branch and bound method (BnB). The proposed algorithms are tested on an instance with $|J| = 33$ nodes that is obtained from practice, a number of problem instances with up to $|J| = 70$ nodes and the obtained solutions are additionally compared with the solutions of the CPLEX solver. Results of the experiments show that the suggested BnB algorithm gives optimal solutions on problem instances with up to $|J| = 40$ nodes. For most of the instances with $50 - 70$ nodes, the proposed heuristic produced solutions with significant gaps form the optimal or best known solutions.

The LTCFLP was further considered by Stanojević et al. in [16]. The authors propose new mathematical formulations for the LTCFLP and compare their effectiveness with existing formulation of the LTCFLP from [9]. All formulations are subject to broad computational experiments by using CPLEX solver on newly dataset of with up to 80 facilities, including two types of facility demands. The authors also present for the first time 51 optimal solutions for considered problem instances with up to 80 potential facility locations. The results of exhaustive computational tests show that new formulations of the LTCFLP outperform the one from [9] in the sense of providing optimal solutions and shorter CPU times for considered dataset.

In this paper we propose a hybrid algorithm (HA) based on implementing an evolutionary approach (EA) in the Variable Neighborhood Search (VNS) frame for solving the LTCFLP. In the evolutionary part of the algorithm we apply the appropriate operators and evolutionary strategies which showed to be successful for solving similar discrete location problems. The solution obtained by EA is used as initial solution for the VNS algorithm. We adopt the VNS concept and fine tune the its parameters to the problem under consideration. Conducted computational experiments clearly indicate that this hybridization showed to be successful for solving the LTCFLP in the sense of both solution quality and the CPU time.

## 2 Mathematical formulation

In this paper, we use the integer formulation presented in [16]. Let $J$ be the set of candidate facility locations. It means that We define $d_{ij}$ as the distance between the location of patient group $i$ and candidate facility location $j$, while $a_i$ represents the number of patients in patient group $i$. An integer number $K > 0$ denotes the maximum number of facilities to be located, and $M > 0$ is a large constant. We introduce decision binary variables $x_{ij} \in \{0, 1\}$ that take the value of 1 if patient group $i$ is allocated to a facility location $j$ and 0 otherwise. Since each patient group is assigned to its nearest established facility, variables $x_{kk}$ are equal to 1, if facility is located at node $k$, 0 otherwise. In this way, it is ensured that patient group in a node where a facility is located is assigned to that particular node (distance is equal to zero). Non-negative variable $L_{max}$ represents the maximum load of an established long-term care facility site.

Using the notation mentioned above, the problem can be written in the following way:

$$min \;\; L_{max} \tag{1}$$

subject to:

$$\sum_{j \in J} x_{ij} = 1 \quad for\ every\ i \in J \tag{2}$$

$$x_{ij} \leq x_{jj} \quad for\ every\ i, j \in J \tag{3}$$

$$\sum_{l \in J} d_{il} x_{il} \leq d_{ij} + M(1 - x_{jj}) \quad for\ every\ i, j \in J \tag{4}$$

$$\sum_{j \in J} x_{jj} \leq K \tag{5}$$

$$\sum_{i \in J} a_i x_{ij} \leq L_{max} \quad for\ every\ j \in J \tag{6}$$

$$x_{ij} \in \{0, 1\} \quad for\ every\ i, j \in J \tag{7}$$

The objective function (1) minimizes the maximum load of established facilities for load balancing. The constraint (2) guarantees that each patient group is allocated to exactly one facility location. The constraint (3) restricts such assignments to be made only to facilities that are already established. Each patient group is assigned to its nearest facility, which is ensured by constraint (4). Constraint (5) guarantees that the total number of established facilities does not exceed a predetermined upper limit $K$. Constraint (6) denotes the lower bound for the objective variable $L_{max}$, which represents the maximum load of established facilities for load balancing. Finally, (7) indicate binary nature of variables $x_{ij}$ respectively.

## 3 A hybrid algorithm for the LTCFLP

For many combinatorial optimization tasks, it has been shown that it is essential to incorporate some improvement method into VNS concept to yield effective optimization tools, see [2], [11], [15], etc. Hybrid VNS-based algorithms usually represent the incorporation of an improvement or repair heuristic into traditional VNS concept, such as: tabu search, greedy heuristic, local search, enumeration method, etc. For the successful application of VNS method, especially for real-size problem instances, it is important to obtain a good quality initial solution solution quickly. In this study we incorporate an evolutionary algorithm into a VNS approach in order to obtain better initial solution that will enable the VNS to solve the LTCFLP efficiently, especially large problem dimensions. The choice of evolutionary heuristic was motivated by obvious potential of evolutionary-based methods that were previously applied for solving other discrete location problems, see [3], [4], [8], etc.

In the initialization part of the hybrid approach, encoding of solutions, the distance measure and the neighborhood structures are defined. The proposed algorithm uses the binary representation of solutions, i.e. each solution is represented by the binary string of length $n = |J|$. Each bit in the code corresponds to one node in the network: bit 1 in the 's code denotes that facility is located at that particular node, while 0 indicates it is not. Since patient groups can be assigned only to established facilities, only the array $x_{kk}$ is obtained from the solution's code

and the values of $x_{ik}, i \neq k$ are determined by comparing corresponding distances $d_{ik}$. The objective value is simply evaluated by comparing loads of established facilities and determining the maximal one.

We say that the distance between two solutions $S$ and $S'$ from the search space $X$ is is equal to $k$ if and only if they differ in $k$ locations. The metric function can be defined as $\varrho(S, S') = |S \backslash S'| = |S'/S|, \forall S, S' \in X$, [2]. We denote with neighborhood $N_k(S)$ the set of solutions such that exactly $k$ locations of facilities from the current solution $S$ are replaced by $k$ others. More precisely, $S' \in N_k(S) \Leftrightarrow \varrho(S, S') = k$. The parameters of stopping criteria for the EA and VNS are set in the initialization stage.

In the evolutionary part of the HA, initial population $P$ of solutions is randomly generated, which ensures good diversibility of the genetic material. For each solution in the population $P$, the objective function is calculated. Evolutionary operators selection, crossover and mutation are iteratively applied until the certain stopping criterion is satisfied. In this implementation, the EA stops when the maximal number $G_{max} = 100$ of EA generations is reached. The best $S$ solution obtained by the EA phase represents the initial solution for the VNS method.

After the evolutionary phase of the HA method, we employ a variant of Variable Neighborhood Search heuristic [2]. We start with a random solution $S'$ from $N_1(x)$. In the shaking procedure exactly $k$ facility locations are exchanged: new $k$ locations to be included are chosen randomly, while $k$ established locations are excluded in such a way that objective function has the best possible value. The next step is to perform a local search in the whole solution space with $S'$ as a starting point in order to obtain an improvement $S''$, if possible. If the obtained local optimum $S''$ is better than the current optimum, we move to the solution $S''$ and we set $k \rightarrow 1$ again. If the new solution is not better than the incumbent, we set $k = k+1$, i.e. we look for improvement by exchanging $k+1$ locations. The VNS steps are iteratively applied until we reach the maximal number of $VNS_{max} = 1000$ iterations.

The basic concept of hybrid evolutionary algorithm HA for solving the LTCFLP is shown below:

```
Hybrid algorithm HA
{
Initialization :
     Define the representation of solutions;
     Neighborhood structures N_k, i = 1, ..., M are induced by distance ϱ;
     Choose the EA and VNS stopping conditions;
Evolutionary part EA for finding a good initial solution S for the VNS part :
    Generate the initial population P;
    Repeat
     {
       For each solution X ∈ P do Objective function(X);
       Selection;
       Crossover;
       Mutation;
      }
     Until EA Stopping criterion
Let p be the number of established facilities in initial solution S;
Let M = min(p, n − p), M ≤ K, p ≤ K;
VNS part of the hybrid algorithm :
  Repeat
```

```
    {
     k=1;
     Repeat
      {
        Shaking : Generate a point S' from the k − th neighborhood of S
        S' ∈ Nk (S) such that exactly k facilities are changed
        Local search : Apply local search with S' as initial solution;
        denote with S'' the so obtained local optimum
        Move or not : If this local optimum is better than the incumbent,
        move there S ← S'' and continue the search in neighborhood N1 (1 ← k),
        otherwise set k ← k + 1
      }
     Until k==M;
     }
  Until VNS Stopping criterion
}
```

### 3.1 Evolutionary part of the HA

Initial EA population, numbering 150 individuals, is randomly generated. In order to provide better quality of the genetic material in the initial population, the probability of generating ones in the initial genetic codes is set to $p = \frac{K}{n}$. Steady-state generation replacement scheme with elitist strategy is used. Only 50 individuals are subject to evolutionary operators and replaced in every EA generation, while the best 100 (elite) individuals are directly passing in the next generation preserving highly fitted genes.

The EA uses the fine grained tournament selection [6], which depends on a real parameter *avgtour* = desired average tournament size. In our implementation, *avgtour* is set to 6.4. The standard one-point crossover is implemented, which exchanges segments of two parent's genetic codes after the randomly chosen crossover point and produces two offspring. The crossover is performed with the rate probability *crossrate* = 0.85. Offspring generated by the crossover operator are subject to mutation with frozen bits [3]. Mutation operator is performed by changing a randomly selected gene in the genetic code (0 to 1, 1 to 0), with basic mutation rate of 0.4/n for non-frozen bits and 1.6/n. for frozen bits. In this way, we provide more diversity of the genetic material and keep the EA away from local optimum trap.

Additional strategies are used in EA part in order to improve its efficiency and prevent the premature convergence. Duplicate individuals are discarded from the population by setting their fitness values to zero, so that selection operator unables them to enter the next generation. Since individuals with the same objective function but different genetic codes may become dominant in the population, we limit their appearance to some constant. Incorrect individuals (the ones with $M > K$ ones in the genetic code) may appear in the population and cause the premature convergence of the EA. These individuals are being corrected by changing $M - K$ ones to zeros from the end of genetic code.

## 4 Computational results

In this section the computational results of HA and comparisons with existing algorithms are presented. All experiments were carried out on an Intel Core i7-860 2.8 GHz with 8GB RAM

Table 1: Results and comparisons of the HA on small-size instance $|J| = 33$

| $\|J\| - K$ | $Opt.Sol$ | $CPU_t$ | $HA_{best}$ | $CPU_{end}(t)$ | $CPU_{start}(t)$ | $agap(\%)$ | $\sigma(\%)$ | $BnB_t$ | $Heur_{best}$ | $Heur_{gap}$ | $Heur_t$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 33-4 | 3610 | 61.56 | opt | 1.61 | 0.56 | 0.0 | 0.0 | 1.14 | 3646 | 1.0 | 0.05 |
| 33-8 | 1993 | 1138.12 | opt | 3.28 | 0.53 | 0.0 | 0.0 | 101.69 | 2001 | 0.4 | 0.09 |
| 33-12 | 1440 | 424.36 | opt | 5.11 | 0.58 | 0.0 | 0.0 | 171.92 | 1440 | 0.0 | 0.17 |
| 33-16 | 1079 | 19.93 | opt | 7.53 | 0.5 | 0.0 | 0.0 | 68.45 | 1127 | 4.4 | 0.27 |
| 33-20 | 1079 | 2.35 | opt | 4.17 | 0.43 | 0.0 | 0.0 | 179.14 | 1079 | 0.0 | 0.25 |
| 33-24 | 1079 | 1.45 | opt | 1.91 | 0.42 | 0.0 | 0.0 | 76.69 | 1079 | 0.0 | 0.30 |
| 33-28 | 1079 | 0.76 | opt | 1.14 | 0.4 | 0.0 | 0.0 | 6.45 | 1079 | 0.0 | 0.25 |

memory under Windows 7 Professional operating system. The HA implementation is coded in $C\sharp$ programming language.

Computational experiments were first performed on a small-size problem instance with $|J| = 33$ nodes, which was derived from real situation in Korea and introduced in [9]. The locations are given by their $(x, y)$ coordinates in the plane, together with forecasted number of patients. The maximal number of facilities to be located - $K$ takes seven values: $(4, 8, 12, 16, 20, 24, 28)$. The results of our HA implementation, BnB method and heuristic approach from [9], together with corresponding total CPU times on a Pentium 3.2 GHz processor are presented in Table 1.

Column headings through Table 1 mean: Instance's parameters: number of nodes $n = |J|$ and $K$; Optimal solution of current instance - $Opt.Sol$ obtained by CPLEX 10.1 solver, taken from [16]; Total CPLEX 10.1 running time in seconds - $CPU_t$; The best value of HA method on the current instance - $HA_{best}$, with mark *opt* in cases when it reached optimal solution; Running time in which the HA reaches $HA_{best}$ for the first time - $CPU_{start}$ in seconds; Total running time of the HA - $CPU_{end}$ in seconds; Average percentage gap - $agap$ of $HA_{best}$ solution from the $OptSol$; Standard deviation - $\sigma$ of $HA_{best}$ solution from the $OptSol$; The total running time of the BnB method - $BnB_t$ in seconds; The best solution of heuristic method $Heur_{best}$; The percentage gap of the solution of heuristic from the optimal one $Heur_{gap}$; The total running time of heuristic - $Heur_t$ in seconds.

As it can be seen from the Table 1, the proposed HA quickly reaches all optimal solutions in maximal 7.53 s of total CPU time. The HA detects optimal solution for the first time in up to 0.4s. These running times are significantly shorter compared to the times of CPLEX 10.1 solver, which was run the same machine as the HA. Based on SPEC fp2006 and fp2000 benchmarks (www.spec.org), the Pentium 3.2 GHz configuration (used for BnB and Heur testings in [9]) has around 1.5 slower performance compared to the one used in this paper. This coefficient was experimentally confirmed through series of additional CPU tests. If we take this fact into account, we conclude that the HA outperforms the BnB method from [9] in the sense of total CPU times. Regarding solution's quality, the HA has obviously better performance compared to the Modified Add-Drop-Interchange heuristic algorithm (Heur). The proposed heuristic doesn't achieve optimal solutions for $K = 4, 8, 16$ and produces average gap of 0.829%.

More detailed comparison of the performance of all three methods could be carried out if we had more available test instances. In the paper [9], the authors also performed experiments on problem instances that were generated randomly by varying the values of $|J|$ and K. They generated test problems with up to $|J| = 70$ nodes and included five levels of K for each problem size. Unfortunately, although we had contacted the authors, these problem instances remained unavailable to us.

In our study, we use modified medium and larger AP instances with $n = 50, 60, 70, 80$ nodes and assume that each node represents potential long-term health facility location or a patient group. These instances are introduced and used in [16] for the first time in the literature. The demand types are denoted as L and T respectively. The values of $K$ were set to integers $20 \leq k \leq |J| - 10$. For detailed description of these instances we refer to [16].

Table 2: Results of the HA on AP instances ($n = 50, 60, 70, 80$)

| Inst | $Opt.Sol$ | $CPU(t)$ | $HA_{best}$ | $CPU_{end}(t)$ | $CPU_{start}(t)$ | $agap(\%)$ | $\sigma(\%)$ |
|---|---|---|---|---|---|---|---|
| 50L-20 | 6084.884 | 44233.39 | opt | 0.28 | 0.18 | 1.658 | 1.897 |
| 50L-30 | 4465.101 | 5.38 | opt | 0.25 | 0.14 | 5.136 | 4.253 |
| 50L-40 | 3495.398 | 3.62 | opt | 0.15 | 0.04 | 0 | 0 |
| 50T-20 | 2292.387 | 968.71 | opt | 0.27 | 0.17 | 5.933 | 3.159 |
| 50T-30 | 1762.632 | 6.62 | opt | 0.27 | 0.1 | 5.867 | 7.051 |
| 50T-40 | 1596.897 | 3.92 | opt | 0.13 | 0.01 | 0 | 0 |
| 60L-50 | 4628.483 | 99.62 | opt | 17.35 | 0.08 | 0.190 | 0.828 |
| 60L-40 | 5533.812 | 111.26 | opt | 45.55 | 7.34 | 0.178 | 0.776 |
| 60L-30 | 6635.676* | t.l. | 6635.676 | 87.06 | 4.81 | 0 | 0 |
| 60L-20 | 9096.214* | t.l. | 9096.214 | 60.87 | 17.05 | 0.123 | 0.405 |
| 60T-50 | 2757.519 | 87.49 | opt | 12.34 | 0.1 | 0.160 | 0.481 |
| 60T-40 | 3366.655 | 109.67 | opt | 41.28 | 8.46 | 0.292 | 0.897 |
| 60T-30 | 4074.371* | t.l. | 4074.371 | 77.38 | 5.1 | 0.063 | 0.274 |
| 60T-20 | 5564.090* | t.l. | 5564.090 | 61.64 | 22.43 | 0.315 | 0.095 |
| 70L-20 | - | - | 10542.096 | 96.82 | 33.5 | 0.674 | 0.601 |
| 70L-30 | 7893.375 * | t.l. | 7893.375 | 157.37 | 22.72 | 0.190 | 0.830 |
| 70L-40 | 6322.830 * | t.l. | 6322.830 | 125.66 | 4.72 | 0.528 | 1.26 |
| 70L-50 | 5473.930 | 647.57 | opt | 94.82 | 5.13 | 0.593 | 1.114 |
| 70L-60 | 4712.006 | 597.72 | opt | 20.75 | 0.08 | 0 | 0 |
| 70T-20 | - | - | 6232.056 | 202.59 | 20.69 | 0.528 | 2.679 |
| 70T-30 | 4418.133 * | t.l. | 4418.133 | 154.11 | 44.98 | 0.238 | 0.566 |
| 70T-40 | 3699.205 * | t.l. | 3699.204 | 143.97 | 14.66 | 0 | 0 |
| 70T-50 | 3248.769 | 625.13 | opt | 43.94 | 1.5 | 0.200 | 0.670 |
| 70T-60 | 2833.775 | 422.1 | opt | 19.82 | 0.53 | 0.092 | 0.320 |
| 80L-20 | - | - | 11810.811 | 249.44 | 57.31 | 2.943 | 0.683 |
| 80L-30 | 8579.026* | 86402.85 | 8579.026 | 242.08 | 70.26 | 0.705 | 0.834 |
| 80L-40 | 6669.918 * | t.l. | 6669.918 | 306.11 | 45.37 | 0.011 | 0.005 |
| 80L-50 | 5842.069 * | t.l. | 5842.069 | 215.12 | 1.08 | 0.351 | 1.130 |
| 80L-60 | 5225.643 | 1164.00 | 5225.643 | 113.95 | 0.96 | 0.249 | 0.800 |
| 80L-70 | 4521.590 | 607.02 | 4521.590 | 30.57 | 0.22 | 0.532 | 0.714 |
| 80T-20 | - | - | 7197.364 | 246.31 | 54.72 | 2.940 | 0.905 |
| 80T-30 | - | - | 5089.139 | 255.35 | 94.59 | 0.716 | 0.614 |
| 80T-40 | - | - | 4189.178 | 258.14 | 38.20 | 1.756 | 0.698 |
| 80T-50 | 3563.480 | 4297.4 | 3563.480 | 200.16 | 14.28 | 0.109 | 0.260 |
| 80T-60 | 3116.192 | 1187.13 | 3116.192 | 137.14 | 14.88 | 1.375 | 1.877 |
| 80T-70 | - | - | 2893.749 | 23.28 | 0.18 | 0.009 | 0.038 |

In Table 2 we present the detailed results of the proposed HA on the instances with $n = 50, 60, 70, 80$ nodes. We also present optimal solutions obtained by CPLEX 10.1 solver and taken from [16] and corresponding CPLEX 10.1 running times. Note that HA and CPLEX 10.1 were run on the same machine. Mark "-" in column *Opt.Sol* through Table 2 means that no optimal solution was obtained by CPLEX, while mark "$*$" denotes that the optimality of the presented solution was not proven due to memory or time limits. From the presented results, it can be seen that the HA quickly reaches all optimal solutions that are previously obtained by CPLEX 10.1 solver. It also reaches all solutions marked with "$*$" for which CPLEX 10.1 solver couldn't prove optimality. In cases of 7 tested instances the HA produced solutions that were unreachable for the CPLEX solver. For instances with $n = 50$ nodes, the total CPU time $CPU_{end}$ was up to 0.28 s, while the $CPU_{start}$ times are even shorter-up to 0.18 s. These CPU times are significantly shorter than the corresponding CPLEX 10.1 running times. The average gap from the optimal solution was $agap \leq 5.933\%$ and standard deviation $\sigma \leq 7.051\%$. For instances with $60 \leq n \leq 80$ nodes the total EA computational time is relatively short $CPU_{end} \leq 306.11s$. The average $CPU_{start}$ times in which the EA reaches optimal solution for the first time are up to 94.59s. The average gap and standard deviation were up to 2.940% and 1.877% respectively. On all tested instances with $50 \leq n \leq 80$ nodes, the CPU times were significantly shorter compared to CPLEX 10.1 solver, in cases when CPLEX gave solutions.

# 5 Conclusion

This paper considers the Long-Term Care Facility Location Problem (LTCFLP) that arises from health care system designing. Encouraged by promising results when applying Variable neigh-

borhood method and evolutionary-based approaches to various location problems, we propose a hybrid algorithm for solving the LTCFLP. The best solution obtained in the evolutionary part of the algorithm are used as initial solution of the VNS method. The HA showed to be very efficient in obtaining previously known optimal solutions on smaller and medium size problem instances. For larger problem dimensions, the applied HA approach produces high-quality solutions and also provides results for instances unsolved before. The presented computational results show the potential of the proposed hybrid method for solving this problem and similar discrete location problems.

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# Decentralized Control of a Platoon of Vehicles Based on a Reduced Measurement Set

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**Abstract**

In this paper the Inclusion Principle is applied to decentralized LQ (linear - quadratic) suboptimal longitudinal control design of a platoon of automotive vehicles. Starting from a linearized platoon state model, input/state overlapping subsystems are identified and extracted by an adequate expansion. An algorithm for approximate LQ optimization of these subsystems is developed in accordance with their hierarchical LBT (lower - block - triangular) structure. A new dynamic platoon controller structure, consisting of a reduced order observer and a static feedback mapping, is obtained by applying the Inclusion Principle to the decentralized overlapping platoon control design in the case when the information from the preceding vehicle is missing. Vehicle controllers obtained after contraction provide a high tracking performance.

**KEYWORDS**

Platoon of vehicles, Inclusion Principle, Overlapping decomposition, LQ suboptimal control, Decentralized control.

## 1. INTRODUCTION

The problem of design of automated highway systems (AHS) has recently attracted a considerable attention among researchers (e.g. Godbole, 1996; Shladover, 1991; Horowitz and Varaiya, 2000). It has been shown that control, communication and computing technologies can be effectively combined into an Intelligent vehicles/Highway System (IVHS) that can significantly increase safety and highway capacity. The IVHS architecture, proposed in (Godbole, 1996 Varaiya, 1993), is based on the notion of platoons, groups of vehicles following the leading vehicle with small intra-platoon separation. Control of platoons of vehicles has been studies from different viewpoints (Shladover, 1991; Swaroop & Hedrick,1996; Stanković et al. 1997, 2000; Khatir E. M., E. J. Davison, 2005). In this paper the Inclusion Principle, which has been found to provide a good theoretical basis for practical solutions in the domain of decentralized control of large-scale systems (Šiljak, 1991), is applied to the design of decentralized overlapping longitudinal control of a platoon of vehicles, in which the local regulators, implemented within each vehicle, are designed starting from local quadratic performance indices expressing a priori desired of the designer.

First, a linearized state model for a string of moving vehicles is derived on the basis of the results presented in (Godbole, 1996; Shladover, 1991; Varaiya, 1993). This state model is then expanded and decomposed into overlapping subsystems by using the methodology of the Inclusion Principle (Šiljak, 1991) (the original model is assumed to represent a restriction of the expanded one). The form of the extracted subsystems, together with the methodology of the inclusion principle applied to both observers and dynamic controllers, allows the design of dynamic controllers for entire platoons, composed of the observers and the corresponding feedback gains. Such controllers are applicable in the situations when the information about velocities and accelerations of the preceding vehicles is missing.

Having in mind the desired control structure resulting from the assumed information structure within platoons, local LQ optimization of the subsystems is not done directly, but by applying a sequential optimization algorithm adapted to LBT (lower- block – triangular) structures (Özgüner & Sivan, 1978; Stanković & Šiljak, 1989). The resulting state feedback, ensuring both LQ suboptimality and desired steady-

state regime, is finally contracted from the expanded space to the original system space, according to the methodology of Inclusion Principle (Šiljak, 1991). Contraction to the original space provides a decentralized controller for the whole platoon. Experimental results are given in order to illustrate the main properties of the proposed methodology and possibility of its extension.

## 2. MODEL FORMULATION

It will be adopted in this paper that i-th automotive vehicle in a close formation platoon consisting of n vehicle can be represented by the following dynamic model:

$$\begin{aligned} \dot{d}_i &= v_{i-1} - v_i \\ \dot{v}_i &= a_i \\ a_i &= f_a^i(z_i - k_1^i v_i^2 - k_2^i) \\ \dot{z}_i &= f_j^i(\alpha_i(u_i - z_i)) \end{aligned} \tag{1}$$

where $d_i = x_{i-1} - x_i$ is distance between two consecutive vehicles $x_{i-1}$ and $x_i$ representing their positions, $v_i$ and $a_i$ are the velocity and acceleration, respectively, $f_a^i(.)$ and $f_j^i(.)$ are static nonlinearities of saturation type (acting on acceleration and jerk, respectively), $\alpha_i$ represents the inverse time-constant of the basic vehicle dynamics, while $u_i$ is the corresponding control input.

Permissible control strategies should essentially be of the decentralized type, having in mind the supposed information structure (Shladover et. al., 1991; Shladover, 1991; Stanković et al. 1997, 2000), i.e. the local control $u_i$ is to be calculated on the basis of the local vehicle state variables $\{d_i, v_i, a_i\}$, together with the information about the velocity and acceleration of the preceding vehicle $\{v_{i-1}, a_{i-1}\}$, which is assumed to be transmitted by appropriate communication channels. Each vehicle is also supplied with the information about the spacing, velocity and acceleration reference command $\{d_r, v_r, a_r\}$. The theory of large scale systems abounds with methodologies for both decentralized design of completely decentralized control structures, e.g. (Šiljak, 1991). One of elegant and powerful methodologies is based on the Inclusion Principle.

In this paper an attempt is done to approach the problem of the design of decentralized platoon controllers on the basis of the Inclusion Principle and decentralized overlapping control, by applying LQ optimization to the design of local state feedback. The discussion given in this paper encompasses important cases in which some of the above mentioned data are missing.

## 3. INCLUSION PRINCIPLE

Consider a pair $(S, \tilde{S})$ of linear time-invariant continuous-time dynamic systems represented by

$$\begin{aligned} S: &\quad \dot{x} = Ax + Bu; \quad y = Cx \\ \tilde{S}: &\quad \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u}; \quad \tilde{y} = \tilde{C}\tilde{x} \end{aligned} \tag{2}$$

where $x(t_0) = x_0$ and $\tilde{x}(t_0) = \tilde{x}_0$. In the above equations $x(t) \in R^n$ and $\tilde{x}(t) \in R^{\tilde{n}}$ are the states, $u(t) \in R^p$ and $\tilde{u}(t) \in R^{\tilde{p}}$ the inputs and $y(t) \in R^q$ and $\tilde{y}(t) \in R^{\tilde{q}}$ the outputs of $S$ and , $\tilde{S}$ respectively. It is assumed that $n \le \tilde{n}$, $p \le \tilde{p}$ and $q \le \tilde{q}$.

Definition 1. The system $\tilde{S}$ includes the system $S$ if there exists a quadruplet of full rank matrices $\{U,V,R,S\}$ satisfying $UV = I_n$, such that for any $x_0$ and $u(t)$ in $S$ the conditions $\tilde{x}_0 = Vx_0$ and $\tilde{u}(t) = Ru(t)$ imply $x(t) = U\tilde{x}(t)$ and $y(t) = S\tilde{y}(t)$ (Šiljak, 1991).

General aspects of the Inclusion Principle are discussed in (Šiljak, 1991). Particular attention is paid to two important special cases of inclusion: to restriction and aggregation. Conditions covering different combinations of states, input and output contractions/expansions are presented in [Stanković et al. 1996]. Our interest is in one particular case of restriction, called disextension in (Iftar & Özgüner, 1990).

Theorem 1. The system $S$ is a restriction, type $b$, (or disextension) of $\tilde{S}$, if there exist full rank matrices $\{V,Q,T\}$ such that

$$\tilde{A}V = VA; \quad \tilde{B} = VBQ; \quad \tilde{C}V = TC. \tag{3}$$

Suppose that static output feedback mapping $\tilde{S}$ and $\tilde{F}$ are introduced:

$$\begin{aligned} F: &\quad u(t) = Hy(t) + v(t) \\ \tilde{F}: &\quad \tilde{u}(t) = \tilde{H}\tilde{y}(t) + \tilde{v}(t) \end{aligned} \tag{4}$$

Definition 2. The controller $\tilde{F}$ for $\tilde{S}$ is contractible to the controller $F$ for $S$ if the closed-loop system $(\tilde{S},\tilde{F})$ includes the closed-loop system $(S,F)$ in the sense of Definition 1.

Theorem 2. The controller $\tilde{F}$ is contractible to the controller $F$ if $S$ is restriction, type $b$, of $\tilde{S}$ and the condition $H = Q\tilde{H}T$ is satisfied.

In the case of the state feedback, when $u(t) = Kx(t) + v(t)$ and $\tilde{u}(t) = \tilde{K}\tilde{x}(t) + \tilde{v}(t)$, the corresponding condition for the feedback gain is $K = Q\tilde{K}V$.

The essence of the application of the Inclusion Principle to the decentralized control design for systems with the overlapping structure lies in the implementation of such a state-input-output expansion which transforms $S$ into $\tilde{S}$ in which subsystems of $S$ appear as disjoint, e.g. (Šiljak, 1991; Stanković et al. 1996 ). For example, if $S$ is defined by (x), where $A = [A_{ij}]$, $B = [B_{ij}]$, $C = [C_{ij}]$, $(i, j = 1,2,3)$, then we can consider, under certain conditions concerning submatrices $A_{13}, A_{31}, B_{13}, B_{31}, C_{13}$ and $C_{31}$, that it is composed of two overlapping subsystems $\tilde{S}_1$ and $\tilde{S}_2$ defined by the following system matrices: $\tilde{A}^1 = [A_{ij}]$, $\tilde{B}^1 = [B_{ij}]$ and $\tilde{C}^1 = [C_{ij}]$, $(i, j = 1,2,3)$ and $\tilde{A}^2 = [A_{jk}], \tilde{B}^2 = [B_{jk}]$, $\tilde{C}^2 = [C_{jk}]$ $(i, j = 1,2,3)$, respectively. Then we look for decentralized (local) controllers $\tilde{F}_1$ and $\tilde{F}_2$, characterized by the gains $\tilde{H}_1$ and $\tilde{H}_2$ (or $\tilde{K}_1$ and $\tilde{K}_2$). Formally, the main point is to find such pairs of matrices $(U,V),(Q,R),$ and $(S,T)$ which ensure both satisfactory decoupling and direct contraction to the original space. A common choice is

$$V = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}; \quad U = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & \beta I & (1-\beta)I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \tag{5}$$

where $0 < \beta < 1$; matrices $R$ and $T$ are analogous to $V$, while $Q$ and $S$ are analogous to $U$ (see Stanković et al. 1996 for details). After expansion, the controller $\tilde{F}$ for $\tilde{S}$ can be designed in a decentralized way by using any convenient methodology; one of frequently used approaches is based on the local subsystem optimization in the LQ sense (Šiljak, 1991; Stanković et al. 1996), where local performance

indices are constructed in accordance with the desired overall system properties (expressed through the global performance index). The resulting gain matrix for $\tilde{S}$ is the block-diagonal form: $H_D = diag\{\tilde{H}_1, \tilde{H}_2\}$ (or $K_D = diag\{\tilde{K}_1, \tilde{K}_2\}$). Contraction to the original space is done, in the case of above defined restriction, by $H = Q\tilde{H}_D T$ (or $K = QH\tilde{K}_D V$). The resulting controller is, in general, only suboptimal, but the original system is stable provided the expanded system is stable (Šiljak, 1991; Stanković et al. 1996).

# 4. DECENTRALIZED SUBOPTIMAL LQ CONTROL OF A PLATOON OF VEHICLES

There are several possibilities for constructing linearized models in the state-space form, depending on the choice of state variables, e.g. (Shladover et. al., 1991; Shladover, 1991). A convenient form follows directly from (1). Assuming for simplicity that is n=3 and that all vehicles have identical models, we obtained following state model S:

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} A_v & 0 & 0 \\ A_d & A_v & 0 \\ 0 & A_d & A_v \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} B_v & 0 & 0 \\ 0 & B_v & 0 \\ 0 & 0 & B_b \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{6}$$

where $X_i^T = \begin{bmatrix} d_i & v_i & a_i \end{bmatrix}$ $(x_0 = 0 \; in \; d_1)$ and

$$A_v = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & +1 \\ 0 & 0 & -\alpha \end{bmatrix}; \; A_d = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \; B_v = \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix}.$$

Notice that both the state and the input matrices are composed of identical diagonal blocks $A_v$ and $B_v$ and that the interconnections between vehicles are modeled by the lower subdiagonal blocks $A_d$. The structure of the above model indicates that it would be possible to consider the platoon as being composed of a sequence of overlapping subsystems. The i-th subsystem is defined by the following state model

$$\mathrm{S_i}: \; \dot{\xi}_i = \begin{bmatrix} A_L & 0 \\ \tilde{A}_d & A_v \end{bmatrix} \xi_i + \begin{bmatrix} B_L & 0 \\ 0 & B_v \end{bmatrix} \begin{bmatrix} u_{i-1} \\ u_i \end{bmatrix} = A_i \xi_i + B_i \zeta_i \tag{7}$$

where

$$A_L = \begin{bmatrix} 0 & +1 \\ 0 & -\alpha \end{bmatrix}; \; \tilde{A}_d^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; B_L = \begin{bmatrix} 0 \\ \alpha \end{bmatrix} \quad \text{and} \quad \xi_i^T = \begin{bmatrix} v_{i-1} & a_{i-1} & d_i & v_i & a_i \end{bmatrix}.$$

The overlapping parts of Si with Si-1 are, obviously, the block AL in Ai and the block BLin Bi, related to $\xi^T{}_{i,1} = \begin{bmatrix} v_{i-1} & a_{i-1} \end{bmatrix}$ and ui-1, respectively, that is the system S is both state and input overlapping (Šiljak, 1991; Shladover, 1991; Stanković et al. 1997, 2000). The subsystems Si can hardly be given any precise physical meaning; notice, however, their state vectors contain exactly the measurements supposed to be available in each vehicle (Shladover et. al., 1991; Shladover, 1991; Stanković et al. 1997, 2000). Their control vectors ζi contain two components: ui which represents the physical control signal in i-th vehicle, and ui-1, which can be considered to represent, together with the corresponding part of the subsystem dynamics, the preceding part of the platoon, as seen by i-th vehicle ( for the second vehicle in platoon this is exactly the leading vehicle dynamics).Therefore, ui depends ion the entire subsystem state, and ui-1 only on the part of the subsystem state vector overlapping with the preceding subsystem.

The methodology of the Inclusion Principle offers a mathematically consistent way of dealing with the problem of decomposition of complex systems with overlapping structure. After expansion, the subsystems in the platoon model appear as disjoint. Application of the LQ methodology based on the definition of local performance indices leads to local state feedback control (depending on the appropriate sets of measurements). Contraction to the original space provides a physically implementable control law.

## 4.1 Leading Vehicle Control

The leading vehicle is supplied with the reference command and uses its own state vector for control design. Formally speaking, if the leading vehicle model is represented by

$$\dot{X}_L = A_L X_L + B_L u_1 \tag{8}$$

where $X_L^T = \begin{bmatrix} v_1 & a_1 \end{bmatrix}$, then the optimal state feedback control law should be found from the conditions for the minimum of the performance index

$$J_L = \int_{t_0}^{\infty} \left[ \left( X_L - X_{1r} \right)^T Q_L \left( X_L - X_{1r} \right) + R_L u_1^2 \right] dt \tag{9}$$

where $X_{1r}^T = \begin{bmatrix} v_r & a_r \end{bmatrix}$ is time-varying reference supplied to the first vehicle, which is known entirely in advance, and $Q_L \geq 0$ and $R_L > 0$ are the corresponding weights. This is, in fact, an LQ optimal tracking problem, which can be solved in a straightforward way ( Anderson and Moore, 1990).

$$\begin{aligned} u &= -K_1 X_L - M_1 X_{1r} \\ K_1 &= R_L^{-1} B_L^T P_L \\ M_1 &= R_L^{-1} B_L^T \left( A_L - B_L K_1 \right)^{-T} Q_L \\ P_L A_L &+ A_L^T P_L - P_L B_L R_L^{-1} B_L^T P_L + Q_L = 0 \end{aligned} \tag{10}$$

The above control law is suboptimal, since the feedforward block is reduced to a constant matrix; this is, however, a nearly optimal solution, having in mind characteristic forms of the reference command signals. A priori choice of the criterion weights can provide different tracking properties. Notice that the static steady state error reduces to zero, having in mind that $A_L$ is singular (Anderson and Moore, 1990).

## 4.2 General Subsystem Control

Control of the second vehicle assumes that the leading vehicle control is appropriately designed. Consequently, control design for the general subsystem model (7) can be decomposed into two parts: first, $u_{i-1}$ is found and the corresponding regulator is implemented and, second, $u_i$ is found for the resulting system by using the complete state feedback. According to (10), we have

$$u_{i-1} = -K_1 \begin{bmatrix} v_{i-1} & a_{i-1} \end{bmatrix}^T - M_1 X_{1r} \tag{11}$$

where $K_1$ and $M_1$ are defined by (10). After implementing (11), one comes to the following subsystem model

$$\dot{\xi}_i = \begin{bmatrix} A_L - B_L K_1 & 0 \\ \tilde{A}_d & A_v \end{bmatrix} \xi_i + \begin{bmatrix} 0 \\ B_v \end{bmatrix} u_i + \begin{bmatrix} -M_1 \\ 0 \end{bmatrix} X_{1r} \tag{12}$$

Now, $u_i$ is found from (12) by minimizing

$$J_L = \int_{t_0}^{\infty} \left[ \left( \xi_i - X_{2r} \right)^T Q_i \left( \xi_i - X_{2r} \right) + R_i u_i^2 \right] dt \tag{13}$$

where $Q_i \geq 0$ and $R_i > 0$, while $X_{2r}^T = \begin{bmatrix} d_r & v_r & a_r \end{bmatrix}$ is the complete set of reference commands. The specific form of state weighting matrix is given in (Shladover, 1991; Stanković et al. 1997, 2000).

An approximately optimal solution, in the sense that all the gains are assumed to be constant, is given by

$$\begin{aligned} u_i &= -K_2 X_i - M_2 X_{2r} - M_3 X_{1r} \\ K_2 &= R_i^{-1} B_i^T P_2 \\ M_2 &= R_i^{-1} B_i^T \left( A_i - B_i K_2 \right)^{-T} Q_i \\ M_3 &= R_i^{-1} B_i^T \left( A_i - B_i K_2 \right)^{-T} P_2 B_M \\ P_2 A_i &+ A_i^T P_2 - P_2 B_i R_i^{-1} B_i^T P_2 + Q_i = 0 \end{aligned} \tag{14}$$

where

$$A_i = \begin{bmatrix} A_L - B_L K_1 & 0 \\ \tilde{A}_d & A_v \end{bmatrix} ; B_i = \begin{bmatrix} 0 \\ B_v \end{bmatrix} ; B_M = \begin{bmatrix} -B_L M_1 \\ 0 \end{bmatrix}$$

The resulting overall subsystem control law possesses a specific structure. The state feedback gain $K_i^T = \begin{bmatrix} K_1^T & K_2^T \end{bmatrix}$ has a lower block triangular (LBT) structure, in accordance with the information supposed to be locally available. The overall feedforward gain matrix $M_i$ for a subsystem, which can be obtained simply from $M_1$, $M_2$ and $M_3$, multiplies essentially $X_{2r}$, since $X_{1r}$ is a subset of $X_{2r}$. It is important to notice that the steady state error is again zero for constant references.

## 4.3 Dynamic Platoon Controller with Decentralized Observers

The previous discussion has been related to static feedback controllers (Shladover, 1991; Stanković et al. 1997, 2000). Starting from the adopted system decomposition, the application of the Inclusion Principle, connected with local sequential optimization, has resulted into a systematic methodology of LQ suboptimal design of the corresponding matrix gain for the entire platoon. It is, however, logical to analyze also the situations in which the information assumed to be available from the preceding vehicle is missing ($v_{i-1}$ and $a_{i-1}$ for i-th vehicle). The decomposition of the platoon to the above defined subsystems, together with the results of the application of the Inclusion Principle to observers and dynamic controllers make it possible to approach this problem in a methodologically consistent way. The corresponding tools at the local level are, obviously, the state observers, which, according to the separation principle, should provide the estimates of the subsystem states to the LQ suboptimal state controllers already designed. Design of the corresponding overall dynamic platoon controller should be then based on the application of the results related to the inclusion of observers and dynamic controllers. Namely, after being successfully designed in the expanded space, the dynamic controllers for the subsystems (consisting of the observer and the feedback gain) have to be contracted to the original space for implementation, ensuring that the inclusion conditions remain satisfied, and that the resulting observers remain essentially decentralized (using only information locally available). The contracted controller, according to the Inclusion Principle, preserves then the main properties of the closed-loop system achieved in the expanded space.

Consider the subsystem model (7) and assume that the available measurements in the i-th vehicle are, in general, $y_i = C\xi_i$, where C is an appropriately defined matrix. We shall consider the case when $C = \begin{bmatrix} 0 \vdots I_3 \end{bmatrix}$, i.e. when the subsystem states $d_i$, $v_i$ and $a_i$ can be locally measured, but both $v_{i-1}$ and $a_{i-1}$ are not available. An estimate $\hat{\xi}_{i,1}$ of $\xi_{i,1} = \begin{bmatrix} v_{i-1} & a_{i-1} \end{bmatrix}^T$ can be obtained by using the standard procedure for constructing reduced order state estimators in the case of perfect measurements of a part of the state vector (Anderson and Moore, 1990; Kwakernaak & Sivan, 1972 ). One obtains directly the following structure:

$$\dot{\hat{\xi}}_{i,1} = A_L\hat{\xi}_{i,1} + B_L u_{i-1} + L\left[\dot{\xi}_{i,2} - \tilde{A}_d\hat{\xi}_{i,1} - A_v\xi_{i,2} - B_v u_i\right], \tag{15}$$

where L is the observer gain ( which can be defined, for example, in accordance with the Kalman filtering methodology. Those $\xi_{i,2}^T = \begin{bmatrix} d_i & v_i & a_i \end{bmatrix}$ is part of the state vector $\xi_i$, that is $\dot{\xi}_{i,2} = A_v\xi_{i,2} + B_v u_i$. We shall assume that the control law for subsystem $S_i$ is now, according to the separation principle:

$$u_{i-1} = -K_1\hat{\xi}_{i,1} - M_1 X_{1r} \tag{16}$$

$$u_i = -K_{21}\hat{\xi}_{i,1} - K_{22}\xi_{i,2} - M_2 X_{2r} - M_3 X_{1r}, \tag{17}$$

where the corresponding matrices are defined in (10) and (11).

## 4.4 Experimental results

Numerous simulations have been undertaken in order to clarify important aspects of the proposed methodology. In the paper some characteristic results for a platoon of 10 vehicles, with the desired interspacing of 1m, are presented. A large variety of transient behaviors can be obtained by varying the free parameters. Figure 1. corresponds to the model with a small time constant (α=2), feedback vectors $K_1 = \begin{bmatrix} 44.7214 & 6.4647 \end{bmatrix}, K_2 = \begin{bmatrix} -17.0102 & -1.6804 & -70.7107 & 48.3832 & 7.0057 \end{bmatrix}$, and the following feedforward parameters: $M_1^{31} = 44.7214$, $M_2^{51} = 62.7459$ and $M_3^{51} = -141.4214$.

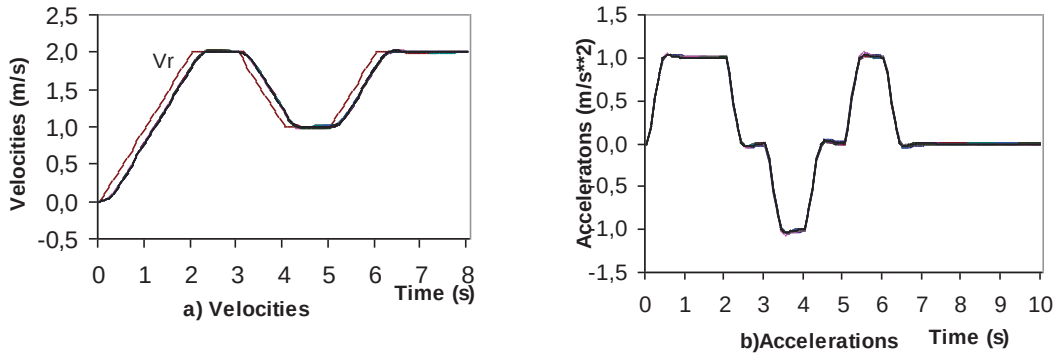


Figure 1. Responses of ten-vehicles platoon (α=2)

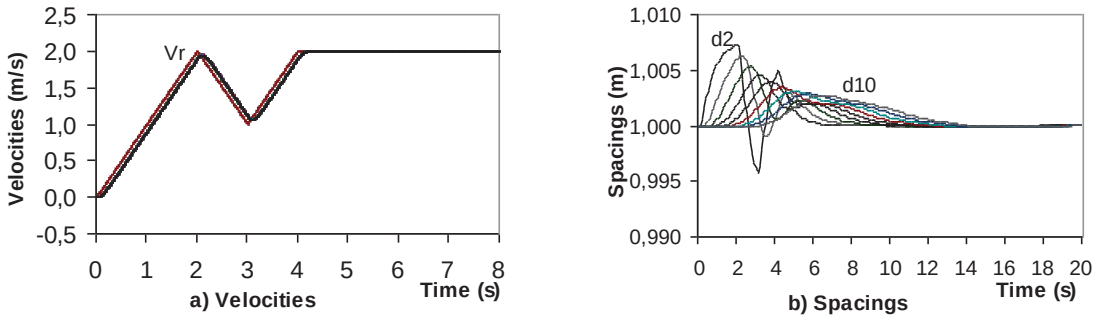


Figure 2. Responses of ten-vehicles platoon (α=10)

Figure 2 gives the time histories for velocities and spacings in the case when $\alpha$=10, where feedback vectors $K_1 = [44.7214 \quad 3.4659]$ and $K_2 = [-17.1266 \quad -1.3476 \quad -70.7107 \quad 48.7929 \quad 4.0424]$, as well as the following feedforward parameters: $M_1^{31} = 44.7214$, $M_2^{51} = 31.6627$ and $M_3^{51} = -70.7107$.

The represented curves are obtained by using dynamic local regulators utilizing local measurements only, i.e. local velocity and acceleration, together with the distance from the preceding vehicle. Decentralized estimation of the states of the subsystems (velocity and acceleration of the preceding vehicles) is done by Kalman filters based on the sets of ideal measurements (Kwakernaak & Sivan, 1972). The achieved control quality is obvious. An extension to the stochastic case is straightforward.

## 5. CONCLUSION

In this paper the Inclusion Principle has been applied to LQ suboptimal control of a platoon of automotive vehicles. Identification of input/state overlapping subsystems and their extraction by an appropriate expansion have enabled an approximate LQ optimization, adapted to the LBT structure of the subsystem model. State estimation is applied in the deterministic context with the goal to reduce the required set of measurements, decreasing in such a way technical complexity of the whole platoon control system.

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# Optimization of Media Plan Efficiency

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**Abstract**

We present a new approach to generation of media plans for TV advertising campaigns. This approach is based on raw data obtained by television audience measurement systems. It provides that as many as possible target viewers are exposed to TV commercial predefined number of times, satisfying additional constraints for budget and daily ratings. We use goal programming for modeling several goals. Iteratively solving mixed integer problems we obtain list of time slots in which TV commercial should be aired, i.e. media plan.

## 1 Introduction

TV advertising is the most significant channel of communication between large companies and audience, providing the fastest propagation of advertising message. In the last 50 years, trading of television advertising time has grown up into a multimillion dollars industry in all countries, regardless of level of its development. Consequently, optimization of TV advertising efficiency has been studied from both commercial and scientific aspects. The necessity for improvement of advertising efficiency in general, was expressed one century ago by John Wanamaker in his statement: "Half the money I spend on advertising is wasted; the trouble is I don't know which half" [9].

Advertising efficiency is expressed by ratio of the increment of revenue appeared because of advertising campaign and advertising cost. However, revenue depends on many psychological factors that influence consumers desire to buy advertising product or service [2]. These factors have to be taken into account when advertising strategy is chosen. Advertising strategy usually considers intensity and dynamics of advertising message exposure. It has to determine who are the target consumers of the product and which are their characteristics.

The main participants in the process of TV advertising are: advertiser, advertising agency, TV station and the audience of viewers – potential consumers of the product. In this paper we influence efficiency of advertising campaign by determining in which commercial break, i.e. time slot, a TV commercial (hereinafter: TVC) will be aired. The selection of time slots in which TVC will be placed is called media plan and it is usually made in agency by media planner. Although many researches on this issue have been done, in practice media planners often relay on their cognitive capabilities, skill and experience. The main goal of this paper is to help media planners to construct more efficient plans such that it influences growth of the advertised product consumption for less costs, taking into account all participants demands and constraints.

A revolution in TV media planning appeared in the beginning of 1980s when television audience measurement (TAM) started to be performed using electronic devices called people-meters. When such devices are installed in statistically significant number of homes, the data: who is watching what channel

at what time, become available. Using these data in combination with program schemes for each channel, it is possible to generate parameters that can be very useful in analysis of the audience and design of media plan.

The following well known parameters can be calculated on the basis of raw data obtained by TAM [13]:

- Rating point represents one percent of a designated group or an entire population that is exposed to a particular advertising vehicle such as TV program.
- Gross rating points (GRPs) are rating points for all viewers covered by TAM.
- Target rating points (TRPs) are an adaptation of gross rating points (GRPs) for a certain advertiser. TRPs adjust GRPs to reflect just those individuals who match the advertiser's target audience.
- Reach is the number or percentage of an advertiser's target audience that is exposed to at least one TVC over an established time frame.
- Reach+$n$ is the number or percentage of target viewers who were exposed to TVC at least $n$ times. In this paper, the acronym NE will be used for the number of exposures of an individual target viewer to TVC.

In Section 2 we describe the problem we solve: an overview of previous researches in this field are given and a brief description of the problem is inserted. In Section 3 there are presented notation and goal programming mathematical model. In Section 4 we describe how we were solving stated problem. Results of some numerical experiments are given. In Section 5 we give several concluding remarks.

## 2 Problem description

Problem of TV advertising has been studied using various approaches and assumptions. Mihiotis and Tsakiris [11] used a mathematical programming approach to achieve the combination of time slots in which TVC will be inserted such that it maximizes the total sum of GRPs subject to the limitation of the available budget. Bollapragada, Bussieck and Mallik [4] analyzed the problem of scheduling on Broadcast Television. They used mixed-integer programming to obtain a schedule which provides as evenly spaced airings of TVCs as possible. Cetin and Esen [7] proposed an integer nonlinear programming model based on the weapon-target assignment problem, for solving media allocation problem including budget allocation. Saha, Pal and Pal [12] used graph coloring approach to model the problem of time slots selection such that the sum of the number of viewers of the selected time slots is maximal.

Goal programming approach has been applied in media planning since 1968 when Charnes et al. [8] introduced goal programming into media selection problem. They observed the impact of message frequencies on the percentage of target segment which is exposed and expressed it as a log-normal distribution. Proposed goal programming model includes constraints related to the percentage of audience segment which must be exposed to TVC at least a given number of times. Bollapragada and Garbiras [5] observed the problem of reordering the initial advertisement schedule to achieve objectives of contracts agreed upon clients and broadcast television. The main goal was to avoid the competing products conflict. The second goal was to minimize perturbation of initial positions in slot. In [3] chance goal programming model was developed by Bhattacharya to solve the problem of selecting optimal number of advertisement in different media (newspapers and TV channels) within the available budget in order to maximize the sum of aggregated TRPs. Jha, Aggarwal and Gupta [10] were investigating the optimal allocation of advertising budget for multiple products advertised through different media (newspapers and websites) in a segmented market. The problem of finding the number of advertisements in different

media in order to maximize the desirable reach has been formulated as a multi-objective problem and solved using goal programming.

Determining the proper NE has been the subject of many marketing studies. Researches in advertising have shown that the number of exposure must exceed some level, so called communication threshold, in order to form a potential consumer's desire for advertised product or service [6]. Otherwise, the viewer can be considered underexposed. On the other hand, high level of exposure, i.e. overexposure would ultimately result in saturation and negative reactions on the message [1]. In that case desire decreases and even aversion toward the product can appear.

The maximal efficiency of an advertising campaign will be achieved if the number of viewers exposed a proper number of times to TVC is maximized. For example, if we have the situation like in Figure 1, we want to maximize value (Reach+3 – Reach+6). However, advertising agencies usually make media plans based on TRPs, which are calculated as average at the target audience level. As a consequence, the situation presented in Figure 1(a) can appear: although the average NE is proper, there can be many viewers who were under or over exposed. If we want NE to be in the certain interval for as many viewers as possible, we have to optimize media plan using individual viewers data collected by TAM systems. Result of such optimization is presented in Figure 1(b).


(a) – improper distribution of Reach+ (b) – proper distribution of Reach+

Figure 1: Audience exposed to a certain number of messages

Data obtained by TAM systems are organized in tables where for each time slot/viewer pair is known whether the viewer saw the time slot or not. For each time slot it is known: to which TV channel it belongs, at what day/time it was aired and what is the cost of inserting TVC in it. Viewers are people in whose homes people-meters are installed. For each viewer we know the characteristics that enable us to determine whether he/she belongs to our target audience or not.

Having in mind the main goal that should be accomplished by media plan and available data, the goal programming appears to be the most suitable for modeling and solving the problem.

## 3 Mathematical model

We introduce several criteria for media plan and a goal for each of them. The primary goal is to provide proper NE of TVC for as many target viewers as possible. In order to achieve it we minimize the difference between targeted and estimated NE for each target viewer. In the case when not all viewers were exposed to TVC targeted number of times in one period, the second goal is to favor these viewers in the next period. This is a kind of diversification mechanism that prevents a certain group of viewers to be neglected while others are overexposed. The third goal is to control target rating points for each day of campaign. The forth goal is to achieve previous three goals with minimal costs.

Instead of observing the entire promotion period as a whole, we split it into sub periods and we optimize media plan for each period sequentially. This approach has two main advantages. First, we significantly reduce computational complexity of the problem. The second advantage of promotion

period partitioning is that in this way we are able to control dynamics of media plan and adjust it to our advertising strategy. The model we developed supports the following controls:

- the NE control for each period – if a viewer was overexposed to TVC in one period, the targeted NE for him will be reduced in the next period;
- the regularity of viewing – if according to the previous rule the targeted NE is reduced for a viewer, we determine what is the minimal NE he should have in order to maintain TVC consumption regular;
- the budget control – we minimize cost, but if it is smaller then the budget for a certain period, is it possible to use it for next periods, completely or partially.

In order to solve this multi-objective optimization problem and to enable tuning the mentioned parameters a mathematical model is developed.

We use the following notation:

**Given parameters**

$P$ set of target viewers covered by TAM;

$D$ set of days of campaign;

$E$ set of time slots during campaign;

$N = \{1, 2, \ldots, n\}$ set of periods;

$D_k$ set of days in $k$-th period, $k \in N$, such that $\bigcup_{k \in N} D_k = D$, $D_r \cap D_s = \emptyset$, $\forall r, s \in N$, $r \neq s$;

$E_l$ set of time slots in $l$-th day, $l \in D$, such that $\bigcup_{l \in D} E_l = E$, $E_r \cap E_s = \emptyset$, $\forall r, s \in D$, $r \neq s$;

$p_{ij}$ expected NE of $i$-th time slot by $j$-th target viewer, $i \in E$, $j \in P$;

$tp_i$ target rating points of $i$-th time slot, $i \in E$;

$tpl_l$,$tpu_l$ lower and upper bound of target rating points for $l$-th day;

$c_i$ cost of inserting a TVC in $i$-th time slot, $i \in E$;

$b_k$ budget for $k$-th period;

$rb$ percent of the surplus of budget from the previous period that can be used, $rb \in [0, 1]$ (typically 0 or 1);

$fq_k$ cumulative targeted NE in $k$-th period ($0 < fq_1 < fq_2 < \cdots < fq_n$);

$mfq_k$ minimal NE in $k$-th period that should be reached even if a viewer was overexposed to TVC in the previous period.

Among the parameters mentioned above, some are given by the advertiser while others are calculated upon the input data obtained by TAM. These raw data can be grouped in a table for certain periods, typically for weeks. We denote elements of the tables $a_{ij}^q \in \{0,1\}$. If viewer $j$ saw time slot $i$ in $q$-th week then $a_{ij}^q = 1$; otherwise $a_{ij}^q = 0$. Collecting such data for a longer period, e.g. $m$ weeks, we can generate probabilities that viewer $j$ will see time slot $i$:

$$p_{ij} = \frac{1}{m} \sum_{q=1}^{m} a_{ij}^q$$

Considering that parameter $p_{ij}$ is related to one precise time slot, it also represents expected number of times that viewer $j$ will see the time slot $i$. Another calculated parameter is TRP. For each time slot $i \in E$ it is calculated from raw input data using formula:

$$tp_i = \frac{100}{m|P|} \sum_{q=1}^{m} \sum_{j \in P} a_{ij}^q$$

Parameters $c_i$, $i \in E$ are determined by advertising agency and they are dependent on several other parameters, typically on GRP of $i$-th time slot, reputation of a particular TV show, prime time coefficient, etc.

On the other hand, parameters given by the advertiser depend on advertising strategy and they are determined by marketing experts.

### Variables

$x_i$ binary decision variables – denote whether the TVC will be inserted in $i$-th time slot ($x_i = 1$) or not ($x_i = 0$), $i \in E$;

$d_j^k$ negative deviation variables for NE – difference between targeted and expected NE of $j$-th target viewer in $k$-th period, $j \in P$, $k \in N$;

$dtpl_l$,$dtpu_l$ negative and positive deviation variables for lower and upper bound, respectively, on TRPs in $l$-th day, $l \in D$.

### Parameters obtained after solving the $(k-1)$-th problem

$FQ_j^{k-1}$ expected NE of $j$-th viewer, $FQ_j^0 = 0$, $FQ_j^{k-1} = \sum_{q=1}^{k-1} \sum_{i \in E_q} p_{ij} x_i$, $\forall j \in P$;

$d_j^{k-1}$ optimal value of deviation variable, $d_j^0 = 0, \forall j \in P$;

$br_{k-1}$ surplus of the budget, $br_0 = 0$, $br_{k-1} = b_{k-1} + rb \cdot br_{k-2} - \sum_{i \in E_{k-1}} c_i x_i$.

### Mathematical model of the $k$-th problem

$$(\min) f_k = M_1 \sum_{j \in P} d_j^k + M_2 \sum_{j \in P} d_j^{k-1} d_j^k + M_3 \sum_{l \in D_k} (dtpl_l + dtpu_l) + \sum_{l \in D_k} \sum_{i \in E_l} c_i x_i$$

subject to

$$\begin{aligned}
&\sum_{l\in D_k}\sum_{i\in E_l} c_i x_i \le b_k + rb\cdot br_{k-1} \\
&\sum_{i\in E_k} p_{ij}x_i + d_j \ge \max\left\{fq^k - FQ_j^{k-1}, fq^{k-1} + mfq^k\right\}, && j\in P \\
&\sum_{i\in E_l} tp_i x_i + dtpl_l \ge tpl_l, && l\in D_k \\
&\sum_{i\in E_l} tp_i x_i - dtpu_l \le tpu_l, && l\in D_k \\
&x_i \in \{0,1\}, && i\in E_l, l\in D_k \\
&d_j^k \ge 0, && j\in P \\
&dtpl_l, dtpu_l \ge 0, && l\in D_k
\end{aligned} \tag{1}$$

Final solution is obtained by solving the sequence of the problems presented by Model (1) for $k = 1,2,\ldots,n$.

Mathematical model (1) represents a goal programming model for the multi-objective problem described above. The terms of the goal function have the following meaning:

- $\sum_{j\in P} d_j^k$ – sum of negative deviations from the targeted NE in $k$-th period – we don't penalize positive deviations, but they are implicitly bounded by minimization of the budget;
- $\sum_{j\in P} d_j^{k-1} d_j^k$ – increasing of weights for deviations of viewers who didn't achieve targeted NE in the previous period;
- $\sum_{l\in D_k} (dtpl_l + dtpu_l)$ – sum of deviations for bounds on daily TRP in $k$-th period;
- $\sum_{l\in D_k}\sum_{i\in E_l} c_i x_i$ – costs for $k$-th period.

When condition $M_1 \gg M_2 \gg M_3 \gg \max_{1\le k\le n}\{b_k\}$ holds, criteria are lexicographically optimized in the order they were introduced in the beginning of this section.

## 4 Solving method and experiments

Unfortunately we were not able to get real life raw data for numerical experiments. Instead, we generated them randomly, taking care they to be realistic. We have tested our model using instances representing different kinds of scenarios that can appear in reality depending on kind of advertised product, size of target audience and advertising strategy. Some experiments were done in order to determine computational characteristics of the problem.

For modeling and solving we used GNU Linear Programming Kit (GLPK), an open source software for solving linear and mixed integer mathematical programming problems. For modeling GLPK supports MathProg modeling language which is a subset (dialect) of well known AMPL language. MathProg can also be referred to as GMPL (GNU Mathematical Programming Language). Since GMPL does not support scripting, we also developed a script written in Java language that enables iterative optimization of sequence of problems described by Model (1).

First results have shown that it is not possible to solve real-life size problems to optimality in reasonable time. Even small examples that had 500 target viewers and 100 time slots per period spent many hours to be solved exactly. Instead, we used GLPK's implementation of Branch and Cut (B&C) algorithm with limited execution time. In that way we were able to obtain an approximate solution in reasonable time.

Experimentally we adjusted parameters for branching and cutting, stating the following rules: branch using hybrid pseudo cost heuristic, backtrack using depth first search, and generating four kinds of cuts: Gomory's mixed integer cut, mixed integer rounding cut, mixed cover cut, and clique cut.

One round of experiments considered computational complexity, possibility to obtain exact solution and quality of approximate solutions. Even the first tests showed that these properties are very numerically dependent. Randomly generated instances with same size were solved in very different time.

In Table 1 there are presented some results that show possibility and efficiency of using time-limited B&C algorithm for obtaining approximate solution. Column Time_1 represents how much time was needed to obtain the first integer (feasible) solution. This time is important because even that solution can be adopted as a final one. Column Time_2 represents the moment when the final improved solution was found and column Time_3 represents the time when the algorithm finished (either limited time expired or algorithm concluded that optimal solution was obtained). If Time_3 is less than 3000 solution obtained after Time_2 seconds is optimal. Otherwise, it can be optimal, but there is no proof of that. All instances were limited to 3000 seconds. For all instances just one period was considered. All three columns represent average values. Label $^*$ means that 25% of solved instances did not finish optimization within 3000 sec. Label $^{**}$ means that none of instances finished optimization within 3000 sec.

| Group of instances | No. of time slots | Number of people-meters | Time_1 [sec] | Time_2 [sec] | Time_3 [sec] |
|---|---|---|---|---|---|
| 1 | 60 | 500 | 1 | 45 | 294 |
| 2 | 60 | 1000 | 2 | 99 | $1353^*$ |
| 3 | 90 | 500 | 2 | 180 | $> 3000^{**}$ |
| 4 | 150 | 1000 | 9 | 904 | $> 3000^{**}$ |

Table 1: Experimental results

The most significant result is that for instances of all sizes the first feasible solution was obtained in very short time. This means that this approach can be used for obtaining approximate solution. In order to estimate the quality of such a solution we can analyze the last two columns. For example, for the first group of instances the optimal solution was found after 45 seconds, but 249 seconds more were needed to prove that it is optimal. In the second group almost 13 times more time was needed to prove the optimality of previously found solution than to find that solution. Even for those instances that were not solved to optimality, the best solution found was obtained in relatively short time. Problem size of fourth group of instances can be considered as real-life, and still the best known solution was found in about 15 minutes.

Considering all previous results, we concluded that this approach can be efficiently used for obtaining near-optimal real-life media plans.

# 5 Conclusion

We have presented a new approach to optimization of efficiency of media plan for TV advertising. This approach relays on raw data obtained by TAM systems. Using these data and iteratively solving goal programming problems, a highly efficient media plan is obtained. Result of application of so obtained media plan is that as many target viewers as possible are exposed to TVC precise number of times. Psychological and marketing researches have shown that proper exposure of TVC produces viewer's desire to consume the advertised product.

Numerical experiments have shown that standard tools for solving mixed integer problems can be used for this problem to obtain near-optimal solutions.

Difficulties in using this approach can appear because of lack of raw TAM data. Our experience has shown that companies that collect such data are not willing to share them with advertising agencies. Perhaps, seeing the possibility to gain mutual advantages from optimization systems that use those data, they will become more opened for cooperation.

The next step in researching this area definitely is to obtain real-life raw TAM data and to compare solutions obtained by this approach and standard solutions offered by advertising agencies. Also, efficiency of that two campaigns should be compared.

### 5.1 Acknowledgments

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# Modeling and Optimization of Job Scheduling in Concrete Pumping Operations

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**Abstract**

The current paper presents a method for modeling and optimization of job scheduling in concrete pumping operations by enterprises in concrete industry. The model is based upon the Vehicle Routing Problem with Time Windows (VRPTW), where a set of vehicles with limits on capacity and travel time are available to service a set of customers with standard demands and earliest and latest time for servicing. In the model a set of the concrete pumps are available for servicing the clients of the enterprise within the earliest and the latest time (time window). These concrete pumps face no limit regarding the capacity and the travel time. The paper presents three different sub models of the same problem. The objective of the first sub model minimizes the total cost of the enterprise for concrete pumping. In the second sub model the objective is to maximize the number of the satisfactory clients when the number of the concrete pumps is lower than necessary. Finally in the third sub model the objective is to minimize the sum of the total difference between the start time of the operations and the desired one. Each enterprise in concrete industry can choose to use one of these sub models in order to optimize concrete operations regarding its unique characteristics.

**KEYWORDS**

Job Scheduling, Vehicle Routing, Concrete Pumping, Concrete Industry.

## 1. INTRODUCTION

In the production process of the concrete industry, despite the high degree of penetration of automation in recent years, the smooth functioning of the larger piece of the production, the transportation and the disposal of the goods is based on the correct decision of trained staff. The fierce competition in the sector of ready mixed concrete doesn't enable companies to have personnel specialized exclusively in one part of the production. The phenomenon of overlapping duties is common for the staff workers and this phenomenon causes the non optimal way they handling daily problems regarding production. The result of this is that precious financial or human resources are wasted and thus periodic results of operations are relatively low when compared with what could be achieved through better organization.
With the optimization of the field of the disposal of goods, enterprises in the concrete industry can significantly increase their efficiency, productivity, while at the same time manage reduce the overall cost of the finished product, creating high benefits for them.

## 2. DESCRIPTION OF THE PROBLEM

In this part we will try to give the necessary additional information needed for formulating the model. We consider as the basic time unit a calendar day. Everyday there are N jobs (tasks) to be completed. On the other hand, there are V concrete pumps, entrusted with the task of handling the tasks. Each task, needs some time in order to be completed, determined by the starting time and the end time (time window). The task assignment on the concrete pumps, must take into account the time window of each task to determine what machine to deliver, while checking that the tasks assigned to each concrete pump don't have cascading time windows.

A basic requirement of the task is to select a concrete pump with appropriate technical features. That is, a suitable length of boom, a desired per hour capacity, the ability to approach and implement the project (eg, concrete pump wider than the entrance, free space is big enough for the concrete pump to spread its legs etc). On the other hand, each concrete pump has technical characteristics that make it capable to serve specific tasks. The technical characteristics of both cases will be quantified, to allow the model analyze the machine's ability to serve each task.

Summarizing the above, there is the case that a concrete pump doesn't undertake a task, so we would say that this crew has that day off. This means that every day we have a typical assignment problem of V concrete pumps in N tasks. We would like to underline here that we have the ability to use a machine more than once in the assignment period so we do not create an initial restriction on the relationship between the number of tasks in relation to the number of machines. We also note an important assumption, that the crews of concrete pumps have uniform abilities and productivity. If this assumption is not the case, it is easy to form the model in order to exclude the assignment of some crews in specific tasks.

Another important issue is that the concrete pumps (not considering the crews), are stressed about the same (because of extend work) in the long term. The reason for that assumption is that all the "economic" concrete pumps of a specific period (which preferred over the rest by the model), in this period will have extend work comparing with the rest. The excessive work will cause extend maintenance work to these concrete pumps which will cause the proportional cost of these concrete pumps to rise, making those machines less attractive by the model in the future. So there is no concern to include in the model the long term stress of the concrete pump. However, the calculation of stress in long term can easily be included in the model, if circumstances require it.

Continuing our analysis, we find that the N jobs of each period can be separated into different categories. This distinction can be made if we take into account the characteristics of each task, the most important of which are:

a) The degree of difficulty of approaching the project with the concrete pump. The specific degree to be considered, as there are projects which are extremely difficult to reach, and approach routes may be gravel, muddy and of very steep slopes.
b) The number of different set-ups that are required within the task. This happens because many times the planning and complexity in the form of projects does not give the possibility to reach all the spots from the same point. The process of set-up is a cumbersome and lengthy process, stressing the concrete pumps.
c) The quality of the ready mixed concrete, which should be pumped up. The need for the separation of this type of project contributes to the fact that there are qualities very difficult to pump up (e.g. super fluid or high cement content), compared with the common types of concrete.

From these criteria we can measure the workload each concrete pump credits for the tasks it serves over the period. This grade can be calculated from the individual criteria listed above, when they scored and then weighted using weights, which can be added up to the unit (for normalization).

Based on the above, it is reasonable for the company to seek for each concrete pump to have a total index of difficulty at the end of planning horizon, which is below an acceptable threshold. Given all the above assumptions, we have essentially a transportation problem (i.e., assignment of work with a bid higher than the unit and with demand equal to the unit for each task per day) for each task. The problem we face is a two-

dimensional problem, meaning that there is a crew-machine agent. Suppose we want to build an agenda for a predetermined period of time and in addition, the number of tasks is N (i, j = 1, 2, ..., N) and the available crews - concrete pumps are K (k = 1, ..., K). Each task is completed only once in a given period.

## 2.1 Theoretical foundation of the VRPTW

The vehicle routing problem can be regarded as a generalization of the traveling salesman problem as first introduced by Dantzig et al. (1959), which in its simplest form seek to determine the shortest route which passes through n given points. The problem of routing with time windows, can be expressed in the form of a graph G (V, A), having a set of nodes V and a set of arcs A as proposed by Cook et al. (1999). Customers are presented as C, denoted by the numbers of nodes 1 to n, while the Warehouse (Depot) is presented in two separate nodes the 0 and n +1. The arrows A represent the dynamic part of the paths between clients and depot. Each arc $A_{ij}$ is relevant is the cost of travel and travel time $t_{ij}$. Node 0 includes only outgoing arcs, node n + 1 includes only incoming arcs. The route for each vehicle is given a directional path in the graph G has a start node 0 and node end node n+1, serving a specific subset of customers who visited along the route. Each vehicle has a loading capacity q and there is a fixed demand for each customer C shown as $d_i$. The total quantity of product to customers served by a particular route can't exceed the total capacity q of the vehicle. The service of each customer i must be commenced within the time window $[a_i, b_i]$, where $a_i$, is the earliest time and $b_i$ the latest time at which the service can start. A vehicle may reach the point of servicing customer i before time a window of time, but then must wait until they reach the years of service $a_i$. Time window exists for the Warehouse (DEPOT) which states that all vehicles must leave the node during this period $[a_0, b_0]$, and should return within the time window $[a_{n+1}, b_{n+1}]$. Given a fleet of T vehicles, the objective is to minimize the total trip cost a total of T paths that meet the requirements of each customer service, which belongs to C. Since the model does not put limits on the routes based on the total time, we can assume without loss of generality that $a_0 = b_0 = a_{n+1} = b_{n+1} = 0$ as proposed by Cook et al. (1999). Also assume that all data, q, $d_i$, $c_{ij}$, $t_{ij}$, $a_i$ and $b_i$, are positive integers. For technical reasons, also all $t_{ij}$ must have a positive value as mentioned by Koln et al. (1997).
Based to what the Koln et al. (1997) mentioned, the triangular inequality, $c_{ij} \leq c_{ih} + c_{hj}$ and $t_{ij} \leq t_{ih} + t_{hj}$, for each h, i, $j \in N$, need not to be met. It is possible to add any scalar at all arc costs $c_{ij}$ without changing the optimal solution, so any instance of VRPTW can be transformed to an instance that satisfies the triangle inequality in cost. This is not true for the time triangle inequality, but it seems that most applications will satisfy this property. This is especially true if $t_i$, includes a service time at i. Koln et al. (1997). The model includes two types of decision variables. The decision variable $x_{ijk}$ (defined for each i, j N, $k \in V$, $i \neq j$, $i \neq n+1$ and $j \neq 0$) equals 1 if the vehicle k, $k \in V$, serving the customer i, then the customer j, and equals 0 otherwise. The decision variable $s_{ik}$ (defined for each i N and each k V) indicates the time a vehicle k, $k \in V$, starts serving the customer i, $i \in C$. If vehicle k does not serve the customer i, its $s_{ik}$, mean nothing. We also assume that $s_{0k} = 0$, for every k, while the $s_{n+1, k}$, indicates the arrival time of vehicle k at the Warehouse. The goal is to design a set of minimum cost paths, one for each vehicle, so that all customers are serviced. The service must be feasible in relation to the carrying capacity of goods vehicles, and windows of time customers served as mentioned by Koln et al. (1997).
An excellent survey on vehicle routing problem with time windows is provided by Solomon et al. (1988).

## 2.2 Mathematical formulation of the suggested model

In this paragraph we include the notation and expressions used in the model. We present a mixed -integer formulation of the vehicle routing problem with time window constraints. Our formulation of the problem is based upon the mathematical formulation of the model used by Kohl et al. (1997), in the analysis of the VRPTW.
The following notation will help in the description of the model. In the mixed integer formulation we use the indices i,j=1,...,N and k=1,...,K.

### 2.2.1 The basic constraints and variables of the model

Parameters:

$V$ = total number of concrete pumps.
$N$ = total number of tasks (0 and N+1 denotes the central starting and finishing point respectively).
$k$ = the symbol for a concrete pump.
$i$ = the symbol for a task.
$t_{ij}$ = total time between customer i and j. It includes the travel time from customer i to customer j.
$m_i$ = total quantity of concrete that must be pumped in the task $i$.
$a_i$ = earliest time allowed for delivery to customer $i$.
$b_i$ = latest time allowed for delivery to customer $i$.
$a_0$ = earliest time allowed for delivery to central starting point before start.
$b_0$ = latest time allowed for delivery to central starting point before start.
$a_n$ = earliest time allowed for delivery to central finishing point after finish.
$b_n$= latest time allowed for delivery to central finishing point after finish.
$c_{ijk}$ = the coefficient that is credited to concrete pump k for assignment in task j after task i.
$\Pi_i$ = the sub total of the concrete pumps that satisfy the needs of the task i.
$S_{ik}$ = the earliest time the vehicle k can start task i.
K = a very big positive number.
$G_i$ = the factor of significance of the client of task i. The factor takes values from 0 to 5.

**Variables:**

$$X_{ijk} = \begin{cases} 1, \text{if the task j is assigned to concrete pump k directly after task i} \\ 0, \text{otherwise} \end{cases}$$

Constraint set (1) states that each task must be assigned to exactly one concrete pump. In this constraint set the factor kϵΠ suggests that the concrete pump that is selected by the model belongs to a subset with the total suitable concrete pumps from the total V regarding the technical specifications.

$$\sum_{j\in N}\sum_{k\in\Pi} X_{ijk} = 1, \forall i \in N \quad (1)$$

Constraint set (2) states that each concrete pump will leave exactly once from the central starting point during the time period of the modeling.

$$\sum_{j\in N} X_{0jk} = 1, \forall k \in V \quad (2)$$

Constraint set (3) states that the concrete pump k will leave task i, if and only if the task i is assigned to this concrete pump.

$$\sum_{i\in N} X_{ihk} - \sum_{j\in N} X_{hjk} = 0, \forall h \in N, \forall k \in V \quad (3)$$

Constraint set (4) states that the concrete pump k will enter the node n+1, which is the central finish point, after the completion of the tasks that is assigned to it.

$$\sum_{i\in N} X_{i,n+1,k} = 1, \forall k \in V \quad (4)$$

Constraint set (5) states that the concrete pump k can't reach the node j before the time $S_{ik}+t_{ij}$ if the task j is assigned to it after the task i. The factor K is a very big positive number. Desrosiers et al. (1995) introduce the nonlinear formulation $x_{ijk}(s_{jk} - s_{ik} - t_{ij}) \le 0$ of constraint set (5). In their model the constraint set (8) is not necessary because there will always exist an optimal solution to the continuous relaxation of the problem which is integer, but since the feasible region is nonconvex, this formulation is not useful in practical computations. Respectively the constraint set (6) states that the time windows for each task are followed.

$$S_{ik} + t_{ij} - K*(1 - X_{ijk}) \le S_{jk} \quad (5)$$

$$a_i \le S_{ik} \le b_i, \forall i \in N, \forall k \in V \quad (6)$$

Constraint set (7) states that the total effort of the concrete pump k doesn't exceed the maximum allowed effort value Wmax.

$$\sum_{j\in N} c_{ijk} * X_{ijk} \le W_{max}, \forall i \in N, \forall k \in V \qquad (7)$$

Finally the constraint set (8) states that the variable X can take only the binary values 0 and 1.

$$X_{ijk} \in \{0,1\}, \forall i \in N, \forall j \in N, \forall k \in V \qquad (8)$$

### 2.2.2 The first sub-model. Minimization of the total cost for concrete pumping operations

The objective of the sub model assigns the tasks to the concrete pumps in a way so as to minimize the total cost of the enterprise. The objective is

$$\text{Minimize } z = \sum_{i\in N}\sum_{j\in N}\sum_{k\in S} m_j * X_{ijk} \qquad (9)$$

The assignments are made so as all the restrictions of the basic model are followed (constraint set (1) – (8)).

### 2.2.3 The second sub-model. Maximization of the satisfaction of the clients

The objective of the second sub model assigns the tasks to the concrete pumps in a way so as to maximize the total satisfaction of the clients of the enterprise. The objective is

$$\text{Maximize } z = \sum_{j\in N} X_{ijk} G_j m_j \qquad (10)$$

The assignments are made so as all the restrictions of the basic model are followed (constraint set (1) – (8)). But except for the constraints in the basic model one more constraint set is used. This is the constraints set (11) that sets the maximum cost the enterprise want to have in order to earn the maximum degree of satisfaction for its clients.

$$\sum_{i\in N}\sum_{j\in N}\sum_{k\in S} m_j * X_{ijk} \le P_{COST} \qquad (11)$$

### 2.2.4 The third sub-model. Minimization of the total difference between the start time of the tasks and the desired one

The objective of the third sub model assigns the tasks to the concrete pumps in a way so as to minimize the total difference between the start time of the tasks and the desired one. The objective of the sub model is

$$\text{Minimize } z = \sum_{i\in N}\sum_{j\in N}\sum_{k\in N} X_{ijk} * \left(S_{ik} + t_{ij} - S_{jk}\right) \qquad (12)$$

The assignments are made so as all the restrictions of the basic model are followed (constraint set (1) – (8)). But except for the constraints in the basic model one more constraint set is used in this sub model. This is the constraints set (11) that sets the maximum cost the enterprise want to have in order to earn the minimum total difference between the start time of the tasks and the desired ones.

## 2.3 Implementation of the first sub-model in LINGO 6.0

This paragraph includes one instance of sub-model 1 built in LINGO 6.0 (LINDO SYSTEMS Inc). The model contains 31.200 binary variables, 40 continuous variables and 904 constraints. It is given below in Figure 1:

Figure 1. Instance of sub-model 1 in LINGO 6.0

```
SETS:
CONCRETE_PUMP:BOOMC,CPC,TDIFFACT_K,VOLUME_K;
! BOOMC = The length of the boom of the concrete pump.;
```

```
! CPC = Cost per m3 concrete that is pumped.;
!TDIFFACT_K = Total difficulty of concrete pump in the period;
!VOLUME_K = Total volume a concrete pump, pumps through period;

TASK:VOLUME,STIME,TDURATION,BOOME,DIFFACT1,DIFFACT2,DIFFACT3;
!VOLUME = The quantity of the concrete (cubic meters) for a task to be pumped in order
to be finished;
! STIME = The time that a concrete pump can start working on a task.;
! TDURATION = The time a task needs in order to be finished.;
! BOOME = The boom length the concrete pump must have in order to be suitable for the task.;
! DIFFACT1 = The difficulty factor a concrete pump credits for reaching the task.;
! The 0 shows small difficulty and 25 shows the greatest difficulty.;
! DIFFACT2 = The difficulty factor a concrete pump credits for the different setups on the task. The 0 shows the smallest
difficulty (1 setup) and 25 shows the biggest difficulty.(above 5 setups);
! DIFFACT3 = The difficulty factor a concrete pump credits for pumping the type of concrete. The 0 shows the smallest
difficulty (Easily pumped concrete)and 25 shows the biggest difficulty.(Difficult pumped concrete);

TXC(TASK,CONCRETE_PUMP):;
TXT(TASK,TASK):;
TXTXC(TASK,TASK,CONCRETE_PUMP):X;

ENDSETS

DATA:
CONCRETE_PUMP = A8 A10 AK AT AL BK BS A3 C4 D5 G7 ML PL GF BV ER LR TR DF A9;
!A concrete pump is written with boom length 5 when it cannot work during the period;
BOOMC = 36 32 36 36 36 42 42 25 44 36 36 36 24 36 44 36 5 5 5 5;
CPC = 3.73 2.65 2.77 2.77 3.73 3.01 3.01 2.65 3.01 2.77 2.77 3.73 3.73 5.43 5.85 3.73 3.01 2.77 2.77 2.77;
TASK = DEPOT T1 T2 T3 T4 T5 T6 T7 T8 T9 T10 T11 T12 T13 T14 T15 T16 T17 T18 T19 T20 T21 T22 T23 T24 T25 T26 T27 T28 T29
T30 T31 T32 T33 T34 T35 T36 T37 T38 T39;
!DEPOT = The place from where the concrete pumps start at the beginning of the period;
!T1-T39 = The tasks of the clients;
STIME = 0 9 11 14 17 8 13 14 11 12.5 15 9 12 14.5 17 7 15 7 13.5 15.5 7 9 11 14 9 12.5 15 9.5 8.5 10.5 15.5 10 12 15.5 9 9
12 11 13.5 15.5;
TDURATION = 0 1 2 1 2 2 2 3 1 2.5 3 2 2 1.5 1 7 4 6 2.5 2.5 5 1 2.5 1.5 3 2.5 2 2.5 1.5 4.5 1.5 1 3 2 2 4 3 2.5 1.5 0.5;
VOLUME = 0 24 18 24 26 63 41 34 30 24 20 12 32 23 16 55 44 88 8 10 46 5 16 7 32 25.5 18 16 5 60 14 24 34.5 40 30 64 30 30
24 20;
BOOME = 0 36 36 36 36 32 32 36 36 36 36 36 36 36 36 36 42 42 42 42 25 25 25 44 44 44 36 36 36 36 36 36 36 24 36 44 36
36 36;
DIFFACT1 = 0 5 10 15 20 15 10 5 10 15 20 25 15 10 15 25 15 10 20 5 10 15 20 15 10 5 15 20 25 20 15 20 15 10 5 15 20 25 20
15;
DIFFACT2 = 0 10 15 20 25 5 15 15 15 15 20 5 10 15 20 10 15 20 25 10 5 15 15 20 10 10 10 15 10 20 10 15 20 10 10 10 15 10
20 15;
DIFFACT3 = 0 10 10 10 5 10 15 20 20 15 10 5 10 15 20 25 15 15 10 20 25 10 5 10 15 15 15 20 10 15 10 5 10 15 15 15 20 10 15
10;
ENDDATA
!-------------------------------------------------------------------;
!Minimize the cost of concrete pumping, that is volume * Cost Per Cubic. There is a minimum cost for 30 cubic meter per
task;
MIN=@SUM(TXTXC(I,J,K)|J#GT#1      #AND#      VOLUME(J)#LT#30:30*CPC(K)*X(I,J,K))+      @SUM(TXTXC(I,J,K)|J#GT#1      #AND#
VOLUME(J)#GE#30:VOLUME(J)*CPC(K)*X(I,J,K));

!Make X binary;
@FOR(TXTXC:@BIN(X););

!A concrete pump when finishes a task can't go to the same task.;
@FOR( TASK( I):
          @FOR( CONCRETE_PUMP( K):
          X( I, I, K) = 0;
          );
);

! A suitable concrete pump must enter a task (except depot) inside the time window.;
@FOR(TASK( J)|J#GT#1:
          @SUM(TXC( I, K)| I #NE# J #AND# BOOMC(K)#GE# BOOME(J)#AND# STIME(I)+TDURATION(I)
          #LE#STIME(J): X( I, J, K))= 1;
);

! A concrete pump must leave the task (except depot) after service.;
@FOR(TXC( H, K)|H #GT# 1:
          @SUM(TASK( I)|I#NE#H:X( I, H, K))=@SUM(TASK(J)|J#NE#H:X( H, J, K));
);

! A concrete pump must enter only once the depot.;
@FOR(CONCRETE_PUMP(K):
          @SUM(TXT( I, J)|J#EQ#1 #AND# I #GT# 1 #AND# I #NE# J:X( I, J, K)) <= 1;
);

! The total volume of concrete pump of all tasks (except depot);
TVOLUME=@SUM(TASK(J)|J#GT#1:VOLUME(J));

! The total difficulty (DIFFACT1) of all tasks (except depot).;
TDIFFACTTOT1=@SUM(TASK(J)|J#GT#1:DIFFACT1(J));

! The total difficulty (DIFFACT2) of all tasks (except depot).;
TDIFFACTTOT2=@SUM(TASK(J)|J#GT#1:DIFFACT2(J));

! The total difficulty (DIFFACT3) of all tasks (except depot).;
TDIFFACTTOT3=@SUM(TASK(J)|J#GT#1:DIFFACT3(J));

! The weighted total difficulty of all tasks.;
TDIFFACTTOT=0.3*TDIFFACTTOT1+0.4*TDIFFACTTOT2+0.3*TDIFFACTTOT3;
```

```
!For each concrete pump with boom length bigger than or equal 24;
@FOR(CONCRETE_PUMP(K)|BOOMC(K)#GE#24:

!The total volume a concrete pump, pumps during the period;
VOLUME_K(K)=@SUM(TXTXC(I,J,K)|J#GT#1 #AND# I#NE#J:X(I,J,K)*VOLUME(J));

!The total difficulty for a concrete pump during the period;
TDIFFACT_K(K)=@SUM(TXTXC(I,J,K)|J#GT#1 #AND# I#NE#J:X(I,J,K)*
(0.3*DIFFACT1(J)+0.4*DIFFACT2(J)+0.3*DIFFACT3(J)));

! Each concrete pump must pump at least the 1/50 of the whole volume of concrete of the period.;
VOLUME_K(K)>=0.02*TVOLUME;

! The total factor of difficulty for each concrete pump must be at most 30% the total difficulty of all tasks.;
TDIFFACT_K(K) <= 30/100*TDIFFACTTOT;
);
END
```

# 3. CONCLUDING REMARKS

Throughout the previous analysis, our purpose was to identify the theoretical background, figure out the mathematical formulation and introduce three sub models for the problem of optimal allocation of tasks in concrete pumping operations in the concrete industry. Relevant enterprises may use each model, considering how it will serve best its current interests. If the company has enough equipment so as to serve all the customers, it can use the first sub model to minimize the cost of customer service. The task scheduling, following the optimal solution will save valuable resources. However, the model has a weakness regarding the number of available concrete pumps. If the number of pumps is not large enough so as to serve all tasks within the time window, this sub model fails to give a feasible solution. Of course it is extremely difficult in advance to determine if the number of available pumps is the crucial in order for the problem to give at least one feasible (hence optimal) solution. The reason is that the required pumps are directly linked to projects which should be serviced throughout the period.

If the number of pumps is not big enough so as to be able to have at least a feasible solution, the company should choose to use one of the other two sub models. It can choose the second sub model if the company believes that it is acceptable to reject servicing some tasks (risk of losing customers from competitors). In this case the model will provide a solution, that chooses the tasks with the larger quantities (the highest income for the company), while taking into account the criticality of the customer for the company. But if the company considers that it would cause great discontent among customers who are not included in the program of the period examined, then they may use the third sub model. In this case the optimal solution will minimize the total delay in the tasks. It is assumed that less frustration will be caused to the customers if the company expresses its inability to service under the original time window in advance. Through this "negotiation" it can be found an alternative second time window, which will also satisfy the customers.

These models will help enterprises to service their clients, choosing the best solution regarding current circumstances. Through this automated process, significant valuable sources can be saved, resulting in efficiency and productivity increase. The optimization of the distribution of products is a great opportunity for enterprises to improve their position on the chessboard of competition in the concrete industry.

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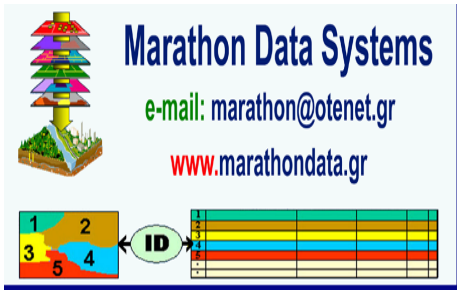




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